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Statistical Investigations of Sarmanov Bivariate Weibull Distribution with Application to Medical Data

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ABSTRACT : The Weibull distribution is often considered the most fundamental and elementary lifespan distribution. This work utilizes the Sarmanov copula and Weibull marginal distribution to construct a bivariate distribution known as the bivariate Weibull Sarmanov distribution (BW-SARD). The statistical characteristics of the BW-SARD are examined, including marginal distributions, product moment, moment generating function, coefficient of correlation between the inner variables, conditional distributions, and conditional expectation. In addition, we calculate various reliability measures, including the hazard rate function, reversed hazard rate function, positive quadrant dependence property, mean residual life function, and vitality function. Model parameter estimation is conducted using the maximum likelihood and Bayesian approaches. Ultimately, genuine data collection is presented, and examined to explore the model, and valuable outcomes are acquired for illustrative intentions.

KEYWORDS: Bivariate Weibull distribution; Sarmanov family; Maximum likelihood estimation; Bayesian estimation; Confidence intervals.; Product moments.

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I. Introduction

The Weibull distribution is highly versatile and capable of fitting various shapes of datasets. The Weibull distribution, similar to the normal distribution, is a unimodal distribution that represents probabilities for continuous data. However, in contrast to the normal distribution, it is capable of accurately representing skewed data. Due to its exceptional versatility, it is capable of accurately modeling data that is skewed in both left and right directions. It can approximately depict different distributions in addition to the normal distribution. Due to its adaptability, analysts utilize it in several contexts, including capability analysis, quality control, engineering, and medical investigations. It is commonly employed in reliability assessments, warranty analysis, and life data to estimate the duration of failure for systems and components. The univariate Weibull distribution (denoted by $W(\gamma;\omega)$) has the following distribution function (DF) and probability density function (PDF), respectively

$$G_Z(z;\gamma,\omega) = 1 - e^{-\left(\frac{z}{\omega}\right)^{\gamma}};\ z > 0, \gamma, \omega > 0,$$

and

$$g_Z(z;\gamma,\omega) = \frac{\gamma}{\omega}\left(\frac{z}{\omega}\right)^{\gamma-1} e^{-\left(\frac{z}{\omega}\right)^{\gamma}};\ z > 0, \gamma, \omega > 0,$$

where γ and ω are the shape and scale parameters, respectively.

One technique for creating bivariate distributions that can be found in the statistical literature is the knowledge of a copula (see Nelsen, 2006). Copulas can help characterize bivariate distributions when they contain an explicit dependency structure. It is a function that joins bivariate DFs with uniform [0,1] margins. Copulas can be applied in this way to look into bivariate distribution studies. The copula function will be selected based on the sort of dependence structure between the two random variables (RVs). Because they make it simple to model and estimate the distribution of random vectors by estimating marginals and copula separately, copulas are useful in high-dimensional statistical applications. For given two marginal univariate distributions $G_{Z_1}(z_1) = P(Z_1 \leq z_1)$ and $G_{Z_2}(z_2) = P(Z_2 \leq z_2)$, a copula $C(u,v)$, and its PDF, i.e., $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$, Sklar (1973) presented the joint DF (JDF) and joint PDF (JPDF), respectively, as follows:

$$G_{Z_1,Z_2}(z_1,z_2) = C\left(G_{Z_1}(z_1), G_{Z_2}(z_2)\right) \qquad (1.1)$$

and

$$g_{Z_1,Z_2}(z_1,z_2) = g_{Z_1}(z_1) g_{Z_2}(z_2) c(G_{Z_1}(z_1), G_{Z_2}(z_2)).$$

The creation of bivariate distributions with specified marginals is one of the pivotal problems in statistical theory and its applications to modeling bivariate data. This is because marginal distributions are typically used as a piece of prior knowledge when modeling bivariate data. Since the creation of the Farlie-Gumbel-Morgenstern (FGM) family in the sixties of the previous century, numerous researchers have created and examined a variety of generalizations about it to enhance the correlation between its components. Among these researchers are Abd Elgawad and Alawady (2022), Abd Elgawad et al. (2021), Alawady et al. (2021, 2022), Barakat and Husseiny (2021), Barakat et al. (2018, 2021a,b), Cambanis (1977), and Husseiny et al. (2022a, 2023). One of the most adaptable and robust extensions of the traditional FGM family of bivariate DFs is the Sarmanov family, denoted by SAR(.), which was suggested and used by Sarmanov (1974) to describe hydrological phenomena. Recently, the superiority of this family over all known extensions of the FGM family has been revealed in a series of works, namely by Abd Elgawad et al. (2024), Alawady et al. (2023), Barakat et al. (2019, 2022a,b), and Husseiny et al. (2022b). The DF and PDF of SAR(η) are given, respectively, by

$$G_{Z_1,Z_2}(z_1,z_2) = G_{Z_1}(z_1) G_{Z_2}(z_2)\left[1 + 3\eta \overline{G}_{Z_1}(z_1)\overline{G}_{Z_2}(z_2) + 5\eta^2 \left(2G_{Z_1}(z_1) - 1\right)(2G_{Z_2}(z_2) - 1)\overline{G}_{Z_1}(z_1)\overline{G}_{Z_2}(z_2)\right]$$

and

$$g_{Z_1,Z_2}(z_1,z_2) = g_{Z_1}(z_1) g_{Z_2}(z_2)\Big[1 + 3\eta(2G_{Z_1}(z_1) - 1)(2G_{Z_2}(z_2) - 1)$$
$$+ \frac{5}{4}\eta^2 (3(2G_{Z_1}(z_1) - 1)^2 - 1)(3(2G_{Z_2}(z_2) - 1)^2 - 1)\Big], |\eta| \leq \frac{\sqrt{7}}{5}, \qquad (1.3)$$

where the survival function $\overline{G}(.) = 1 - G(.)$. In situations when the marginals are uniform, the correlation coefficient is η. Due to this, the correlation coefficient ρ for this family's lowest and highest values are -0.529 and 0.529, respectively (see Balakrishnan and Lai, 2009; page 74). In this study, a bivariate distribution known as the bivariate Weibull Sarmanov distribution (BW-SARD) is created using the SAR copula and Weibull marginal distributions.

The subsequent sections of this work are structured in the following manner. In Section 2, we acquire BW-SARD. Section 3 examines important statistical characteristics of the BW-SARD, including marginal distributions, product moment, moment generating function, coefficient of correlation between the inner variables, conditional distributions, and conditional expectation. Section 4 focuses on examining reliability measures, including the hazard rate function, reversed hazard rate function, positive quadrant dependence property, mean residual life, and vitality functions. In Section 5, the parameters of the model are estimated using both maximum likelihood (ML) and Bayesian approaches. In Section 6, we examine facts from the real world as an illustration. Ultimately, the paper is concluded in Section 7.

II. Proposed Model

Let $Z_1 \sim W(\gamma_1; \omega_1)$ and $Z_2 \sim W(\gamma_2; \omega_2)$. Thus, according to (1.1), the JDF of bivariate Weibull distribution based on SAR copula, denoted by BW-SARD $(\gamma_1, \omega_1; \gamma_2, \omega_2)$, is as follows

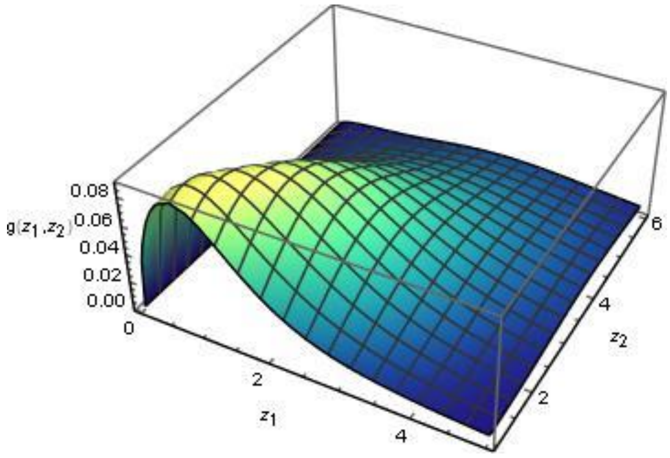
$$G_{Z_1,Z_2}(z_1,z_2) = \left(1 - e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 - e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\left[1 + 3\eta e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right.$$

$$+\,5\eta^2\left(1-2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1-2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\,e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\Bigg] \tag{2.1}$$

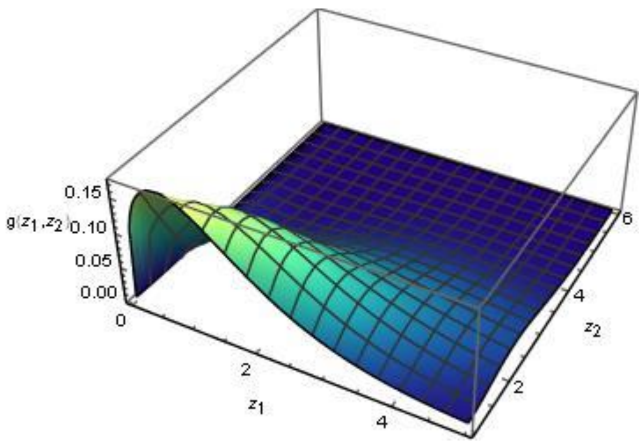
and according to (1.2), the corresponding JPDF is given by

$$g_{Z_1,Z_2}(z_1,z_2)=\frac{\gamma_1\gamma_2}{\omega_1\omega_2}\left(\frac{z_1}{\omega_1}\right)^{\gamma_1-1}\left(\frac{z_2}{\omega_2}\right)^{\gamma_2-1}e^{-\left(\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}+\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}\right)}\Bigg[1+3\eta\left(1-2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\Bigg(1-2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\Bigg)+\frac{5\eta^2}{4}\left(3\left(1-2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)\left(3\left(1-2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)^2-1\right)\Bigg] \tag{2.2}$$

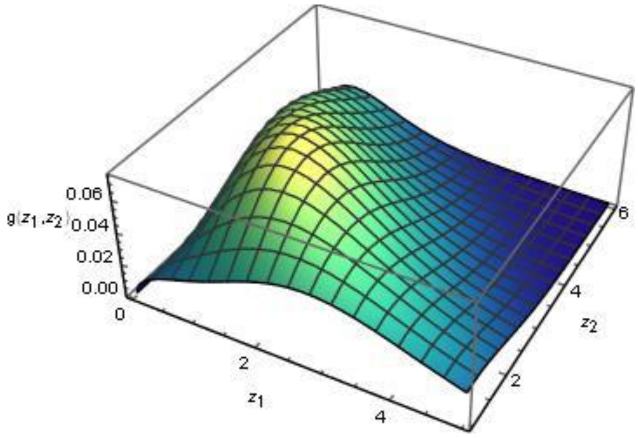
Figures 1 shows the plots 3-dimension for the JPDF of BW-SARD with different values of $\gamma_1, \omega_1, \gamma_2, \omega_2$, and η. The diversity in the surface shapes of this family for different values of its parameters reflects the various values of skewness and kurtosis that the family provides. This confirms the efficiency of this family in describing various categories of data.



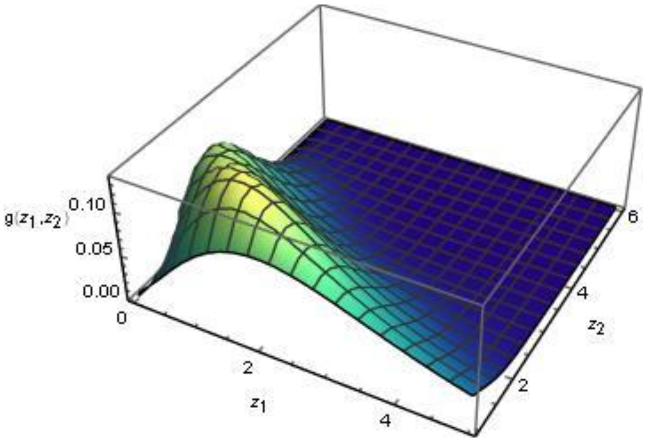
(a) $\gamma_1=1.5, \omega_1=2.5, \gamma_2=1.8, \omega_2=3.5, \eta=0.4$



(b) $\gamma_1=1.5, \omega_1=2.5, \gamma_2=1.8, \omega_2=3.5, \eta=-0.4$



(c) $\gamma_1 = 1.3, \omega_1 = 2.5, \gamma_2 = 1.9, \omega_2 = 2, \eta = 0.5$



(d) $\gamma_1 = 1.3, \omega_1 = 2.5, \gamma_2 = 1.9, \omega_2 = 2, \eta = -0.5$

Figure 1: JPDF for BW-SARD

III. Statistical Properties

In this section, we give some important statistical properties of the BW-SARD such as marginal distributions, conditional distributions, conditional expectation, moment generating function, product moment, coefficient of correlation between the inner variables, and the positive quadrant dependence property.

1) The marginal distributions

$$g_{Z_1}(z_1;\gamma_1,\omega_1) = \frac{\gamma_1}{\omega_1}\left(\frac{z_1}{\omega_1}\right)^{\gamma_1-1} e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\ ;\ z_1 > 0, \gamma_1, \omega_1 > 0,$$

$$g_{Z_2}(z_2;\gamma_2,\omega_2) = \frac{\gamma_2}{\omega_2}\left(\frac{z_2}{\omega_2}\right)^{\gamma_2-1} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\ ;\ z_2 > 0, \gamma_2, \omega_2 > 0,$$

2) Conditional distribution and their expectation

The conditional PDF of Z_2 given $Z_1 = z_1$ is given as follows

$$g_{Z_2|Z_1}(z_2|z_1) = \frac{\gamma_2}{\omega_2}\left(\frac{z_2}{\omega_2}\right)^{\gamma_2-1} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\left[1 + 3\eta\left(1 - 2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 - 2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\right.$$

$$+\frac{5\eta^2}{4}\left(3\left(1-2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)\left(3\left(1-2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)^2-1\right)\Bigg],$$

and the conditional DF is

$$G_{Z_2|Z_1}(z_2|z_1)=\left(1-e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\left[1+3\eta e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\left(2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}-1\right)\right.$$
$$\left.+\frac{5}{4}\eta^2\left(3\left(1-\ 2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)\left(4\left(1-e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)^2-6\left(1-e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)+2\right)\right].$$

Consequently, the BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$ regression curve for Z_2 given $Z_1=z_1$ is

$$E(Z_2|Z_1=z_1)=\omega_2\Gamma(1+\tfrac{1}{\gamma_2})\left[1+3\eta\left(1-2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1-2^{\frac{-1}{\gamma_2}}\right)+\frac{5}{2}\eta^2\left(3\left(1-\ 2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)^2-1\right)\left(2\times3^{\frac{-1}{\gamma_2}}-3\times2^{\frac{-1}{\gamma_2}}+1\right)\right],$$

where the conditional expectation is non-linear with respect to z_1.

3) Moment generating function

By using (2.2) the moment generating function of Z_1 and Z_2 is given by

$$M_{Z_1,Z_2}(t_1,t_2)=\sum_{n=0}^{\infty}\frac{(t_1\omega_1)^n}{n!}\sum_{m=0}^{\infty}\frac{(t_2\omega_2)^m}{m!}\Gamma(1+\tfrac{n}{\gamma_1})\Gamma(1+\tfrac{m}{\gamma_2})\left(1+3\eta(1-2^{\frac{-n}{\gamma_1}})(1-\ 2^{\frac{-m}{\gamma_2}})+5\eta^2\left(2\times3^{\frac{-n}{\gamma_1}}-3\times2^{\frac{-n}{\gamma_1}}+1\right)\left(2\times3^{\frac{-m}{\gamma_2}}-3\times2^{\frac{-m}{\gamma_2}}+1\right)\right).$$

Thus, it is easy to show that the product moments of the BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$ family is given by

$$E(Z_1^rZ_2^s)=\omega_1^r\omega_2^s\Gamma(1+\tfrac{r}{\gamma_1})\Gamma(1+\tfrac{s}{\gamma_2})\left(1+3\eta\left(1-2^{\frac{-r}{\gamma_1}}\right)\left(1-2^{\frac{-s}{\gamma_2}}\right)\right.$$
$$\left.+5\eta^2\left(2\times3^{\frac{-r}{\gamma_1}}-3\times2^{\frac{-r}{\gamma_1}}+1\right)\left(2\times3^{\frac{-s}{\gamma_2}}-3\times2^{\frac{-s}{\gamma_2}}+1\right)\right),r,s=1,2,\dots \tag{3.1}$$

Thus, by using (3.1) at $r=s=1$, we get

$$E(Z_1Z_2)=\omega_1\omega_2\Gamma(1+\tfrac{1}{\gamma_1})\Gamma(1+\tfrac{1}{\gamma_2})\left(1+3\eta\left(1-2^{\frac{-1}{\gamma_1}}\right)\left(1-2^{\frac{-1}{\gamma_2}}\right)\right.$$
$$\left.+5\eta^2\left(2\times3^{\frac{-1}{\gamma_1}}-3\times2^{\frac{-1}{\gamma_1}}+1\right)\left(2\times3^{\frac{-1}{\gamma_2}}-3\times2^{\frac{-1}{\gamma_2}}+1\right)\right).$$

Therefore, the coefficient of correlation between Z_1 and Z_2 is

$$\rho_{Z_1,Z_2}=\frac{3\eta\left(1-2^{\frac{-1}{\gamma_1}}\right)\left(1-2^{\frac{-1}{\gamma_2}}\right)+5\eta^2\left(2\times3^{\frac{-1}{\gamma_1}}-3\times2^{\frac{-1}{\gamma_1}}+1\right)\left(2\times3^{\frac{-1}{\gamma_2}}-3\times2^{\frac{-1}{\gamma_2}}+1\right)}{\sqrt{\left(\frac{\Gamma(1+\frac{2}{\gamma_1})}{(\Gamma(1+\frac{1}{\gamma_1}))^2}-1\right)\left(\frac{\Gamma(1+\frac{2}{\gamma_2})}{(\Gamma(1+\frac{1}{\gamma_2}))^2}-1\right)}}. \tag{3.2}$$

We notice that $\rho_{Z_1,Z_2}=0$ when $\eta=0$, this implies that Z_1 and Z_2 are independent.

Table 1 shows the correlation, ρ_{Z_1,Z_2}, for BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$, by using (3.2). The result of this table manifests that the maximum and minimum values of ρ_{Z_1,Z_2} from BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$, are 0.511894, and -0.511767, respectively. Generally, the value of ρ_{Z_1,Z_2} increases with η increases. In contrast, the value of ρ_{Z_1,Z_2} decreases when η decreases.

Table 1: The coefficient of correlation, ρ_{z_1,z_2}, in BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$

$\eta=0.2$			$\eta=0.3$			$\eta=0.4$			$\eta=0.529$		
ρ_{z_1,z_2}	γ_1	γ_2	ρ_{z_1,z_2}	γ_1	γ_2	ρ_{z_1,z_2}	γ_1	γ_2	ρ_{z_1,z_2}	γ_1	γ_2
0.177354	1.4	1.4	0.263791	1.2	1.5	0.380777	2	4	0.477099	1.2	1.7
0.180084	1.4	1.6	0.266401	1.2	1.7	0.374941	2	7	0.474078	1.2	1.5
0.181799	1.4	1.8	0.268403	1.2	2	0.371214	2	10	0.478688	1.2	2
0.188622	1.9	2	0.289645	3	4	0.384616	3	5	0.492449	1.5	1.8
0.190372	1.9	2.9	0.288524	3	5	0.38304	3	6	0.494058	1.5	2
0.193363	3	3	0.286838	5	6	0.382654	5	6	0.503961	2	2
0.193113	3	4	0.285946	5	7	0.383855	5	5	0.504281	3.1	1.9
0.191123	5	6	0.282178	8	9	0.381507	5	7	0.505931	3.1	2
0.190508	5	7	0.281117	8	11	0.38048	7	6	0.510453	3.1	2.5
0.189066	5	10	0.28029	8	13	0.378525	7	8	0.511586	3.1	2.9
0.187935	7	10	0.279347	9	14	0.377693	7	9	0.511684	3.1	3
0.187584	7	11	0.27904	9	15	0.377621	8	8	**0.511748**	**3.18**	**3.18**

$\eta=-0.2$			$\eta=-0.3$			$\eta=-0.4$			$\eta=-0.529$		
ρ_{z_1,z_2}	γ_1	γ_2	ρ_{z_1,z_2}	γ_1	γ_2	ρ_{z_1,z_2}	γ_1	γ_2	ρ_{z_1,z_2}	γ_1	γ_2
-0.17197	1.4	1.4	-0.25053	1.2	1.5	-0.38188	2	4	-0.443162	1.2	1.7
-0.17564	1.4	1.6	-0.25549	1.2	1.7	-0.37960	2	7	-0.432836	1.2	1.5
-0.17815	1.4	1.8	-0.26037	1.2	2	-0.37731	2	10	-0.45372	1.2	2
-0.18681	1.9	2	-0.28980	3	4	-0.38528	3	5	-0.469267	1.5	1.8
-0.18983	1.9	2.9	-0.28890	3	5	-0.38398	3	6	-0.475208	1.5	2
-0.19327	3	3	-0.28592	5	6	-0.38103	5	6	-0.49255	2	2
-0.19318	3	4	-0.28484	5	7	-0.38269	5	5	-0.501871	3.1	1.9
-0.19072	5	6	-0.27958	8	9	-0.37954	5	7	-0.503761	3.1	2
-0.19002	5	7	-0.27821	8	11	-0.37772	7	6	-0.509245	3.1	2.5
-0.18842	5	10	-0.27717	8	13	-0.37476	7	8	-0.510941	3.1	2.9
-0.18685	7	10	-0.27587	9	14	-0.37360	7	9	-0.511159	3.1	3
-0.18644	7	11	-0.27548	9	15	-0.37338	8	8	**-0.511767**	**3.35**	**3.35**

IV. Reliability Function

Sreelakshmi (2018) introduced the relationship between copula and reliability copula which is defined as follows

$$\Re(z_1,z_2)=1-G_{Z_1}(z_1)-G_{Z_2}(z_2)+C\big(G_{Z_1}(z_1),G_{Z_2}(z_2)\big).$$

The following is the reliability function for BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$

$$\Re(z_1,z_2)=e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\left[1+3\eta\left(1-e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1-e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\right.$$

$$\left.+\,5\eta^2\left(1-2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1-2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\left(1-e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1-e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\right].$$

1) Hazard rate function

The bivariate hazard rate function at a point (z_1,z_2) is defined, according to Basu (1971), by $\mathbb{H}(z_1,z_2)=\frac{g_{Z_1,Z_2}(z_1,z_2)}{\Re(z_1,z_2)}$. Thus, we get

$$\mathbb{H}(z_1,z_2)=\frac{\frac{\gamma_1\gamma_2}{\omega_1\omega_2}\left(\frac{z_1}{\omega_1}\right)^{\gamma_1-1}\left(\frac{z_2}{\omega_2}\right)^{\gamma_2-1}\left[1+3\eta\beta_1\beta_2+\frac{5\eta^2}{4}(3\beta_1^2-1)(3\beta_2^2-1)\right]}{\left[1+\frac{3}{4}\eta(1+\beta_1)(1+\beta_2)+\frac{5}{4}\eta^2\beta_1\beta_2(1+\beta_1)(1+\beta_2)\right]}, \tag{4.1}$$

where $\beta_j=\left(1-2e^{-\left(\frac{z_j}{\omega_j}\right)^{\gamma_j}}\right),\;\; j=1,2.$

One of the key constraints of Basu (1971) is that $\mathbb{H}(z_1,z_2)$, as defined by (4.1), is not a vector quantity, as defined by $\Re^2\rightarrow\Re$. The bivariate hazard rate function was created in vector form by Johnson et al. (1975) and Sreelakshmi (2018) to get around this restriction.

$$\mathbb{H}(z_1,z_2)=\left(\frac{-\partial\ln\Re(z_1,z_2)}{\partial z_1},\frac{-\partial\ln\Re(z_1,z_2)}{\partial z_2}\right), \tag{4.2}$$

where, $\Re$ denotes the bivariate reliability function for SAR copula. For the FGM copula Vaidyanathan (2016) studied the elements in the vector (4.2). For the BW-SARD copula, we can, after simple algebra, get the following relations:

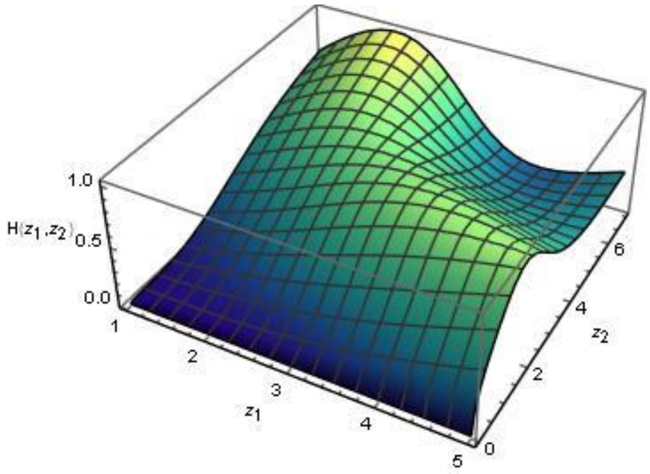
$$\frac{-\partial\ln\Re(z_1,z_2)}{\partial z_1}=\frac{\gamma_1}{\omega_1}\left(\frac{z_1}{\omega_1}\right)^{\gamma_1-1}\left(1-\frac{3\alpha+5\alpha^2\beta_2(2\beta_1+1)}{\left(\frac{(1-\beta_1)(1+\beta_2)}{4}\right)^{-1}+(3\alpha+5\alpha^2\beta_1\beta_2)\left(\left(\frac{(1-\beta_1)}{2}\right)^{-1}-1\right)}\right)$$

and

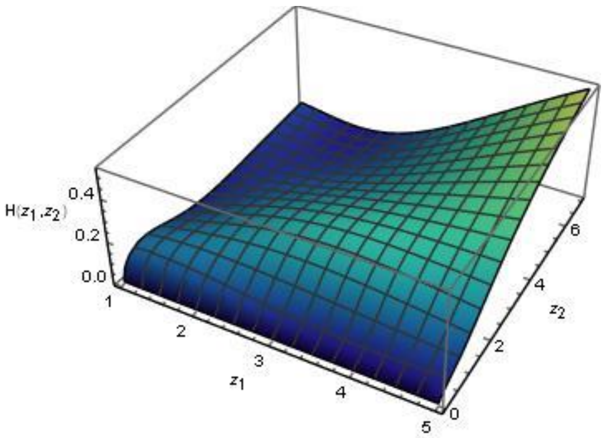
$$\frac{-\partial\ln\Re(z_1,z_2)}{\partial z_2}=\frac{\gamma_2}{\omega_2}\left(\frac{z_2}{\omega_2}\right)^{\gamma_2-1}\left(1-\frac{3\alpha+5\alpha^2\beta_1(2\beta_2+1)}{\left(\frac{(1-\beta_2)(1+\beta_1)}{4}\right)^{-1}+(3\alpha+5\alpha^2\beta_1\beta_2)\left(\left(\frac{(1-\beta_2)}{2}\right)^{-1}-1\right)}\right).$$

The vector hazard rate function of BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$ is obtained by substituting (4.3) and (4.4) in (4.2).

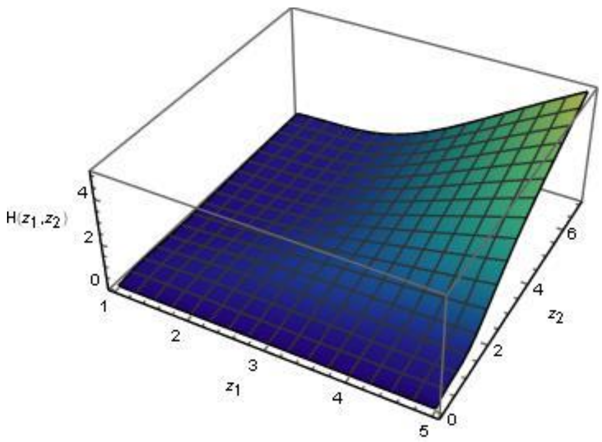
Figures 2 depicts 3D plots of the joint hazard rate function (JHRF) of a BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$ for various parameter values.



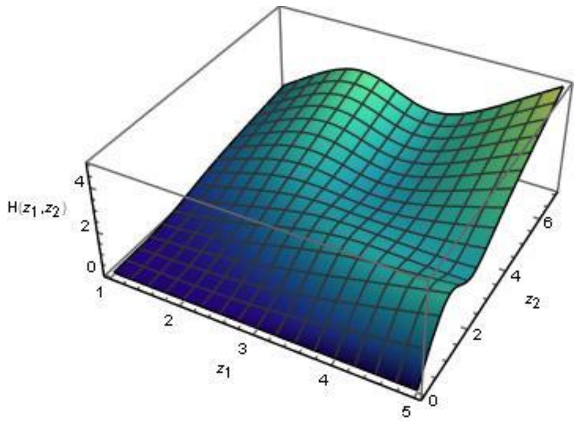
(a) $\gamma_1 = 1.8, \omega_1 = 3.5, \gamma_2 = 1.5, \omega_2 = 2.5, \eta = -0.5$



(b) $\gamma_1 = 1.8, \omega_1 = 3.5, \gamma_2 = 1.5, \omega_2 = 2.5, \eta = 0.5$



(c) $\gamma_1 = 2.5, \omega_1 = 3, \gamma_2 = 1.9, \omega_2 = 2, \eta = 0.4$



(d) $\gamma_1 = 2.5, \omega_1 = 3, \gamma_2 = 1.9, \omega_2 = 2, \eta = -0.4$

Figure 2: JHRF for BW-SARD

2) Reversed hazard rate function

The reversed hazard (RH) function at a point (z_1, z_2) is defined as $R\mathbb{H}(z_1, z_2) = \frac{g_{Z_1,Z_2}(z_1,z_2)}{G_{Z_1,Z_2}(z_1,z_2)}$. Thus, we get

$$R\mathbb{H}(z_1,z_2) = \frac{\frac{\gamma_1\gamma_2}{\omega_1\omega_2}\left(\frac{z_1}{\omega_1}\right)^{\gamma_1-1}\left(\frac{z_2}{\omega_2}\right)^{\gamma_2-1} e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\left[1+3\eta\beta_1\beta_2+\frac{5\eta^2}{4}(3\beta_1^2-1)(3\beta_2^2-1)\right]}{\left(1-e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1-e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\left[1+3\eta e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}} +5\eta^2\beta_1\beta_2 e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right]}.$$

3) Positive quadrant dependence

Our goal in this section is to examine the positive quadrant dependence property, denoted by PQD, (negative quadrant dependence, denoted by NQD) of the RVs Z_1 and Z_2 associated with the BW-SARD(γ_1, ω_1; γ_2, ω_2). As a type of RV dependence, PQD was introduced by Lehmann (1966). This kind of dependence describes the joint behavior of two RVs when they are both large (or small). More specifically, two RVs are PQD if there is at least as much chance that they are both tiny at the same time as there would be if they were independent. The following Theorem reveals these properties for the BW-SARD($\gamma_1, \omega_1; \gamma_2, \omega_2$).

Theorem 1 The BW-SARD($\gamma_1, \omega_1; \gamma_2, \omega_2$) is PQD (NQD) for the positive (negative) value of η.

Proof. Consider

$$\zeta(z_1, z_2; \eta) = P(Z_1 > z_1, Z_2 > z_2) - P(Z_1 > z_1)P(Z_2 > z_2) = \mathfrak{R}(z_1, z_2) - \mathfrak{R}(z_1)\mathfrak{R}(z_2)$$

$$= e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}} \left[1 + 3\eta\left(1 - e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 - e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right) + 5\eta^2\left(1 - 2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\right.$$

$$\left.\times\left(1 - 2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\left(1 - e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 - e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\right] - e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}$$

$$= e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}} e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\left(1 - e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 - e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\left[3\eta + 5\eta^2\left(1 - 2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 - 2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\right]$$

$$= \delta(z_1, z_2)\left[3\eta + 5\eta^2\left(1 - 2e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\right)\left(1 - 2e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\right)\right]$$

$$= \delta(z_1,z_2)\left[3\eta + 5\eta^2(2G_{Z_1}(z_1;\gamma_1,\omega_1) - 1)(2G_{Z_2}(z_2;\gamma_2,\omega_2) - 1)\right]$$

$$= 5\eta\delta(z_1,z_2)\left[0.6 + \eta(2G_{Z_1}(z_1;\gamma_1,\omega_1) - 1)(2G_{Z_2}(z_2;\gamma_2,\omega_2) - 1)\right] = \eta\Delta(z_1,z_2;\eta),$$

where $\delta(z_1,z_2) = G(z_1)G(z_2)\Re(z_1)\Re(z_2)$ and $\Delta(z_1,z_2;\eta) = 5\delta(z_1,z_2)[0.6 + \eta(2G_{Z_1}(z_1;\gamma_1,\omega_1) - 1)\,(2G_{Z_2}(z_2;\gamma_2,\omega_2) - 1)]$.

On one hand, the function $\delta(z_1,z_2)$, for all values of z_1 and z_2 is always non-negative, because the DF and reliability function take values ranging from zero to one. On the other hand, we have $|2G_{Z_i}(z_i;\gamma_i,\omega_i) - 1)| \le 1, i = 1,2$, and $|\eta| \le 0.529$. Therefore, we get

$$5[0.6 - 0.529] \le \Delta(z_1,z_2) \le 5[0.6 + 0.529],$$

which implies $\zeta(z_1,z_2) \ge 0$, if $\eta \ge 0$, and $\zeta(z_1,z_2) \le 0$, if $\eta \le 0$. This completes the proof of the theorem.

Remark 4.1 *It is worth mentioning that, the RVs Z_1 and Z_2 are PQD if and only if $\phi(Z_1)$ and $\psi(Z_2)$ are PQD for any increasing functions $\phi(.)$ and $\psi(.)$. This indicates that PQD is a property of the underlying copula (Sarmanov copula) and is not influenced by the marginals (cf. Joe, 1997).*

4) Mean residual life

The mean residual life (MRL) refers to the expected remaining lifespan of a unit after it has already lasted for a given duration of time t. The MRL function, similar to the PDF or the characteristic function, fully characterizes a distribution with a finite mean. It can be used to derive the distribution by an inversion formula (cf. Guess and Proschan, 1988). The MRL is utilized for both parametric and nonparametric modeling. Actuaries utilize the Minimum Reserve Liability MRL method to determine the appropriate rates and benefits for life insurance. Researchers in the biomedical field examine survivorship studies using the MRL method.

Shanbag and Kotz (1987) introduced the concept of the MRL for vector-valued RVs as

$$\mathfrak{m}(z_1,z_2) = (\mathfrak{m}_1(z_1,z_2), \mathfrak{m}_2(z_1,z_2)), \tag{4.3}$$

where

$$\mathfrak{m}_1(z_1,z_2) = E(Z_1 - z_1 | Z_1 \ge z_1, Z_2 \ge z_2)$$

and

$$\mathfrak{m}_2(z_1,z_2) = E(Z_2 - z_2 | Z_1 \ge z_1, Z_2 \ge z_2).$$

The expressions for $\mathfrak{m}_1(z_1,z_2)$ and $\mathfrak{m}_2(z_1,z_2)$ in BW-SARD$(\gamma_1,\omega_1;\gamma_2,\omega_2)$ are obtained as

$$\mathfrak{m}_1(z_1,z_2) = \frac{\omega_1\Gamma(1+\frac{1}{\gamma_1})\left(1+3\eta\beta_2\left(1-2^{\frac{-1}{\gamma_1}}\right)+\frac{5}{2}\eta^2(3\beta_2^2-1)\left(2\times 3^{\frac{-1}{\gamma_1}}-3\times 2^{\frac{-1}{\gamma_1}}+1\right)\right)}{e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\left[1+\frac{3}{4}\eta(1+\beta_1)(1+\beta_2)+\frac{5}{4}\eta^2\beta_1\beta_2(1+\beta_1)(1+\beta_2)\right]}$$

and

$$\mathfrak{m}_2(z_1,z_2) = \frac{\omega_2\Gamma(1+\frac{1}{\gamma_2})\left(1+3\eta\beta_1\left(1-2^{\frac{-1}{\gamma_2}}\right)+\frac{5}{2}\eta^2(3\beta_1^2-1)\left(2\times 3^{\frac{-1}{\gamma_2}}-3\times 2^{\frac{-1}{\gamma_2}}+1\right)\right)}{e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\left[1+\frac{3}{4}\eta(1+\beta_1)(1+\beta_2)+\frac{5}{4}\eta^2\beta_1\beta_2(1+\beta_1)(1+\beta_2)\right]}$$

Substituting (4.6) and (4.7) in (4.5) yields BW-SARD's MRL.

5) Vitality function

The vitality function is a useful tool for modeling data related to the duration of life. Kupka and Loo (1989) extensively investigated it in the context of their studies on the process of aging. Kotz and Shanbhag (1980) utilized this concept to generate different characterizations of lifetime distributions. The vitality function provides a more straightforward evaluation of the failure pattern as it is measured by an extended average lifespan, whereas the hazard rate indicates the probability of abrupt death within a given lifespan. The vitality function linked to a non-negative RV Z is defined as $\mathfrak{m}(z) = E(Z|Z > z)$. The bivariate vitality function of random vector (Z_1, Z_2) is defined on a positive domain as a binomial vector as

$$\mathbb{V}(z_1, z_2) = \big(\mathbb{V}_1(z_1, z_2), \mathbb{V}_2(z_1, z_2)\big), \tag{4.4}$$

where

$$\mathbb{V}_1(z_1, z_2) = E(Z_1|Z_1 \geq z_1, Z_2 \geq z_2)$$

and

$$\mathbb{V}_2(z_1, z_2) = E(Z_2|Z_1 \geq z_1, Z_2 \geq z_2).$$

For more details, see Sankaran and Nair (1991). Also, $\mathbb{V}_i(z_1, z_2)$ is related to $\mathfrak{m}_i(z_1, z_2)$ by

$$\mathbb{V}_i(z_1, z_2) = z_i + \mathfrak{m}_i(z_1, z_2), \quad i = 1,2. \tag{4.5}$$

Here $\mathbb{V}_1(z_1, z_2)$ computes the expected lifetime to the first component as the sum of current age z_1 and the average lifetime remaining to it, assuming the second component has survived past age z_2. $\mathbb{V}_2(z_1, z_2)$ has a similar interpretation. Using (4.6) and (4.7) in (4.9), we obtain $\mathbb{V}_1(z_1, z_2)$ and $\mathbb{V}_2(z_1, z_2)$ of BW-SARD(γ_1, ω_1; γ_2, ω_2) as:

$$\mathbb{V}_1(z_1, z_2) = z_1 + \frac{\omega_1 \Gamma(1+\frac{1}{\gamma_1})\left(1+3\eta\beta_2\left(1-2^{\frac{-1}{\gamma_1}}\right)+\frac{5}{2}\eta^2(3\beta_2^2-1)\left(2\times 3^{\frac{-1}{\gamma_1}}-3\times 2^{\frac{-1}{\gamma_1}}+1\right)\right)}{e^{-\left(\frac{z_1}{\omega_1}\right)^{\gamma_1}}\left[1+\frac{3}{4}\eta(1+\beta_1)(1+\beta_2)+\frac{5}{4}\eta^2\beta_1\beta_2(1+\beta_1)(1+\beta_2)\right]}$$

and

$$\mathbb{V}_2(z_1, z_2) = z_2 + \frac{\omega_2 \Gamma(1+\frac{1}{\gamma_2})\left(1+3\eta\beta_1\left(1-2^{\frac{-1}{\gamma_2}}\right)+\frac{5}{2}\eta^2(3\beta_1^2-1)\left(2\times 3^{\frac{-1}{\gamma_2}}-3\times 2^{\frac{-1}{\gamma_2}}+1\right)\right)}{e^{-\left(\frac{z_2}{\omega_2}\right)^{\gamma_2}}\left[1+\frac{3}{4}\eta(1+\beta_1)(1+\beta_2)+\frac{5}{4}\eta^2\beta_1\beta_2(1+\beta_1)(1+\beta_2)\right]}.$$

From (4.10) and (4.11), the vitality function of BW-SARD can be obtained using (4.8).

V. Different Estimation Methods

In this section, we discuss two estimation methods for estimating the unknown parameters of the BW-SARD($\gamma_1, \omega_1; \gamma_2, \omega_2$), the ML and Bayesian estimation.

1) The maximum likelihood estimation

The ML approach is a prevalent and consequential statistical technique. Through the utilization of the ML technique, one can acquire estimates of parameters that possess advantageous statistical qualities, including consistency, asymptotic unbiasedness, efficiency, and asymptotic normality. In order to acquire parameter estimates using the ML approach, it is essential to calculate the parameter estimates that maximize the probability of the sample data. The log likelihood function $\ln L$ is obtained by using the PDF given in (2.2).

$$\ln L = n\ln(\frac{\gamma_1}{\omega_1}) + (\gamma_1 - 1)\sum_{i=1}^{n}\ln(\frac{z_{i,1}}{\omega_1}) - \sum_{i=1}^{n}(\frac{z_{i,1}}{\omega_1})^{\gamma_1} + n\ln(\frac{\gamma_2}{\omega_2}) + (\gamma_2 - 1)\sum_{i=1}^{n}\ln(\frac{z_{i,2}}{\omega_2}) - \sum_{i=1}^{n}(\frac{z_{i,2}}{\omega_2})^{\gamma_2} + \sum_{i=1}^{n}\ln[1 + 3\eta(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}}) + \frac{5\eta^2}{4}(3(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})^2 - 1)(3(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}})^2 - 1)].$$

We obtain the following normal equations by partially differentiating $\ln L$ concerning the vector of parameters $\varphi = (\gamma_1, \omega_1, \gamma_2, \omega_2, \eta)$ and equating them to zero. The following are those derivatives:

$$\frac{\partial \ln L(\varphi)}{\partial \gamma_l} = \sum_{i=1}^{n} \frac{3\eta e^{-\left(\frac{z_{i,1}}{\omega_1}\right)^{\gamma_1}}\left(\frac{z_{i,1}}{\omega_1}\right)^{\gamma_1}\ln\left(\frac{z_{i,1}}{\omega_1}\right)\left[2\beta_{i,j} + 5\eta\beta_{i,l}(3\beta_{i,j}^2 - 1)\right]}{\left[1 + 3\eta\beta_{i,l}\beta_{i,j} + \frac{5\eta^2}{4}(3\beta_{i,l}^2 - 1)(3\beta_{i,j}^2 - 1)\right]}$$

$$+ \frac{n}{\gamma_l} + \sum_{i=1}^{n}\ln\left(\frac{z_{i,l}}{\omega_l}\right) - \sum_{i=1}^{n}\left(\frac{z_{i,l}}{\omega_l}\right)^{\gamma_l}\ln\left(\frac{z_{i,l}}{\omega_l}\right),$$

$$\frac{\partial \ln L(\varphi)}{\partial \omega_l} = \sum_{i=1}^{n} \frac{\frac{3\eta\gamma_l}{\omega_l} e^{-\left(\frac{z_{i,1}}{\omega_1}\right)^{\gamma_1}}\left(\frac{z_{i,1}}{\omega_1}\right)^{\gamma_1}\left[2\beta_{i,j} + 5\eta\beta_{i,l}(3\beta_{i,j}^2 - 1)\right]}{\left[1 + 3\eta\beta_{i,l}\beta_{i,j} + \frac{5\eta^2}{4}(3\beta_{i,l}^2 - 1)(3\beta_{i,j}^2 - 1)\right]}$$

$$- \frac{n}{\omega_l} + \frac{1 - \gamma_l}{\omega_l} + \sum_{i=1}^{n}\frac{\gamma_l}{\omega_l}\left(\frac{z_{i,l}}{\omega_l}\right)^{\gamma_l},$$

and

$$\frac{\partial \ln L(\varphi)}{\partial \eta} = \sum_{i=1}^{n} \frac{3\beta_{i,l}\beta_{i,j} + \frac{5\eta}{2}(3\beta_{i,l}^2 - 1)(3\beta_{i,j}^2 - 1)}{\left[1 + 3\eta\beta_{i,l}\beta_{i,j} + \frac{5\eta^2}{4}(3\beta_{i,l}^2 - 1)(3\beta_{i,j}^2 - 1)\right]},$$

where $\beta_{i,k} = (1 - 2e^{-(\frac{z_{i,k}}{\omega_k})^{\gamma_k}}),\ k = l, j;\ l, j = 1,2$, and $l \neq j$.

2) Bayesian estimation

The Bayesian estimate approach is a robust method for inferring unknown parameters using observed data. The Bayes Theorem, a fundamental idea in probability theory, allows for incorporating of new information to revise the likelihood of a hypothesis. Incorporating past knowledge into the estimation process, this approach offers several advantages compared to the conventional machine learning approach. Furthermore, it can evaluate the level of uncertainty associated with each parameter. In order to proceed, we must choose a prior PDF and hyperparameter values that accurately represent our beliefs about the data. For the parameters γ_l and $\omega_l,\ l = 1,2$, we select gamma-independent priors, specifically,

$$\Pi_i(\gamma_l) \propto \gamma_l^{a_l - 1} e^{-b_l\gamma_l},\ \gamma_l > 0, a_l, b_l > 0,\ l = 1,2,$$

and

$$\Pi_{ii}(\omega_l) \propto \omega_l^{c_l - 1} e^{-d_l\omega_l},\ \omega_l > 0, c_l, d_l > 0,\ l = 1,2.$$

While the copula parameter η has uniform prior distribution. The prior JPDF is given by

$$\Pi(\varphi) \propto \gamma_1^{a_1 - 1} e^{-b_1\gamma_1}\omega_1^{c_1 - 1} e^{-d_1\omega_1}\gamma_2^{a_2 - 1} e^{-b_2\gamma_2}\omega_2^{c_2 - 1} e^{-d_2\omega_2}.$$

One can use the likelihood method's estimate and variance-covariance matrix to find out how to elicit the hyper-parameters of the independent joint prior.. When representing the obtained hyper-parameters, the mean and variance of the gamma prior can be utilized. For more information, refer to Gupta and Kundu

(2002), Dey et al. (2016), and Hamdy and Almetwally (2023). The parameters γ_l and ω_l where $l = 1{,}2$, of BW-SARD should be well-known and positive. The likelihood function is given by

$$L(\varphi) = (\frac{\gamma_1}{\omega_1})^n \prod_{i=1}^{n} (\frac{z_{i,1}}{\omega_1})^{(\gamma_1-1)} e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}} (\frac{\gamma_2}{\omega_2})^n \prod_{i=1}^{n} (\frac{z_{i,2}}{\omega_2})^{(\gamma_2-1)} e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}} \prod_{i=1}^{n} [1 + 3\eta(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}}) + \frac{5\eta^2}{4}(3(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})^2 - 1)(3(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}})^2 - 1)].$$

The corresponding posterior density is given as follows:

$$\Pi(\varphi|z_1, z_2) = \gamma_1^{n+a_1-1} \omega_1^{-n+c_1-1} \gamma_2^{n+a_2-1} \omega_2^{-n+c_2-1} e^{-b_1\gamma_1 - d_1\omega_1 - b_2\gamma_2 - d_2\omega_2} \prod_{i=1}^{n} (\frac{z_{i,1}}{\omega_1})^{(\gamma_1-1)} e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}} (\frac{\gamma_2}{\omega_2})^n \times \prod_{i=1}^{n} (\frac{z_{i,2}}{\omega_2})^{(\gamma_2-1)} e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}} \prod_{i=1}^{n} [1 + 3\eta(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}}) + \frac{5\eta^2}{4}(3(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})^2 - 1)(3(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}})^2 - 1)].$$

The marginal posterior distributions $\Pi(\gamma_l|\omega_l, \eta, z_1, z_2)$, $\Pi(\omega_l|\gamma_l, \eta, z_1, z_2)$, and $\Pi(\eta|\gamma_l, \omega_1, z_1, z_2)$ of the parameters γ_l and $\omega_l, l = 1{,}2$, may be found by integrating out the nuisance parameters from the posterior distribution $\Pi(\varphi|z_1, z_2)$ as follows:

$$\Pi(\gamma_l|z_1, z_2) \propto \gamma_l^{n+a_l-1} e^{-b_l\gamma_l} \prod_{i=1}^{n} (\frac{z_{i,l}}{\omega_l})^{(\gamma_l-1)} e^{-(\frac{z_{i,l}}{\omega_l})^{\gamma_l}} \prod_{i=1}^{n} [1 + 3\eta(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}}) + \frac{5\eta^2}{4}(3(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})^2 - 1)(3(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}})^2 - 1)],$$

$$\Pi(\omega_l|z_1, z_2) \propto \omega_l^{-n+c_l-1} e^{-d_l\omega_l} \prod_{i=1}^{n} (\frac{z_{i,l}}{\omega_l})^{(\gamma_l-1)} e^{-(\frac{z_{i,l}}{\omega_l})^{\gamma_l}} \prod_{i=1}^{n} [1 + 3\eta(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}}) + \frac{5\eta^2}{4}(3(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})^2 - 1)(3(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}})^2 - 1)],$$

and

$$\Pi(\eta|z_1, z_2) \propto \prod_{i=1}^{n} [1 + 3\eta(1 - 2e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}}) + \frac{5\eta^2}{4}(3(1 - e^{-(\frac{z_{i,1}}{\omega_1})^{\gamma_1}})^2 - 1)(3(1 - 2e^{-(\frac{z_{i,2}}{\omega_2})^{\gamma_2}})^2 - 1)], where \;\; l = 1{,}2.$$

VI. Application of Real Data

In this analysis, we will investigate a practical dataset that has two variables. Both serum creatinine (SrCr) and the duration of diabetes are considered. We are assessing the potential consequences arising from the patient's preexisting diabetes. This study forecasts the potential difficulties that may arise by categorizing the data into two groups: diabetic nephropathy groups (with SrCr levels of 1.4mg/dl) and nondiabetic nephropathy groups (also with SrCr values of 1.4mg/dl). Between January 2012 and August 2013, the pathology reports of these individuals were collected from the pathology laboratory of Dr. Lal. The data has been analyzed and examined by Grover et al. (2014). The bivariate data set is given in Table 2. We fit the Weibull distribution to the duration of diabetes and serum creatinine separately. The ML estimates of the scale and shape parameters $(\gamma_i, \omega_i), i = 1{,}2$, are $(3.37951, 19.0555)$ and $(9.02796, 1.82242)$, respectively, $\eta = 0.0622486$. As an illustration of the data, Figure 3 provides a basic statistical analysis.

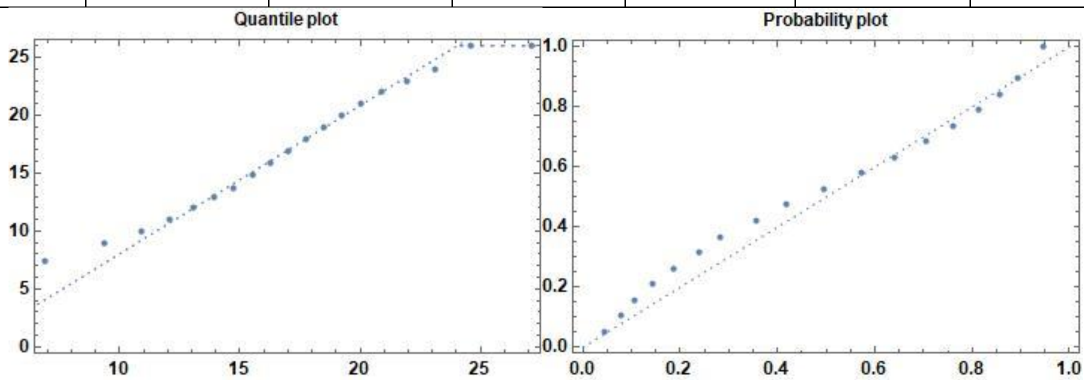
Table 3 discusses the negative log-likelihood values $-\ln L$, and different measures, namely Akaike information criterion (AIC), corrected AIC (AICc), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and consistent AIC (CAIC). The ML estimates of parameters for the BW-SARD distribution are calculated, $\hat{\gamma}_1 = 3.37951, \hat{\omega}_1 = 19.0555, \hat{\gamma}_2 = 9.02796, \hat{\omega}_2 = 1.82242, \hat{\eta} = 0.0622486$. Table 3 clearly shows that, in terms of data fit, the BW-SARD model performs better than the bivariate generalized exponential Sarmanov distribution (BGE-SARD) and the bivariate exponential Sarmanov distribution (BE-SARD).

Table 2: Diabetic nephropathy of two components

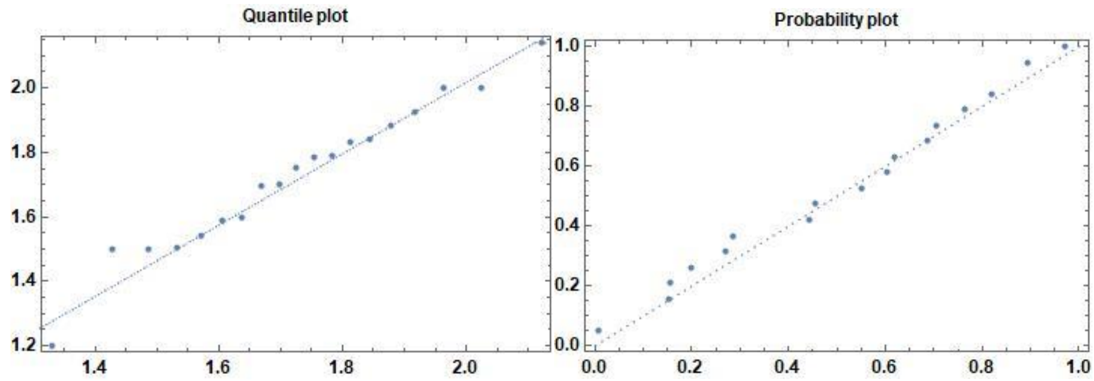
No.	Duration of diabetes	Serum creatinine	No.	Duration of diabetes	Serum creatinine
1	7.4	1.925	11	18	1.832
2	9	1.5	12	19	1.59
3	10	2	13	20	1.7833
4	11	1.6	14	21	1.2
5	12	1.7	15	22	1.792
6	13	1.7533	16	23	1.5
7	13.75	1.54	17	24	1.5033
8	14.92	1.694	18	26	2
9	15.8286	1.8843	19	26.6	2.14
10	16.9333	1.8433			

Table 3: $-\ln L$, AIC, AICc, BIC, HQIC, CAIC

Models	$-\ln L$	AIC	AICc	BIC	HQIC	CAIC
BW-SARD	57.7031	125.406	130.022	130.128	126.205	130.022
BGE-SARD	60.9082	131.816	136.432	136.539	132.616	136.432
BE-SARD	98.8	203.6	205.2	206.433	204.079	205.2



(a) Duration of diabetes



(b) Serum creatinine

Figure 3: Classical representation of diabetic nephropathy data

VII. Conclusion

Bivariate Weibull distribution (BW-SARD) based on the Sarmanov copula is proposed in this work. It is a major and original contribution to the area of bivariate modeling in the context of analysis of bivariate data. Maximum and minimum values of the coefficient of correlation between Z_1 and Z_2 for the proposed distribution BW-SARD are 0.512, and -0.512, respectively. With its modest correlation value, this model can handle bivariate data with weak and moderate correlations, unlike models based on the FGM family. Additionally covered have been dependability metrics including the vitality function, mean residual life function, and hazard rate function. Moreover, the proposed model was demonstrated to satisfy the PQD(NQD) characteristic depending on the sign of the shape. Finally, the significance and adaptability of the BW-SARD were investigated using a bivariate data set.

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