Parametric Inference for Inverted Exponentiated Rayleigh Distribution under General Progressive Type II Censoring Scheme

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Abstract:

A parametric inference of point and interval estimates for inverted exponentiated Rayleigh distribution with general progressive Type-II censoring scheme is studied using both classical and Bayesian estimation. The maximum likelihood method is used for estimating the unknown two-parameter inverted exponentiated Rayleigh distribution and some lifetime parameters such as survival and hazard rate functions in presence of general progressive Type-II censoring sampling. The approximate confidence intervals through Fisher information matrix, highest posterior density credible intervals for the two-parameter inverted exponentiated Rayleigh distribution, and any function of them, are constructed. In addition, Bayesian estimates of the unknown parameters are computed using two different loss functions squared error loss and linear exponential loss functions. A Markov Chain Monte Carlo method technique is applied to carry out Bayesian estimation procedure. Finally, a numerical simulation is carried out to assess the performance of the proposed methods and real data is analyzed using the suggested inference methods.

Keywords: General progressive Type II censoring scheme, Inverted exponentiated Rayleigh distribution, Maximum likelihood estimation, Bayesian estimation, Markov Chain Monte Carlo.

1 Introduction

Statistical analysis plays a significant role in practical applications such as engineering, medical, economic, agricultural, and other fields. No scientific study is complete without performing statistical analysis to support the obtained results. In practice, there are challenges in obtaining the life times of all units in an experiment for several reasons, including the long lifespan of the units, financial costs, and the loss of some experimental units due to incidental causes. Thus, the concept of making statistical inferences using incomplete samples has become a very important topic in statistics in general, and particularly in life studies.

There are many strategies in reliability studies and life tests to address such challenges, such as the censoring data for units under natural conditions. Censoring can be applied based on a prefixed time or prefixed number of failures or sometimes a combination of both prefixed time and number of failures. Depending on these criteria, there are different types of censoring schemes. Among the most common censoring systems in practical experiments are Type-I censoring and Type-II censoring. Both Type-I and Type-II censoring schemes have certain disadvantages. In Type-I censoring, may get very few failures or the number of failures may be zero, while in Type-II censoring, the experimental time may be very large. Therefore, estimating the unknown parameters can not be done efficiently, Lawless (2011). To enable the experiment designer to remove some units from the life test at a specific time during the experiment, aiming to accelerate the test or reduce time and costs, another type of censoring was introduced, known as progressive censoring, such that the experiment designer can remove some units at pre-specified time points (progressive Type-I) or remove units at each failure (progressive Type-II) But there is a major drawback of progressive Type-II censoring scheme that it may take a large time to complete an experiment.

General progressive Type-II censoring (GPTIIC) scheme is a useful and more general scheme in which a specific fraction of individuals at risk may be removed from the study at each of several ordered failure times. It generalizes the progressive censoring scheme. This scheme allows removing surviving or live units from the experiment at various stages is an attractive feature as it will potentially save a lot for the experiment designer in terms of cost and time. This scheme of censoring was generalized by Balakrishnan and Sandhu (1996) as follows: At time zero, n randomly selected units are placed on a life test; the first r failure times, $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ are not observed; at the time of $(r+1)-th$, R_{r+1} units of surviving units are removed from the test randomly; at the time of $(r+2)-th$, R_{r+2} units of $(n-2-r-R_{r+1})$ surviving units are removed from the test randomly and so on. Finally, at the time of the m-th failure, the test is terminated and the remaining R_m $(R_m = n-m-R_{r+1}-R_{r+2}-\ldots-R_{m-1})$ surviving units are removed from the test. Therefore, $x_{(r+1)} \leq x_{(r+2)} \leq \ldots \leq x_{(m)}$ are the lifetimes of completely observed units to fail, and $R_{r+1}, R_{r+2}, \ldots, R_m$ are the numbers of units removed from the test at these failure times. The R_i's, m and r are pre-specified integers which must satisfy the conditions: $0 \leq r < m \leq n, 0 \leq R_i \leq n-i$, for $i = r+1, \ldots, m-1$. The resulting $(m-r)$ ordered values $x_{(r+1)}, x_{(r+2)}, \ldots, x_{(m)}$ are referred to as GPTIIC. The likelihood function, in this case, is given by

$$L(\theta) = C[F(x_{(r+1)}]^r \prod_{i=r+1}^{m} f(x_{(i)})[1 - F(x_{(i)})]^{R_i}, \quad \textbf{(1)}$$

where

$$C = \binom{n}{r}(n-r)\prod_{j=r+2}^{m}(n - \sum_{i=r+1}^{j-1} R_i - j + 1),$$

C is a constant which doesn't depend on parameters.

There are some special cases for which the GPTIIC can reduces to other censoring types If $r = 0$, $R_i > 0, i = r+1, \ldots, m-1$, then $m \leq n$, the GPTIIC scheme reduces to the progressive Type-II censoring scheme. If $r = 0$ and $R_i = 0$, for $i = r+1, \ldots, m-1$, then $R_m = n - m$, this scheme is reduces to conventional Type-II censoring scheme. If $r = 0$ and $R_i = 0$, for $i = r+1, \ldots, m$, then $m = n$, this scheme is reduces to the case of no censoring (complete sample case), where all n usual are observed. If $r \neq 0$ and $R_i = 0$, for $i = r+1, \ldots, m-1$, then $R_m = n - m$, which corresponds to the case of Type-II doubly censored sample.

Many authors have been discussed inference under GPTIIC scheme using different lifetime distributions, among others, Kang and Cho (1997) obtained the approximate maximum likelihood estimator (AMLE) of the scale parameter of the one parameter exponential distribution under GPTIIC sample. Fernandez (2004) derived the maximum likelihood (ML) estimation and Bayesian estimation for exponential parameters with GPTIIC. Estimation problems for Burr XII and Rayleigh distributions were studied in Kim (2006) and Kim and Han (2009) using GPTIIC schemes. Abd-Elrahman and Sultan (2007) discussed the problem of estimating the parameters and reliability function of the two-parameter Weibull model on the basis of a GPTIIC sample by using ML estimation and Bayesian approaches. Soliman (2008) derived the maximum likelihood estimators (MLEs) and Bayesian estimators of the parameters as well as some survival time parameters for Pareto model based on GPTIIC scheme. More details on GPTIIC can be found in El-Din et al. (2014), Jang et al. (2014), Xiuyun and Zaizai (2016), Abu-Moussa and El-Din (2018), Ma and Gui (2019), Wang and Gui (2021) and Lv et al. (2024).

This paper, discusses the problem of parameters estimation using inverted exponentiated Rayleigh (IER) distribution under GPTIIC scheme. This distribution was introduced in the literature by Ghitany et al. (2014) and it is a particular member of a general class of inverse exponentiated distribution. In reliability and life testing, IER distribution finds wide application in analyzing data from experiments across various fields, such as physics, medicine, biology, and engineering science. The probability density function (PDF), cumulative distribution function (CDF), reliability function (SF), and hazard rate function (HF) of a lifetime random variable X has IER distribution with shape parameter ϑ and scale parameter θ are given, respectively, by

$$f(x;\vartheta,\theta) = 2\vartheta\theta x^{-3} e^{-\frac{\theta}{x^2}}(1 - e^{-\frac{\theta}{x^2}})^{\vartheta-1}, \quad x > 0, \quad \vartheta,\theta > 0, \quad \textbf{(2)}$$

$$F(x;\vartheta,\theta) = 1 - (1 - e^{-\frac{\theta}{x^2}})^{\vartheta}, \quad x > 0, \quad \vartheta,\theta > 0, \quad \textbf{(3)}$$

$$S(x;\vartheta,\theta) = (1 - e^{-\frac{\theta}{x^2}})^{\vartheta}, \quad x > 0, \quad \vartheta,\theta > 0, \quad \textbf{(4)}$$

and

$$H(x;\vartheta,\theta) = 2\vartheta\theta x^{-3} e^{-\frac{\theta}{x^2}}(1 - e^{-\frac{\theta}{x^2}})^{-1}, \quad x > 0, \quad \vartheta,\theta > 0. \quad \textbf{(5)}$$

The plots of PDF and HF with various values of parameters are shown in Figure (1).

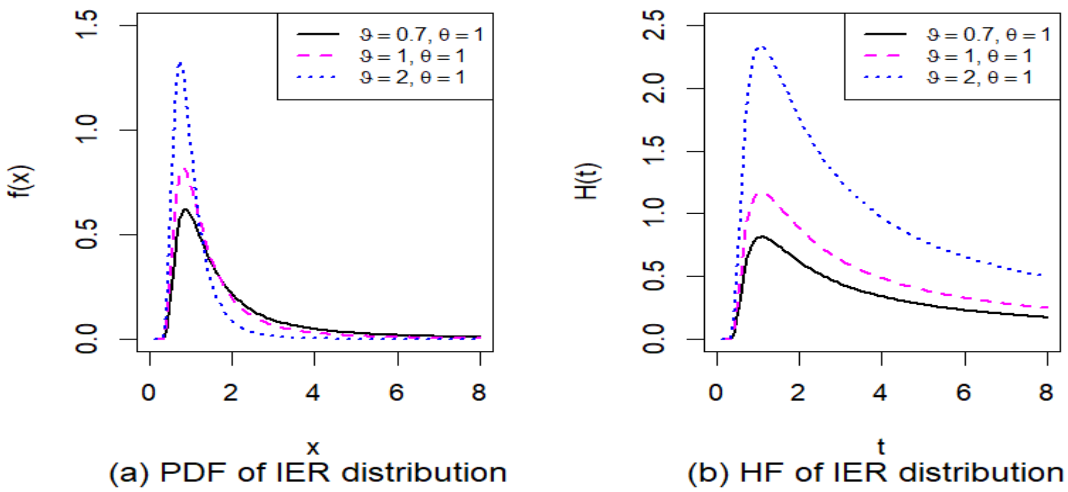

Figure 1. Plots of PDF and HF of IER distribution.

This distribution exhibits behavior similar to several well-known models, such as the log-normal distribution, the inverse Weibull distribution, and the generalized inverse exponential distribution. There are several studies presented by many researchers focusing on estimating the parameters of IER distribution when observations are complete or censored. For example, Rastogi and Tripathi (2014) used progressive Type-II censoring to compute the ML and Bayesian estimators for the unknown parameters as well as the reliability and hazard functions of IER distribution and also constructed interval estimation for the unknown parameters. Kohansal (2017) derived the MLEs, Bayesian estimators and some confidence intervals for stress-strength reliability of IER distribution based on progressive Type-II censoring scheme. Kayal et al. (2018) derived the ML estimates for the parameters of IER distribution under hybrid Type-I censored using the Expectation–Maximization (EM) algorithm and Bayesian estimations as well as obtained the predictive estimates and prediction intervals of censored observations under hybrid Type-I censored. Maurya et al. (2019a) and Maurya et al. (2019b) derived different point and interval estimators for the unknown parameters of IER distribution using ML and Bayesian methods of estimation as well as obtained prediction estimates and prediction intervals of censored observations in classical and Bayesian context based on progressively first-failure censoring and progressive Type II censoring schemes, respectively. Gao and Gui (2019) obtained the ML and Bayesian estimators for the unknown parameters of IER distribution and also constructed interval estimation for these parameters in the context of progressively first-failure censoring. Gao et al. (2020) proposed the pivotal inference methods to estimate the two unknown parameters of IER distribution based on progressively censored scheme. Panahi and Moradi (2020) studied IER distribution under adaptive progressively hybrid censoring and derived ML and Bayesian estimates of the unknown parameters. Mahto and Tripathi (2020) estimated the multicomponent stress-strength reliability for IER distribution under progressive Type II censoring. Fan and Gui (2022) conducted a study on the statistical inference of IER distribution, utilizing joint progressively Type-II censoring and derived the ML estimates using EM algorithm, Bayesian estimates and some confidence intervals of the unknown parameters. More papers have discussed IER distribution, such as Anwar et al. (2023), Chalabi (2023), Elshahhat et al. (2023), Hashem et al. (2023), Wang et al. (2024), Tashkandy et al. (2024), and Lodhi et al. (2024).

In this paper, estimation problems for the two-parameter IER distribution are discussed using GPTIIC and is organized as follows: In section 2, the maximum likelihood estimation and approximate confidence intervals are presented. In section 3 Bayesian estimates are obtained under symmetric loss functions (squared error (SE)) and asymmetric loss functions (linear exponential (LINEX)) using Markov Chain Monte Carlo (MCMC) technique and the highest posterior density (HPD) credible intervals are provided. In section 4, A Monte Carlo simulation study is carried out to compare the performance of different estimates. In section 5, A real data set is introduced and analyzed to investigate the model. Finally, conclusions of this paper have been drawn in section 6.

2 Maximum Likelihood Method

Let $(x_{(r+1)},\dots,x_{(m)})$ denotes a GPTIIC sample from IER distribution with censoring scheme $(R_{(r+1)},\dots,R_{(m)})$. Using equations (1), (2) and (3), the likelihood function of ϑ and θ can be written in the following form

$$L(\vartheta,\theta)=C[2\vartheta\theta]^{m-r}[1-y_{(r+1)}^{\vartheta}]^{r}\prod_{i=r+1}^{m}x_{(i)}^{-3}e^{-\theta x_{(i)}^{-2}}y_{(i)}^{(\vartheta(R_i+1)-1)}, \quad (6)$$

where $y_{(r+1)}=(1-e^{-\theta x_{(r+1)}^{-2}})$ and $y_{(i)}=(1-e^{-\theta x_{(i)}^{-2}})$. The nature logarithm of the likelihood function (6), denoted by ℓ, can be derived in the following form

$$\ell=\ln C_3+(m-r)\ln(2)+(m-r)\ln(\vartheta)+(m-r)\ln(\theta+r\ln(-y_{(r+1)}^{\vartheta})$$
$$+\sum_{i=r+1}^{m}[-3\ln(x_{(i)})-\theta x_{(i)}^{-2}+(\vartheta(R_i+1)-1)\ln(y_{(i)})]. \quad \mathbf{(7)}$$

To compute the MLEs of ϑ and θ, we calculate the first partial derivatives of (7) with respect to ϑ and θ and equating to zero as follows

$$\frac{m-r}{\hat{\vartheta}}-\frac{r y_{(r+1)}^{\hat{\vartheta}}\ln(y_{(r+1)})}{1-y_{(r+1)}^{\hat{\vartheta}}}+\sum_{i=r+1}^{m}(R_i+1)\ln(y_{(i)})=0 \quad \mathbf{(8)}$$

and

$$\frac{m-r}{\hat{\theta}}-\frac{r\hat{\vartheta}e^{-\hat{\theta}x_{(r+1)}^{-2}}y_{r+1}^{\vartheta-1}}{x_{(r+1)}^{2}(1-y_{(r+1)}^{\hat{\vartheta}})}-\sum_{i=r+1}^{m}\frac{1}{x_{(i)}^{2}}+\sum_{i=r+1}^{m}(\hat{\vartheta}(1+R_i)-1)\frac{e^{-\hat{\theta}x_{(i)}^{-2}}}{x_{(i)}^{2}y_{(i)}}=0, \mathbf{(9)}$$

It's clear that equations (8) and (9) are not in explicit form, therefore, the estimated values for parameters ϑ and θ can be obtained using numerical techniques. Furthermore, once the estimates $\hat{\vartheta}$ and $\hat{\theta}$ are obtained, using the invariance property of MLE, the MLEs $\hat{S}(t)$ and $\widehat{H}(t)$ of $S(t)$ and $H(t)$ can be obtained, respectively, for a given mission time t, by replacing ϑ and θ with their MLEs $\hat{\vartheta}$ and $\hat{\theta}$ in (4) and (5).

$$\hat{S}(t) = (1 - e^{-\frac{\hat{\theta}}{t^2}})^{\hat{\vartheta}}, \qquad t > 0,$$

and

$$\widehat{H}(t) = 2\hat{\vartheta}\hat{\theta}t^{-3}e^{-\frac{\hat{\theta}}{t^2}}(1 - e^{-\frac{\hat{\theta}}{t^2}})^{-1}, \qquad t > 0.$$

Two-sided approximate confidence intervals (ACIs) for the unknown parameters ϑ and θ of IER distribution are constructed using the asymptotic normal approximation of the MLEs. The variance-covariance matrix is obtained by inverting the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood functions. In the present situation, it seems appropriate to approximate the expected values using their ML estimates, Cohen (1965). Accordingly, the approximate variance covariance matrix is as

$$I^{-1}(\hat{\vartheta}, \hat{\theta}) \cong \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \vartheta^2} & -\frac{\partial^2 \ell}{\partial \vartheta \partial \theta} \\ -\frac{\partial^2 \ell}{\partial \theta \partial \vartheta} & -\frac{\partial^2 \ell}{\partial \theta^2} \end{bmatrix}^{-1}_{(\theta=\hat{\theta}, \vartheta=\hat{\vartheta})} = \begin{bmatrix} \hat{\sigma}^2_{\hat{\vartheta}} & \hat{\sigma}_{\hat{\vartheta},\hat{\theta}} \\ \hat{\sigma}_{\hat{\theta},\hat{\vartheta}} & \hat{\sigma}^2_{\hat{\theta}} \end{bmatrix} \qquad \textbf{(10)}$$

where

$$\frac{\partial^2 \ell}{\partial \vartheta^2} = -\frac{(m-r)}{\vartheta^2} - \frac{r y^{\vartheta}_{(r+1)}(\ln(y_{(r+1)}))^2}{\left(1-y^{\vartheta}_{(r+1)}\right)^2},$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{(m-r)}{\theta^2} - \frac{r\vartheta y_{(r+1)}^{\vartheta-1} e^{-\theta x_{(r+1)}^{-2}}}{x_{(r+1)}^4} [\frac{(\vartheta-1) y_{(r+1)}^{-1} e^{-\theta x_{(r+1)}^{-2}} - 1}{1-y_{(r+1)}^{\vartheta}} + \frac{\vartheta y_{(r+1)}^{\vartheta-1} e^{-\theta x_{(r+1)}^{-2}}}{(1-y_{(r+1)}^{\vartheta})^2}]$$

$$-\sum_{i=r+1}^{m} \frac{(\vartheta(R_i+1)-1) e^{-\theta x_{(i)}^{-2}}}{x_{(i)}^4} \left[\frac{1}{y_{(i)}} + \frac{e^{-\theta x_{(i)}^{-2}}}{y_{(i)}^2}\right],$$

and

$$\frac{\partial^2 \ell}{\partial \vartheta \partial \theta} = \frac{\partial^2 \ell}{\partial \theta \partial \vartheta} = -\frac{r y_{(r+1)}^{\vartheta-1} e^{-\theta x_{(r+1)}^{-2}}}{x_{(r+1)}^2} \left(\frac{\vartheta \ln(y_{(r+1)})}{(1-y_{(r+1)}^{\vartheta})^2} + \frac{1}{\left(1-y_{(r+1)}^{\vartheta}\right)}\right)$$

$$+\sum_{i=r+1}^{m} (R_i+1) \frac{e^{-\theta x_{(i)}^{-2}}}{x_{(i)}^2 y_{(i)}}.$$

The $100(1-\alpha)$ % two-sided ACIs for the two parameter IER distribution ϑ and θ based on GPTIIC, can be constructed as:

$$\hat{\vartheta} \mp z_{\frac{\alpha}{2}} \sqrt{\hat{\sigma}_{\hat{\vartheta}}^2} \quad \text{and} \quad \hat{\theta} \mp z_{\frac{\alpha}{2}} \sqrt{\hat{\sigma}_{\hat{\theta}}^2}$$

where $\hat{\sigma}_{\hat{\vartheta}}^2$ and $\hat{\sigma}_{\hat{\theta}}^2$ are the entries on the main diagonal of the approximate variance covariance matrix (10), and $z_{\frac{\alpha}{2}}$ is the upper percentile of the standard normal distribution. To construct the ACIs of $S(t)$ and $H(t)$ of IER distribution based on GPTIIC, the delta method is considered to obtain the approximate estimates of the variances of $\hat{S}(t)$ and $\widehat{H}(t)$ (Greene (2012)). Hence, according to the delta method, the variances $\sigma_{\hat{S}(t)}^2$ and $\sigma_{\widehat{H}(t)}^2$ of $\hat{S}(t)$ and $\widehat{H}(t)$, can be approximated, respect

$$\hat{\sigma}_{\hat{S}(t)}^2 = [\nabla \hat{S}(t)]^{\mathbf{T}} \mathrm{I}^{-1}(\hat{\vartheta}, \hat{\theta}) [\nabla \hat{S}(t)]$$

and

$$\hat{\sigma}_{\widehat{H}(t)}^2 = [\nabla \widehat{H}(t)]^{\mathbf{T}} \mathrm{I}^{-1}(\hat{\vartheta}, \hat{\theta}) [\nabla \widehat{H}(t)],$$

where $[\nabla \hat{S}(t)]^{\mathbf{T}}$ represents the transpose of $[\nabla \hat{S}(t)]$, $[\nabla \hat{S}(t)]$ and $[\nabla \widehat{H}(t)]$ are, respectively, the gradient (vector of first partial derivatives) of $S(t)$ and $H(t)$ with respect to ϑ and θ, obtained at $\vartheta=\hat{\vartheta}$ and $\theta=\hat{\theta}$,

$[\hat{S}(t)]^{\mathbf{T}}=[\partial S(t)/\partial\vartheta,\partial S(t)/\partial\theta]_{(\vartheta=\hat{\vartheta},\theta=\hat{\theta})}$

and

$[\widehat{H}(t)]^{\mathbf{T}}=[\partial H(t)\partial\vartheta,\partial H(t)\partial\theta]_{(\vartheta=\hat{\vartheta},\theta=\hat{\theta})}$.

Hence, the $100(1-\alpha)\%$ two-sided ACIs of $S(t)$ and $H(t)$, are given, respectively, by

$$\hat{S}(t)\mp z_{\alpha/2}\sqrt{\hat{\sigma}^2_{\hat{S}(t)}} \quad \text{and} \quad \widehat{H}(t)\mp z_{\alpha/2}\sqrt{\hat{\sigma}^2_{\widehat{H}(t)}}$$

where $z_{\alpha/2}$ is the upper percentile of the standard normal distribution.

3 Bayesian Estimation

Bayesian estimations of the two-parameter IER distribution ϑ and θ as well as SF and HF, will be obtained against symmetric and asymmetric loss functions such as SE and LINEX loss functions based on GPTIIC scheme. Assume the prior distributions for the unknown parameters ϑ and θ are independent Gamma distributions as follows

$$\pi(\vartheta)=\frac{b^a}{\Gamma(\vartheta)}\vartheta^{a-1}e^{-b\vartheta}, \quad a,b>0, \tag{11}$$

$$\pi(\theta)=\frac{d^c}{\Gamma(\theta)}\theta^{c-1}e^{-d\theta}, \quad c,d>0, \tag{12}$$

where all the hyper-parameters a,b,c and d, are assumed to be non-negative and known. The joint prior density of the unknown parameters ϑ and θ is given by

$$\pi(\vartheta,\theta)\propto\vartheta^{a-1}e^{-b\vartheta}\theta^{c-1}e^{-d\theta}. \tag{13}$$

Combining (6) and (13), the posterior density of the unknown parameters ϑ and θ can be written as

$$\pi(\vartheta,\theta|\underline{x})=\frac{1}{A^*}L(\vartheta,\theta)\pi(\vartheta,\theta), \tag{14}$$

where A^* is the normalizing constant

$$A^* = \int_0^\infty \int_0^\infty L(\vartheta,\theta)\pi(\vartheta,\theta)d\vartheta d\theta. \quad \textbf{(15)}$$

The selection of a loss function is a crucial aspect of Bayesian estimation methods. Numerous loss functions have been introduced in the literature to represent different types of loss frameworks. Among these, the symmetric SE loss function is one of the most commonly used and is widely applied in inference. Using posterior distribution (14) based on GPTIIC, Bayesian estimators of any function of $\vartheta, \theta, S(t)$ and $H(t)$ respectively, of IER distribution, say $\underline{\theta} = \vartheta, \theta, S(t), H(t)$ under SE loss function is the posterior expectation, as follows

$$\tilde{\underline{\theta}}_{\text{SE}} = E(\underline{\theta}|\underline{x}) = \frac{1}{A^*}\int_0^\infty \int_0^\infty \underline{\theta}\ L(\vartheta,\theta)\pi(\vartheta,\theta)d\vartheta d\theta, \quad \textbf{(16)}$$

where $E(.)$ denotes the posterior expectation. Under LINEX loss function, Bayesian estimators of $\underline{\theta} = \vartheta, \theta, S(t)$, $H(t)$ of IER distribution can be derived as

$$\tilde{\underline{\theta}}_{\text{LINEX}} = -\frac{1}{b^*}\ln(E(e^{-b^*\underline{\theta}}|\underline{x})), b^* \neq 0, \quad \textbf{(17)}$$

where b^* is constant. The sign and magnitude of b^* represent the direction and degree of symmetry.

Since the ratio of multiple integrals given in (16) and (17) cannot be obtained in a simple closed form, therefore, Bayesian estimates of $\vartheta, \theta, S(t)$ and $H(t)$ can be evaluated numerically by using computer facilities and numerical techniques. So, MCMC method is used to obtain an approximate value for these integrals as well as construct HPD credible intervals, for this purpose, the conditional posterior distribution of each parameter is obtained using the posterior distribution. The conditional posteriors are as follows

$$\pi(\vartheta|\theta,\underline{x}) = \sum_{s=0}^{r} \binom{r}{s} (-1)^s \vartheta^{m-r+a-1} e^{-\vartheta(b-s\ln(y_{(r+1)})-\sum_{i=r+1}^{m}(R_i+1)\ln(y_{(i)}))}$$

$$\sim Gamma\ \ Distribution(m-r+a, A_\vartheta), \qquad \textbf{(18)}$$

and

$$\pi(\theta|\vartheta,\underline{x}) = \theta^{m-r+c-1} e^{-\theta(d+\sum_{i=r+1}^{m} x_{(i)}^{-2})} [1-y_{(r+1)}^{\vartheta}]^r$$

$$\prod_{i=r+1}^{m} y_{(i)}^{\vartheta(R_i+1)-1}, \qquad \textbf{(19)}$$

where $A_\vartheta = (b - s\ln(y_{(r+1)}) - \sum_{i=r+1}^{m} (R_i+1)\ln(y_{(i)}))$. From (18), the unknown parameter ϑ follows gamma distribution. Thus, samples of θ can be easily generated using any gamma generating routine. Since the conditional probability (19) is not a well-known distribution, Metropolis–Hastings (M-H) sampler will be used to generate values of θ following the distribution (19). Therefore, M-H algorithm with a normal proposal distribution is applied. Thus, M-H within Gibbs sampling steps are applied to generate random samples from the conditional posterior densities as follows:

Step 1. Start with an initial values $\vartheta^{(0)}$ and $\theta^{(0)}$.

Step 2. Set $J = 1$.

Step 3. Generate $\vartheta^{(J)}$ from $Gamma(m-r+a, A_\vartheta)$.

Step 4. Generate $\theta^{(J)}$ from $\pi(\theta^{(J-1)}|\vartheta^{(J)},\underline{x})$ using the following M-H algorithm:

(a) Generate a proposal θ^* from the proposal distribution (the Normal distribution $N(\theta^{(J-1)}, \sigma_\theta^2)$

(b) Evaluate the acceptance probabilities by

$$D(\theta^{(J-1)}, \theta^*) = \min(1, \frac{\pi(\theta^*|\vartheta^{(J)},\underline{x})}{\pi(\theta^{(J-1)}|\vartheta^{(J)},\underline{x})}),$$

(c) Generate a w from a uniform $U(0,1)$ distribution.

(d) If $w \leq D(\theta^{(J-1)}, \theta^*)$, accept the proposal and set $\theta^{(J)} = \theta^*$, else set $\theta^{(J)} = \theta^{(J-1)}$.

Step 5. Using $\vartheta^{(J)}$ and $\theta^{(J)}$ Bayesian estimators of SF and HF of IER distribution as in (4) and (5), for a given mission time t, are given, respectively, as

$$S^{(J)}(t) = (1 - e^{-\frac{\theta^{(J)}}{x^2}})^{\vartheta^{(J)}}, \qquad t \geq 0$$

and

$$H^{(J)}(t) = 2\vartheta^{(J)}\theta^{(J)}x^{-3}e^{-\frac{\theta^{(J)}}{x^2}}(1 - e^{-\frac{\theta^{(J)}}{x^2}})^{-1}, t \geq 0.$$

Step 6. Set $J = J + 1$.

Step 7. Repeat steps 3-6 for $\mathcal{M}$ times.

Step 8. Removing the first $\mathcal{M}_0$ number of iterative values, Bayesian estimators under SE loss function of $\underline{\theta} = \vartheta, \theta, S(t), H(t)$ are derived as

$$\tilde{\underline{\theta}}_{\text{SE}} = \frac{1}{\mathcal{M} - \mathcal{M}_0} \sum_{J=\mathcal{M}_0+1}^{\mathcal{M}} \underline{\theta}^{(J)}$$

Step 9. Bayesian estimators under LINEX can be computed as

$$\tilde{\underline{\theta}}_{\text{LINEX}} = -\frac{1}{b^*} \ln\left(\frac{1}{\mathcal{M} - \mathcal{M}_0} \sum_{J=\mathcal{M}_0+1}^{\mathcal{M}} e^{-b^* \underline{\theta}^{(J)}}\right)$$

Step 10. To construct the HPD credible intervals of $\underline{\theta} = \vartheta, \theta, S(t), H(t)$ sort the remaining values of $\mathcal{M} - \mathcal{M}_0$ in ascending order to be $\vartheta_{(1)}, \vartheta_{(2)}, \ldots, \vartheta_{(\mathcal{M}-\mathcal{M}_0)}, \theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(\mathcal{M}-\mathcal{M}_0)}, S_{(1)}(t), S_{(2)}(t), \ldots, S_{(\mathcal{M}-\mathcal{M}_0)}(t)$, and $H_{(1)}(t), H_{(2)}(t), \ldots, H_{(\mathcal{M}-\mathcal{M}_0)}(t)$. The $100(1-\alpha)\%$ HPD credible interval for the unknown parameters or any function of them as in $\underline{\theta}$, is given by

$$\left(\underline{\theta}_{(\epsilon^*)}, \underline{\theta}_{(\epsilon^* + \mathcal{M}^*)}\right), \quad \epsilon^* = 1, 2, \ldots, (\mathcal{M} - \mathcal{M}_0) - \mathcal{M}^*,$$

where $\mathcal{M}^* = [(1-\alpha) \times (\mathcal{M} - \mathcal{M}_0)]$ and ϵ^* is selected when the following equation is satisfied:

$$\underline{\theta}_{(\epsilon^*+\mathcal{M}^*)} - \underline{\theta}_{(\epsilon^*)} = \min_{1\leq \epsilon\leq(\mathcal{M}-\mathcal{M}_0)-\mathcal{M}^*} (\underline{\theta}_{(\epsilon+\mathcal{M}^*)} - \underline{\theta}_{(\epsilon)}).$$

Then, following Chen and Shao (1999), the HPD credible intervals of $\underline{\theta}$ can be obtained by choosing the interval which has the shortest length.

4 Numerical Applications

The aim is to evaluate the performance of the different estimation methods discussed in the previous sections. Monte Carlo simulation is conducted to examine the behavior of the proposed methods and assess the statistical performance of the estimators under GPTIIC. All calculations will be carried out using $\mathcal{R}$ programming language, version 4.4.1.

To study the performance of our approach, 1,000 GPTIIC samples from IER distribution are simulated with the following assumptions:

1. Assume the following selected values of parameters of IER distribution: $(\vartheta, \theta) = (1.5,1)$.

2. Different values of (n, m, r) are selected such as: $n = 20$(small), 50(medium) and 100 (large) and $m = 13,15,35,45,50,$ and 70, while $r = 2,4,5,10,15$ and 20.

3. Assume that the following censoring schemes (C.S) will be used to remove the remaining units:

C.S I: $R = (n - m, 0^*(m - r - 1))$,

C.S II: $R = (0^*(\frac{m-r-1}{2}), n - m, 0^*(\frac{m-r-1}{2}))$,

C.S III: $R = (0^*(m - r - 1), n - m)$,

where $(0^*(m - r - 1))$ means that 0 is repeated $(m - r - 1)$ times.

Based on the generated data, ML estimates and associated 95% ACIs are computed. For Bayesian estimation method, Bayesian estimates using a strategy combining M-H within Gibbs algorithm is used to generate 10,000 MCMC samples, discarding the first 2,000 samples as (burn-in period). Based on the remaining 8,000 samples, Bayesian estimates using SE and LINEX loss functions (with $b^* = \mp 0.002$) and 95% HPD credible intervals are computed. Accordingly hyper parameters of gamma prior for Bayesian estimation are assigned as $a = b = c = d = 0.0001$. The process is replicated 1,000 times to compute average estimates (means) and average length of confidence intervals. The mean squared errors (MSEs) are also computed to compare the performance of the ML and Bayesian estimates in Monte Carlo simulation study. Further, the average interval lengths (AILs) and coverage probabilities (CPs) of 95% ACIs/HPD credible intervals are computed to compare the performance of the interval estimates.

Further, the ML and Bayesian estimates of the SF and HF are obtained where the corresponding true values of the survival parameters $S(t)$ and $H(t)$ for specified time $t = 0.80$, are taken as $S(0.80) = 0.7026855$ and $H(0.80) = 1.553909$, respectively. The average ML and Bayesian estimates of $\vartheta, \theta, S(t)$ and $H(t)$ are presented in Table (1). In addition, the AILs and CPs of $\vartheta, \theta, S(t)$ and $H(t)$ are listed in Table (2). The following conclusions are drawn based on the numerical results.

• The performance of Bayesian estimates for the parameters under SE and LINEX are better than those based on the MLEs.

• When n and m increase, the MSE of all estimations decreases.

• For fixed n and r as m increases the MSEs of ML and Bayesian estimates decrease.

• For fixed n, m and r, in sence of MSEs Scheme I is smaller than Scheme

II and III while for $S(t)$, Scheme III is smaller than Scheme I and II.

• When $m-r$ and n increase, ML estimates become notably more accurate, where $m-r$ and n represent the sizes of the observed and complete samples, respectively.

• In most cases, Bayesian estimates of all unknown parameters under LINEX loss function with $b^* = +0.002$ have the lowst MSE values among all the various estimates.

• It is observed that HPD credible intervals have shorter average lengths than ACIs and according to CPs the ACIs have better performance.

• Based on the obtained results in this study and because of the need to deal with small samples in life testing, we recommend to use Bayesian estimators in place of ML estimators.

Table 1: **Average estimates values and mean square error of the ML and Bayesian estimates based on GPTIIC Schemes at different values of (n, m, r) for $\vartheta = 1.5, \theta = 1$.**

n(m,r)	C.S	Par	MLE		SE		LINEX			
							$b^* = -0.002$		$b^* = 0.002$	
			Av.Es	MSE	Av.Es	MSE	Av.Es	MSE	Av.Es	MSE
20(13,2)	I	ϑ	2.09476	2.08723	1.35775	0.51968	1.35842	0.52125	1.35707	0.51813
		θ	1.18217	0.20325	0.91512	0.10965	0.91523	0.10967	0.91501	0.10962
		$S(0.80)$	0.70966	0.00968	0.70810	0.00609	0.70811	0.00609	0.70809	0.00609
		H(0.80)	1.65542	0.33719	1.32976	0.33719	1.32997	0.20769	1.32956	0.20763
	II	ϑ	2.35135	7.44839	1.33569	0.82120	1.33674	0.82707	1.33465	0.81548
		θ	1.18160	0.24114	0.85805	0.12385	0.85817	0.12387	0.85793	0.12384
		$S(0.80)$	0.69932	0.00984	0.69935	0.00583	0.69936	0.00583	0.69934	0.00583
		H(0.80)	1.73897	0.49194	1.32530	0.49194	1.32552	0.23032	1.32508	0.23022
	III	ϑ	2.63785	13.3356	1.31293	0.95031	1.31456	0.96508	1.31135	0.93678
		θ	1.22929	0.28899	0.82987	0.12673	0.82999	0.12674	0.82974	0.12673
		$S(0.80)$	0.70864	0.00886	0.70271	0.00469	0.70272	0.00469	0.70270	0.00469
		H(0.80)	1.73171	0.52375	1.27889	0.52376	1.27911	0.24501	1.27866	0.24490
20(15,2)	I	ϑ	1.96354	1.24294	1.37618	0.38733	1.37668	0.38809	1.37569	0.01545
		θ	1.14386	0.13817	0.91675	0.08432	0.91685	0.08434	0.91664	0.00695
		$S(0.80)$	0.70473	0.00885	0.69875	0.00604	0.69875	0.00604	0.69874	0.00604
		H(0.80)	1.66243	0.29184	1.39241	0.29184	1.39259	0.17874	1.39223	0.17867
	II	ϑ	2.10391	2.64634	1.38303	0.67159	1.38386	0.67613	1.38221	0.66721

		θ	1.16418	0.18586	0.89784	0.10808	0.89794	0.10809	0.89773	0.10806
		S(0.80)	0.70690	0.00850	0.69920	0.00560	0.69920	0.00560	0.69919	0.00560
		H(0.80)	1.66394	0.32738	1.36441	0.32738	1.36459	0.19521	1.36422	0.19514
	III	ϑ	2.18089	3.88148	1.34920	0.65813	1.35015	0.66311	1.34827	0.65329
		θ	1.19074	0.21288	0.88199	0.10405	0.88210	0.10407	0.88188	0.10404
		S(0.80)	0.71216	0.00837	0.70070	0.00519	0.70071	0.00519	0.70070	0.00519
		H(0.80)	1.65263	0.31020	1.33539	0.31020	1.33557	0.18356	1.33521	0.18352
20(13,4)	I	ϑ	2.30080	6.74027	0.72770	0.68385	0.72796	0.68364	0.72743	0.68406
		θ	1.21005	0.25240	0.57058	0.22838	0.57068	0.22831	0.57047	0.22844
		S(0.80)	0.70953	0.01048	0.69842	0.00344	0.69842	0.00344	0.69841	0.00344
		H(0.80)	1.67604	0.38714	0.97519	0.38714	0.97536	0.39102	0.97503	0.39130
	II	ϑ	3.07826	65.0288	0.66033	0.80980	0.66062	0.80971	0.66004	0.80989
		θ	1.26314	0.39400	0.50779	0.28344	0.50789	0.28336	0.50769	0.28351
		S(0.80)	0.71220	0.00990	0.69543	0.00267	0.69543	0.00267	0.69542	0.00267
		H(0.80)	1.70997	0.60528	0.91520	0.60528	0.91536	0.46451	0.91504	0.46480
	III	ϑ	3.16393	45.3623	0.59796	0.88423	0.59821	0.88401	0.59772	0.88445
		θ	1.31084	0.43938	0.46193	0.32794	0.46203	0.32786	0.46184	0.32801
		S(0.80)	0.71350	0.00997	0.69224	0.00247	0.69225	0.00247	0.69223	0.00248
		H(0.80)	1.73519	0.46509	0.86607	0.46509	0.86623	0.51383	0.86593	0.51417
20(15,4)	I	ϑ	2.06778	2.08164	0.88218	0.51073	0.88246	0.51064	0.88192	0.51082
		θ	1.19240	0.23753	0.66997	0.17287	0.67008	0.17283	0.66987	0.17291
		S(0.80)	0.71456	0.01000	0.69329	0.00437	0.69330	0.00437	0.69329	0.00437
		H(0.80)	1.60407	0.25180	1.10145	0.25180	1.10160	0.26754	1.10131	0.26772
	II	ϑ	2.20830	3.89938	0.83588	0.56760	0.83616	0.56751	0.83560	0.56770
		θ	1.20422	0.24443	0.62552	0.19443	0.62562	0.19438	0.62542	0.19448
		S(0.80)	0.71335	0.00898	0.68899	0.00375	0.68899	0.00375	0.68898	0.00375
		H(0.80)	1.64272	0.30899	1.07655	0.30899	1.07669	0.29172	1.07640	0.29191
	III	ϑ	2.41933	6.95743	0.79235	0.62403	0.79266	0.62407	0.79204	0.62400
		θ	1.27544	0.36722	0.60538	0.21478	0.60549	0.21474	0.60528	0.21483
		S(0.80)	0.72496	0.01072	0.69151	0.00388	0.69152	0.00388	0.69150	0.00388
		H(0.80)	1.60128	0.32392	1.02873	0.32392	1.02887	0.32949	1.02859	0.32970

Par-Parameter, C.S-Censoring Scheme, Av.Es-Average estimates

Continue Table 1

50(35,10)	I	ϑ	1.71684	0.32537	0.74492	0.59611	0.74498	0.59602	0.74485	0.59619
		θ	1.08084	0.06760	0.60456	0.17519	0.60460	0.17517	0.60452	0.17522
		S(0.80)	0.70630	0.00387	0.69681	0.00169	0.69682	0.00169	0.69681	0.00169
		H(0.80)	1.59454	0.08667	1.06826	0.08667	1.06831	0.25716	1.06820	0.25726
	II	ϑ	1.71826	0.38609	0.66447	0.71977	0.66453	0.71969	0.66441	0.71987
		θ	1.07804	0.06764	0.54239	0.22496	0.54242	0.22494	0.54235	0.22499
		S(0.80)	0.70751	0.00359	0.69443	0.00144	0.69444	0.00144	0.69443	0.00144
		H(0.80)	1.58748	0.09093	1.00447	0.09093	1.00452	0.32148	1.00441	0.32158
	III	ϑ	1.84044	0.72197	0.62019	0.79441	0.62024	0.79432	0.62013	0.79449
		θ	1.11559	0.09639	0.50323	0.26153	0.50326	0.26150	0.50319	0.26156

		S(0.80)	0.71089	0.00373	0.69146	0.00133	0.69146	0.00133	0.69145	0.00133
		H(0.80)	1.60357	0.10915	0.96769	0.10915	0.96774	0.36257	0.96763	0.36268
50(45,10)	I	ϑ	1.64882	0.21620	0.94108	0.35189	0.94115	0.35184	0.94102	0.35195
		θ	1.06776	0.05519	0.71340	0.10446	0.71343	0.10444	0.71337	0.10447
		S(0.80)	0.70935	0.00354	0.68894	0.00221	0.68895	0.00221	0.68894	0.00221
		H(0.80)	1.56181	0.06796	1.23561	0.06796	1.23566	0.12877	1.23557	0.12882
	III	ϑ	1.65018	0.20057	0.91817	0.37194	0.91824	0.37188	0.91811	0.37200
		θ	1.06403	0.05274	0.69417	0.11373	0.69420	0.11371	0.69413	0.11375
		S(0.80)	0.70745	0.00348	0.68587	0.00221	0.68588	0.00221	0.68588	0.00221
		H(0.80)	1.57053	0.06538	1.22732	0.06538	1.22736	0.13121	1.22727	0.13126
	III	ϑ	1.66124	0.24950	0.89955	0.39763	0.89961	0.39756	0.89948	0.39769
		θ	1.06595	0.06291	0.67817	0.12616	0.67821	0.12614	0.67814	0.12618
		S(0.80)	0.70701	0.00375	0.68340	0.00237	0.68340	0.00237	0.68340	0.00237
		H(0.80)	1.56950	0.06846	1.21656	0.06846	1.21661	0.13863	1.21651	0.13869
50(35,20)	I	ϑ	1.86805	0.88355	0.06295	2.06526	0.06295	2.06526	0.06295	2.06526
		θ	1.13732	0.13769	0.00360	0.99281	0.00360	0.99281	0.00360	0.99281
		S(0.80)	0.71405	0.00530	0.63743	0.00431	0.63743	0.00431	0.63742	0.00431
		H(0.80)	1.57943	0.10661	0.15635	0.10661	0.15636	1.95399	0.15635	1.95403
	II	ϑ	1.96314	1.26799	0.06150	2.06942	0.06150	2.06942	0.06150	2.06942
		θ	1.17256	0.18923	0.00320	0.99361	0.00320	0.99361	0.00320	0.99362
		S(0.80)	0.71978	0.00572	0.63860	0.00416	0.63860	0.00416	0.63859	0.00416
		H(0.80)	1.56374	0.10378	0.15288	0.10378	0.15288	1.96367	0.15287	1.96370
	III	ϑ	2.06183	2.36433	0.06055	2.07215	0.06055	2.07215	0.06055	2.07215
		θ	1.19039	0.22687	0.00289	0.99423	0.00289	0.99423	0.00289	0.99423
		S(0.80)	0.72177	0.00641	0.63940	0.00406	0.63941	0.00406	0.63940	0.00406
		H(0.80)	1.55640	0.10956	0.15061	0.10956	0.15062	1.96998	0.15061	1.97001
50(45,20)	I	ϑ	1.69827	0.36853	0.38070	1.25676	0.38072	1.25672	0.38068	1.25680
		θ	1.10024	0.09379	0.21970	0.61299	0.21972	0.61297	0.21969	0.61303
		S(0.80)	0.71528	0.00469	0.61446	0.00856	0.61447	0.00856	0.61446	0.00856
		H(0.80)	1.53905	0.07130	0.77156	0.07130	0.77160	0.61993	0.77151	0.62005
	II	ϑ	1.69273	0.33112	0.36753	1.28619	0.36755	1.28615	0.36752	1.28623
		θ	1.09793	0.09759	0.20704	0.63286	0.20706	0.63283	0.20702	0.63289
		S(0.80)	0.71461	0.00495	0.61212	0.00899	0.61213	0.00899	0.61212	0.00899
		H(0.80)	1.53668	0.07282	0.75324	0.07281	0.75328	0.64879	0.75320	0.64891
	III	ϑ	1.74415	0.40873	0.35961	1.30371	0.35962	1.30367	0.35959	1.30375
		θ	1.11904	0.10995	0.19782	0.64693	0.19784	0.64690	0.19781	0.64695
		S(0.80)	0.71686	0.00496	0.60967	0.00935	0.60967	0.00935	0.60966	0.00935
		H(0.80)	1.54289	0.06984	0.74368	0.06984	0.74372	0.66334	0.74364	0.66347

Continue Table 1

100(50,5)	I	ϑ	1.60365	0.13344	1.29151	0.11259	1.29159	0.11257	1.29143	0.11260
		θ	1.03209	0.02274	0.92217	0.02306	0.92219	0.02306	0.92215	0.02306
		S(0.80)	0.70219	0.00192	0.70905	0.00158	0.70906	0.00158	0.70905	0.00158
		H(0.80)	1.58781	0.05667	1.40732	0.05667	1.40737	0.06042	1.40728	0.06044

	II	ϑ	1.60924	0.17502	1.20400	0.16166	1.20409	0.16164	1.20390	0.16169
		θ	1.03135	0.02422	0.88186	0.03084	0.88188	0.03084	0.88184	0.03084
		S(0.80)	0.70285	0.00168	0.71178	0.00135	0.71178	0.00135	0.71177	0.00135
		H(0.80)	1.58577	0.06289	1.35043	0.06289	1.35048	0.07959	1.35038	0.07961
	III	ϑ	1.68649	0.31552	1.13329	0.21434	1.13341	0.21430	1.13317	0.21438
		θ	1.05472	0.03589	0.84389	0.04310	0.84392	0.04309	0.84387	0.04311
		S(0.80)	0.70477	0.00143	0.71300	0.00107	0.71300	0.00107	0.71299	0.00107
		H(0.80)	1.60161	0.07031	1.30063	0.07031	1.30068	0.09829	1.30058	0.09833
100(70,5)	I	ϑ	1.56809	0.08147	1.34728	0.07413	1.34734	0.07413	1.34722	0.07414
		θ	1.02065	0.01851	0.93513	0.01921	0.93514	0.01921	0.93511	0.01921
		S(0.80)	0.70123	0.00178	0.70203	0.00153	0.70203	0.00153	0.70203	0.00153
		H(0.80)	1.57823	0.04240	1.46142	0.04240	1.46146	0.04093	1.46139	0.04094
	II	ϑ	1.58637	0.10212	1.32442	0.08868	1.32449	0.08867	1.32435	0.08869
		θ	1.02684	0.02031	0.92348	0.02148	0.92349	0.02148	0.92346	0.02148
		S(0.80)	0.70213	0.00145	0.70291	0.00120	0.70291	0.00120	0.70291	0.00120
		H(0.80)	1.58114	0.03724	1.44446	0.03724	1.44449	0.03924	1.44443	0.03924
	III	ϑ	1.61777	0.14997	1.30668	0.10973	1.30676	0.10973	1.30659	0.10974
		θ	1.04086	0.02749	0.91394	0.02638	0.91396	0.02638	0.91392	0.02639
		S(0.80)	0.70452	0.00153	0.70348	0.00123	0.70349	0.00123	0.70348	0.00123
		H(0.80)	1.58087	0.04154	1.42804	0.04154	1.42807	0.04488	1.42800	0.04489
100(50,15)	I	ϑ	1.62778	0.18499	0.64621	0.74144	0.64625	0.74138	0.64618	0.74150
		θ	1.04005	0.03023	0.58717	0.17920	0.58719	0.17918	0.58715	0.17921
		S(0.80)	0.70322	0.00197	0.72422	0.00123	0.72422	0.00123	0.72422	0.00123
		H(0.80)	1.58837	0.05853	0.95495	0.05853	0.95498	0.37090	0.95491	0.37098
	II	ϑ	1.69701	0.26652	0.52485	0.95840	0.52488	0.95835	0.52482	0.95846
		θ	1.05837	0.03422	0.48900	0.26771	0.48902	0.26769	0.48898	0.26773
		S(0.80)	0.70243	0.00180	0.72521	0.00105	0.72521	0.00105	0.72520	0.00105
		H(0.80)	1.61781	0.06791	0.84450	0.06791	0.84453	0.51168	0.84447	0.51177
	III	ϑ	1.76756	0.50545	0.42677	1.15677	0.42679	1.15673	0.42675	1.15682
		θ	1.07296	0.05567	0.39946	0.36585	0.39948	0.36583	0.39944	0.36587
		S(0.80)	0.70408	0.00174	0.72582	0.00092	0.72583	0.00092	0.72583	0.00092
		H(0.80)	1.62236	0.08673	0.74342	0.08673	0.74345	0.66356	0.74339	0.66365
100(70,15)	I	ϑ	1.58184	0.09947	0.92576	0.35117	0.92580	0.35113	0.92572	0.35121
		θ	1.02955	0.02460	0.73072	0.08429	0.73073	0.08428	0.73070	0.08429
		S(0.80)	0.70327	0.00182	0.70231	0.00106	0.70231	0.00106	0.70230	0.00106
		H(0.80)	1.57249	0.03814	1.21325	0.03814	1.21328	0.13172	1.21322	0.13175
	II	ϑ	1.58490	0.12191	0.84986	0.44226	0.84990	0.44222	0.84983	0.44231
		θ	1.03412	0.02590	0.68395	0.11036	0.68397	0.11035	0.68393	0.11037
		S(0.80)	0.70601	0.00155	0.70226	0.00083	0.70226	0.00083	0.70226	0.00083
		H(0.80)	1.56225	0.03697	1.15922	0.03697	1.15925	0.16982	1.15919	0.16986
	III	ϑ	1.62676	0.16536	0.79575	0.51276	0.79578	0.51271	0.79571	0.51280
		θ	1.04865	0.03056	0.64262	0.13712	0.64264	0.13711	0.64260	0.13714
		S(0.80)	0.70690	0.00150	0.69883	0.00075	0.69884	0.00075	0.69883	0.00075
		H(0.80)	1.57376	0.03961	1.12567	0.03961	1.12570	0.19603	1.12564	0.19608

Table 2: AILs and CPs of $\vartheta, \theta, S(0.80)andH(0.80)$ using ML and Bayesian estimates based on GPTIIC schemes at different values of (n, m, r) for $\vartheta = 1.5, \theta = 1$.

n(m,r)	C.S	Par	ACI		HPD	
			AIL	CP	AIL	CP
20(13,2)	I	ϑ	4.67658	95.50	3.02555	95.16
		θ	1.61732	95.40	1.69182	95.66
		S(0.80)	0.38487	97.70	0.45669	98.43
		H(0.80)	2.24231	95.30	2.24803	95.70
	II	ϑ	7.43606	97.30	3.03429	95.09
		θ	1.78938	96.10	1.67180	95.53
		S(0.80)	0.38883	97.70	0.45374	98.34
		H(0.80)	2.65327	95.60	2.11424	95.56
	III	ϑ	9.44232	97.40	3.43435	95.07
		θ	1.90693	95.00	1.65464	95.43
		S(0.80)	0.36851	97.90	0.41173	98.31
		H(0.80)	2.75132	95.10	2.05053	95.44
20(15,2)	I	ϑ	3.95181	95.90	2.57033	95.21
		θ	1.34422	95.70	1.46749	95.80
		S(0.80)	0.36879	98.30	0.43265	98.44
		H(0.80)	2.07549	95.60	2.08373	95.80
	II	ϑ	5.06593	96.90	3.35439	95.14
		θ	1.56337	94.70	1.54857	95.66
		S(0.80)	0.36112	97.40	0.42712	98.16
		H(0.80)	2.20211	95.70	2.15133	95.77
	III	ϑ	5.80618	97.50	3.16815	95.13
		θ	1.64765	95.00	1.65527	95.61
		S(0.80)	0.35691	97.60	0.41143	98.24
		H(0.80)	2.14974	96.10	1.89597	95.73
20(13,4)	I	ϑ	7.14357	98.40	1.72999	95.12
		θ	1.78984	95.00	1.29404	95.00
		S(0.80)	0.40062	98.60	0.39095	98.39
		H(0.80)	2.39273	96.20	1.66313	95.58
	II	ϑ	18.5856	98.80	1.41858	95.09
		θ	2.23500	95.90	1.16164	95.00
		S(0.80)	0.38858	97.50	0.40976	98.26
		H(0.80)	2.98921	97.30	1.60904	95.55
	III	ϑ	15.9618	98.60	1.40298	95.09
		θ	2.29609	95.10	1.17259	95.00
		S(0.80)	0.38929	96.40	0.41774	98.07
		H(0.80)	2.57841	94.70	1.49259	95.52
20(15,4)	I	ϑ	4.66873	96.20	1.83546	95.18

		θ	1.75616	94.90	1.40653	95.31
		S(0.80)	0.38944	97.40	0.41959	98.20
		H(0.80)	1.95813	95.60	1.67880	95.82
	II	ϑ	5.82287	97.40	1.78999	95.17
		θ	1.76582	95.40	1.30893	95.19
		S(0.80)	0.36931	97.10	0.39447	98.20
		H(0.80)	2.15205	96.00	1.63362	95.70
	III	ϑ	7.26737	97.10	1.84814	95.15
		θ	2.11693	95.00	1.42009	95.06
		S(0.80)	0.39650	97.50	0.42779	98.00
		H(0.80)	2.22435	96.40	1.58025	95.72

Par -Parameter.

Continue Table 2

50(35,10)	I	ϑ	2.06914	95.80	1.03229	95.62
		θ	0.96914	95.60	0.88962	96.09
		S(0.80)	0.24359	97.70	0.27599	97.86
		H(0.80)	1.14354	95.90	1.00550	96.26
	II	ϑ	2.28160	95.60	0.96234	95.58
		θ	0.97298	95.80	0.84327	95.96
		S(0.80)	0.23428	97.30	0.27619	97.84
		H(0.80)	1.17529	96.40	1.01945	96.24
	III	ϑ	3.05320	95.20	0.91575	95.49
		θ	1.13008	95.50	0.81454	95.85
		S(0.80)	0.23739	97.20	0.25957	97.81
		H(0.80)	1.28099	97.00	1.05069	96.19
50(45,10)	I	ϑ	1.72767	95.00	1.15654	95.76
		θ	0.88217	95.90	0.94387	96.22
		S(0.80)	0.23175	97.30	0.29248	97.80
		H(0.80)	1.02193	95.70	1.05437	96.36
	II	ϑ	1.65474	95.50	1.11588	95.81
		θ	0.86492	95.80	0.85596	96.26
		S(0.80)	0.23063	97.50	0.28738	97.88
		H(0.80)	1.00068	95.90	1.02653	96.41
	III	ϑ	1.85411	95.70	1.11735	95.72
		θ	0.94907	95.40	0.89616	96.13
		S(0.80)	0.23948	96.90	0.28056	97.76
		H(0.80)	1.02430	96.00	1.04501	96.36
50(35,20)	I	ϑ	3.39211	96.10	0.12484	95.32
		θ	1.35195	95.30	0.02679	95.00
		S(0.80)	0.28208	96.60	0.30172	97.86
		H(0.80)	1.27663	96.40	0.30634	95.35
	II	ϑ	3.97585	95.60	0.10844	95.36
		θ	1.56608	94.80	0.02351	95.00

		S(0.80)	0.28913	96.20	0.27741	97.86
		H(0.80)	1.26282	96.20	0.26854	95.36
	III	ϑ	4.86857	96.30	0.11128	95.39
		θ	1.71233	96.00	0.01946	95.00
		S(0.80)	0.30491	97.40	0.28573	97.89
		H(0.80)	1.29807	95.70	0.27140	95.38
50(45,20)	I	ϑ	2.25029	97.20	0.54346	95.77
		θ	1.13494	96.20	0.51589	95.02
		S(0.80)	0.26401	97.10	0.30629	97.59
		H(0.80)	1.04563	96.50	0.81910	96.43
	II	ϑ	2.12644	95.80	0.53645	95.73
		θ	1.16343	95.60	0.52407	95.00
		S(0.80)	0.27198	97.00	0.31448	97.59
		H(0.80)	1.05615	97.20	0.84005	96.42
	III	ϑ	2.31729	96.40	0.51592	95.71
		θ	1.21377	95.40	0.47755	95.00
		S(0.80)	0.27054	96.80	0.29603	97.57
		H(0.80)	1.03552	97.20	0.83665	96.44

Continue Table 2

100(50,5)	I	ϑ	1.37378	95.40	1.43360	95.89
		θ	0.57784	96.30	0.69661	96.58
		S(0.80)	0.17195	97.80	0.22499	97.90
		H(0.80)	0.92415	95.90	1.07527	96.36
	II	ϑ	1.58379	96.10	1.42440	95.74
		θ	0.59780	96.60	0.72693	96.65
		S(0.80)	0.16106	97.70	0.19705	97.85
		H(0.80)	0.97556	96.00	1.06344	96.31
	III	ϑ	2.07801	95.90	1.57376	95.53
		θ	0.71142	96.60	0.81175	96.54
		S(0.80)	0.14803	97.60	0.19032	97.88
		H(0.80)	1.02297	96.00	1.09308	96.20
100(70,5)	I	ϑ	1.08709	95.70	1.21559	96.17
		θ	0.52745	96.60	0.69232	96.74
		S(0.80)	0.16543	97.40	0.22112	97.87
		H(0.80)	0.80191	96.40	0.97314	96.56
	II	ϑ	1.20665	96.30	1.32740	95.97
		θ	0.54895	96.70	0.69118	96.65
		S(0.80)	0.14924	97.60	0.19740	97.70
		H(0.80)	0.74925	96.60	0.91202	96.58
	III	ϑ	1.44687	96.10	1.44697	95.76
		θ	0.63016	96.00	0.77086	96.59
		S(0.80)	0.15332	97.60	0.19307	97.74
		H(0.80)	0.79232	96.50	0.95219	96.51

100(50,15)	I	ϑ	1.61065	95.80	0.80402	95.80
		θ	0.66355	96.80	0.64463	96.55
		$S(0.80)$	0.17390	97.90	0.20081	97.89
		$H(0.80)$	0.93918	96.90	0.87527	96.46
	II	ϑ	1.87148	95.40	0.69805	95.74
		θ	0.68847	96.50	0.61638	96.33
		$S(0.80)$	0.16637	97.60	0.19254	97.94
		$H(0.80)$	0.99084	95.90	0.76408	96.35
	III	ϑ	2.58327	95.20	0.55044	95.68
		θ	0.87999	96.00	0.56397	96.13
		$S(0.80)$	0.16348	96.90	0.17746	97.90
		$H(0.80)$	1.12340	96.60	0.70832	96.27
100(70,15)	I	ϑ	1.19453	95.70	0.93090	96.06
		θ	0.60409	96.00	0.63807	96.55
		$S(0.80)$	0.16737	97.70	0.20505	97.85
		$H(0.80)$	0.76242	96.30	0.81767	96.56
	II	ϑ	1.32824	95.90	0.84578	95.99
		θ	0.61677	96.70	0.65678	96.61
		$S(0.80)$	0.15386	97.00	0.19387	97.83
		$H(0.80)$	0.75336	97.00	0.77578	96.53
	III	ϑ	1.51536	95.30	0.85788	95.91
		θ	0.65847	96.30	0.64258	96.53
		$S(0.80)$	0.15121	97.80	0.18611	97.78
		$H(0.80)$	0.77665	95.80	0.76233	96.55

5- Real Data

Real data will be analyzed to compare the performance of different estimates and confidence intervals for the non-Bayesian (ML) and Bayesian methods of unknown parameters $\vartheta, \theta, S(t)$ and $H(t)$ of IER distribution based on GPTIIC scheme.

A real-life data set, originally reported by Efron (1988) , comprises the survival times of 44 patients with head and neck cancer who were treated with a combination of radiotherapy and chemotherapy(RT+CT). This data set is analyzed to illustrate the application of the proposed methods. The survival times for these cancer patients are detailed in Table (3). This data has also been recently analyzed by Maurya et al.(2019a) and Maurya et al.

(2019b), who fitted IER distribution to this real data set and proved that IER distribution is a good model for fitting this data. For computational convenience, the original data is transformed by divide it on hundred.

Table 3: Survival times of head and neck cancer patients

0.1220	0.2356	0.2374	0.2587	0.3198	0.3700	0.4135	0.4738
0.5546	0.5836	0.6347	0.6846	0.7447	0.7826	0.8143	0.8400
0.9200	0.9400	1.1000	1.1200	1.1900	1.2700	1.3000	1.3300
0.4000	1.4600	1.5500	1.5900	1.7300	1.7900	1.9400	1.9500
0.0900	2.4900	2.8100	3.1900	3.3900	4.3200	4.6900	5.1900
0.3300	7.2500	8.1700	17.7600				

Using the data sets from Table (3) various choices of m, r and $R_i, i = r + 1, \dots, m$ to illustrate the proposed methods, three different groups of GPTIIC data with corresponding censoring schemes were generated and presented in Table (4). These groups were randomly drawn from the parent sample as follows:

Table 4: Three different generated GPTIIC samples (head neck cancer)

Sample	m	r	R	GPTIIC Samples						
	37	3	(7,0*33)	0.2587,	0.6846,	0.7447,	0.7826,	0.8143,	0.8400,	0.9294,
				0.9400,	1.1000,	1.1200,	1.1900,	1.2700,	1.3000,	1.3300,
				1.4000,	1.4600,	1.5500,	1.5900,	1.7300,	1.7900,	1.9400,
				1.9500,	2.0900,	2.4900,	2.8100,	3.1900,	3.3900,	4.3200,
				4.6900,	5.1900,	6.3300,	7.2500,	8.1700,	17.7600	
	37	3	(0*12,1,0*12,2,0*8,4)	0.2587,	0.3198,	0.3700,	0.4135,	0.4738,	0.5546,	0.5836,
				0.6347,	0.6846,	0.7447,	0.7826,	0.8143,	0.8400,	0.9400
				1.1000,	1.1200,	1.1900,	1.2700,	1.3000,	1.3300,	1.4000,
				1.4600,	1.5500,	1.5900,	1.7300,	1.7900,	2.0900,	2.4900,
				2.8100,	3.1900,	3.3900,	4.3200,	4.6900,	5.1900	
	37	3	(0*33,7)	0.2587,	0.3198,	0.3700,	0.4135,	0.4738,	0.5546,	0.5836,
				0.6347,	0.6846,	0.7447,	0.7826,	0.8143,	0.8400,	0.9294,
				0.9400 ,	1.1000,	1.1200,	1.1900,	1.2700,	1.3000,	1.3300,
				1.4000,	1.4600,	1.5500,	1.5900,	1.7300,	1.7900,	1.9400,
				1.9500,	2.0900,	2.4900,	2.8100,	3.1900,	3.3900	

Here, (0*8), means that the censoring scheme employed is (0,0,0,0,0,0,0,0)

Using the data sets from Table (4), the ML and Bayesian estimates of the unknown parameters ϑ and θ, as well as, the reliability characteristics $S(t)$ and $H(t)$ at given mission time $t = 0.50$, are computed and listed in Table (5). The initial values for the unknown parameters for running

MCMC sampler algorithm were taken to be their ML estimates. Moreover, two-sided 95% ACI and HPD credible intervals with their lengths are calculated and listed in Table (6).

Using SE and LINEX (for $(b^* = -0.5, 0.5)$) loss functions, Bayesian estimates are obtained under the hyper parameters the $a = b = c = d = 0.0001$. Using MCMC algorithm, $10{,}000$ MCMC samples are generated and then the first $1{,}000$ iterations (burn-in period) have been discarded from the generated sequence. All necessary computational algorithms are implemented using $\mathcal{R}$ statistical programming language, version 4.4.1, with two key statistical packages: the 'coda' package for performing MCMC Bayesian estimations as proposed by Plummer et al. (2006), and the 'maxLik' package, which utilizes the Newton-Raphson method for maximization in computations, as proposed by Henningsen and Toomet (2011).

Based on Tables (5&6), it can be seen from the estimated results of point and interval estimates of the unknown parameters ϑ and $\theta,$ as well as $S(t)$ and $H(t)$ that Bayesian estimates have performed better than the ML estimates. Additionally, it is observed that the length of the HPD credible intervals is less than the corresponding length of the ACIs. Furthermore, it is observed that for $\vartheta, \theta,$ and $S(t)$, the performance of the estimates in sample 3 is better than in the other two samples (1 and 2) while for $H(t)$ the performance of the estimates in sample 1 is better than in the other two samples (2 and 3).

Table 5: Point estimates of $\vartheta, \theta, S(t)$, and $H(t)$ using ML and Bayesian estimates for real data set based on GPTIIC under various censoring schemes.

Sample	Par	MLE	SE	LINEX	
1				$b^* = -0.5$	$b^* = 0.5$
	ϑ	0.35579	0.30995	0.31095	0.30896
	θ	0.17771	0.15397	0.15468	0.15327
	S(0.50)	0.78629	0.78009	0.78078	0.77940
	H(0.50)	0.97676	0.89017	0.89617	0.88429
2	ϑ	0.43186	0.37132	0.37290	0.36977
	θ	0.16479	0.14091	0.14144	0.14039
	S(0.50)	0.73011	0.72759	0.72832	0.72685
	H(0.50)	1.22023	1.09614	1.10540	1.08711
3	ϑ	0.35440	0.30663	0.30767	0.30560
	θ	0.14922	0.12771	0.12819	0.12723
	S(0.50)	0.75321	0.75035	0.75102	0.74968
	H(0.50)	1.03639	0.93236	0.93904	0.92583

Par -Parameter

Table 6: The 95%two-sided ACIs/HPD credible intervals of $\vartheta, \theta, S(t)$ and $H(t)$ for real data set based on GPTIIC Scheme.

Sample	Par	ACI			HPD		
		lower	Upper	AIL	lower	upper	AIL
1	ϑ	0.22019	0.49140	0.27120	0.19732	0.44004	0.24271
	θ	0.06473	0.29070	0.22598	0.05458	0.25677	0.20220
	S(0.50)	0.68287	0.88972	0.20686	0.67312	0.87566	0.20254
	H(0.50)	0.25099	1.70252	1.45152	0.58687	1.18782	0.60094
2	ϑ	0.26013	0.60359	0.34346	0.21965	0.52202	0.30236
	θ	0.06726	0.26231	0.19506	0.05202	0.22945	0.17743
	S(0.50)	0.62238	0.83786	0.21548	0.61612	0.82689	0.21076
	H(0.50)	0.40824	2.03222	1.62399	0.73461	1.47260	0.73798
3	ϑ	0.21566	0.49315	0.27749	0.18988	0.43680	0.24692
	θ	0.05560	0.24283	0.18722	0.04877	0.21918	0.17041
	S(0.50)	0.65146	0.85495	0.20349	0.64360	0.84530	0.20171
	H(0.50)	0.39599	1.67679	1.28080	0.62599	1.25387	0.62788

Par -Parameter, AIL-Average interval length.

Moreover, as a further illustration, the trace plots for all parameters in MCMC trace with their histograms for each parameter and the convergence of MCMC estimation for ϑ and θ of GPTIIC using MCMC are shown in

Figure (2). It is evident that MCMC algorithm sampler converges well. In each histogram plot, it shows that all the generated posterior estimates have been well approximated the theoretical posterior density functions. It also shows that discarding the first 1,000 samples as burn-in is an appropriate size to remove the effect of the initial guesses.

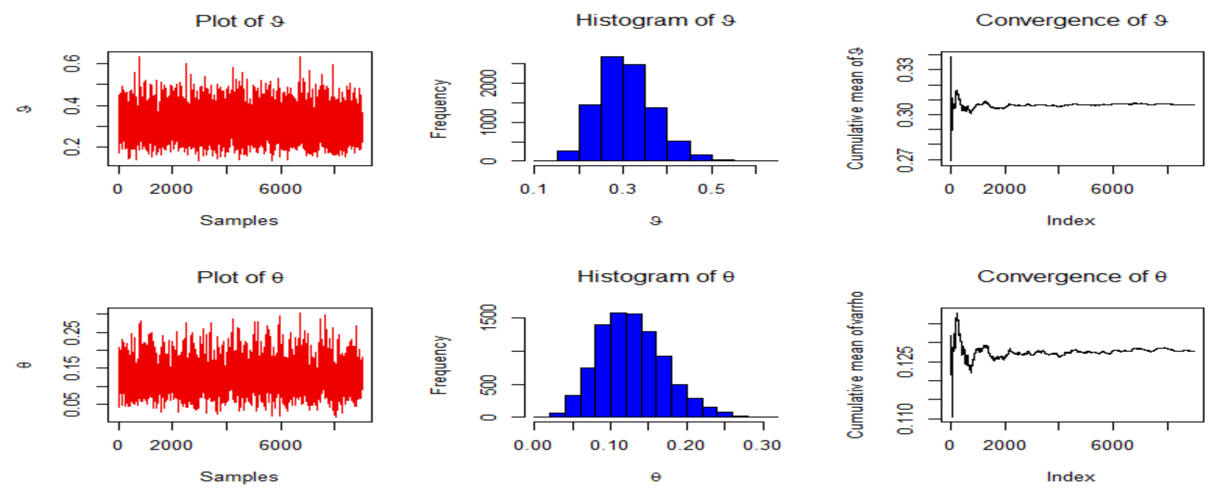

Figure 2: Trace plots , histograms and convergence for ϑ and θ using MCMC algorithm.

6 Conclusions

The object of this paper is to study different estimates of the unknown model parameters of IER distribution under GPTIIC scheme. Both ML and Bayesian estimation methods are used to obtain the estimates and Bayesian estimates are obtained based on the symmetric and asymmetric loss functions under the assumption of independent gamma priors using MCMC method. Approximate confidence intervals for the unknown parameters are constructed based on asymptotic theory and also approximate confidence intervals for survival characteristics are constructed using the delta method. Monte Carlo simulation study is performed to compare the performance of the estimates in terms of the MSEs. It is observed that Bayesian estimates under LINEX loss function perform better than the other estimates. Further, a real data set is considered for illustrative purposes.

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