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Calculating the entropy and number of spanning trees of a complex network model

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Abstract: In real-world situations, complex networks are prevalent. Free-scale networks, small-world networks, and fractals are examples of complex networks. In this paper, we generalize the models presented for El Atik and Ma. We discuss some topological properties of the proposed model like the clustering coefficient and the diameter. Also, the entropy and the number of spanning trees are significant measures related to the reliability and communication aspects of the network. Therefore, we calculate analytically the entropy and number of spanning trees of the model, which clarifies that the results of El Atik et al. are unerring whereas the given results of Ma and Yao are erroneous.

Keywords: Complex network, Clustering coefficient, Number of spanning trees.

2020 AMS Subject Classifications: 05C05, 05C50.

1 Introduction

Natural phenomena are easily expressed with the aid of networks. A complex network is a distinguished category of graph networks which is ramified with unpredictable topological properties. Researchers have primarily studied complex networks with a focus on free-scale networks, small-world networks, and fractals. A free-scale network [35] means that its degree distribution goes along with a power law which can be expressed mathematically as

$$P(d) \sim d^{-\delta}, \quad 2 < \delta < 3.$$

A small-world network [17] is characterized by its large clustering coefficient, a lower diameter, and a small average path length. While a fractal network [34] is distinguished by a hierarchical property called "self-similarity" allowing it to replicate its structure and dynamics. The study of these networks has intensively been observed and discussed as it has numerous applications in various fields like mathematics, computer science,

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biology, biochemistry, physics even social communications [18, 22, 29]. For more applications of complex networks, see [1, 2, 5–7, 9, 11, 14, 31].
The problem of finding the topological measurements of a network and their effect on the dynamical processes of the network is still considered a significant challenge. Sub-graphs of the underlying network play important roles in characterizing the structure of the network and identifying the dynamical features operating on it. Spanning trees are one of the most fundamental categories among a range of sub-graphs. Reliability [8], transport [33], and connectivity [24] are only a few of the applications.
The number of spanning trees of a network is associated to the resistance [5] among the vertices of the network, an essential invariant for random walks [3] that has found a variety of applications in many fields such as physics, biology, and computer science. Furthermore, the number of spanning trees of a network is also related to connectivity, a crucial chemistry index according to the work of Kan et al. [24] who counted the spanning trees of some molecular graphs.
The number of spanning trees $\tau(G)$ of a graph G is algebraically evaluated by using Kirchhoff's matrix-tree theorem, which shows that this number can be found in a polynomial time and is equal to any cofactor of the Laplacian matrix of the graph. In particular, let $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$ be the Laplacian eigenvalues of a graph G of order n, then $\tau(G)$ is given by

$$\tau(G) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i(G) \tag{1}$$

For complex networks, applying Kirchhoff's matrix-tree theorem to enumerate $\tau(G)$ is a highly cost and tedious task [12, 13, 16]. Most researchers have provided some procedures to bypass the tiresome and complicated calculations of large determinants. Some authors [15, 20, 21] gave upper bounds to estimate this number. Whereas, several research works [23, 26, 28] presented an exact solution for $\tau(G)$ analytically.
This paper is constructed as follows. Section 2 is demonstrated to present some definitions and results from graph theory that would be used throughout the paper. In Section 3, we investigate the topological characteristics of a class of small-world $F(t)$ introduced in [19, 27] including the average degree, clustering coefficient, and diameter. The entropy and number of spanning trees of $F(t)$ are also discussed and coincided with the results obtained in [19] which enhances their validity. In Section 4, the generalization of model $F(t)$ is proposed and its topological properties are studied. As a consequence, we find that model $\mathbb{F}(t)$ has no small-world feature because of vanishing its clustering coefficient. We give an exact formulas for the entropy and number of spanning trees of $\mathbb{F}(t)$. Section 5 is devoted to a brief summary.

2 Preliminaries

We mention some necessary concepts and facts from graph theory. (For more details, see [4, 10, 32].) Let $G = (V, E)$ be a graph whose set of vertices and set of edges are V and E respectively. Let $n = |V|$ and $e = |E|$ be the number of vertices of G "known as graph-order" and the number of edges of G "also known as graph-size" respectively. The diameter $D(G)$ of graph G is a topological measure for the structure of G that equals the biggest distance between a pair of vertices of G. Suppose that $v \in V$ with degree d_v. Then, the clustering coefficient of v, denoted by $\mathscr{C}_v$, is another topological measure of G which characterizes how the neighbors of v are likely connected. The clustering coefficient $\mathscr{C}_v$ of v is the portion of the entire number of edges e_v existing between the neighbors of v to the number $\frac{d_v(d_v-1)}{2}$ of all possible edges between them. That is to say,

$$\mathscr{C}_v = \frac{2e_v}{d_v(d_v-1)}. \tag{2}$$

Definition 1. *The clustering coefficient $\overline{\mathscr{C}}$ of G is the average value of the clustering coefficients of the whole vertices of G, i.e.,*

$$\overline{\mathscr{C}} = \frac{\sum\limits_{v\in V} \mathscr{C}_v}{n}. \tag{3}$$

Definition 2. *The diameter D of G is the greatest shortest path between any pair of vertices of G.*

A sub-graph $G' = (V', E')$ of G is called a spanning sub-graph of G if $V' = V$ and $E' \subseteq E$.

Definition 3. *A spanning tree T of G is a spanning sub-graph which is a tree.*

The number of spanning trees $\tau(G)$ of a graph G is a vital topological parameter of G which measures the reliability, diffusion properties and connectivity of G.
For a cycle C of length ℓ, we have the following facts:

–$D = \lfloor \frac{\ell}{2} \rfloor$
–$\overline{\mathscr{C}} = \begin{cases} 1 & \text{if } \ell = 3, \\ 0 & \text{otherwise.} \end{cases}$
–$\tau(C) = \ell$.

Let C_i be a cycle of length ℓ_i, for $i = 1, 2, ..., I$. Assume that $\Gamma = \bigcup\limits_{i=1}^{I} C_i,\ i \in \mathbb{Z}_+$ is a graph constructed by connecting finite number of cycles C_i where the intersection of any two cycles of its components is one vertex at most (see Figure 1).

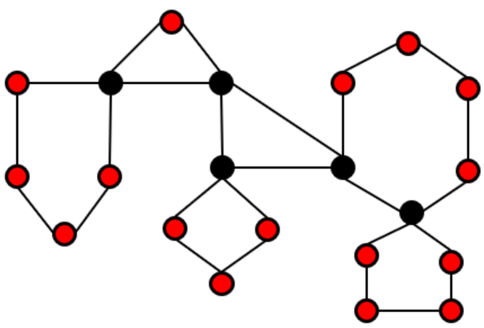

Fig. 1: An example of Γ-graph with $\tau(\Gamma) = 5400$.

Theorem 1. *(See [30].) The number of spanning trees of a Γ-graph is the product of the number of spanning trees of each cycle of Γ. That is,*

$$\tau(\Gamma) = \prod_{i=1}^{I} \tau(C_i) = \prod_{i=1}^{I} \ell_i, \tag{4}$$

where C_i is a cycle of length ℓ_i.

Define $\mathscr{P}$ and $\mathscr{C}$ to be the set of paths of length 2 and the set of cycles of length 3 respectively. Two operations will be defined as:

–Operation 1: In operation 1, each edge $e \in E$ would be replaced with a path $P \in \mathscr{P}$ and this operation can be denoted by $g_1 : E \to \mathscr{P}$.

–Operation 2: In operation 2, each vertex $v \in V$ would be mapped to a cycle $C \in \mathscr{C}$ and this operation can be denoted by $g_2 : V \to \mathscr{C}$.

For example, let us apply the above operations to a cycle of length 3. One could obtain the results depicted in Figure 2.

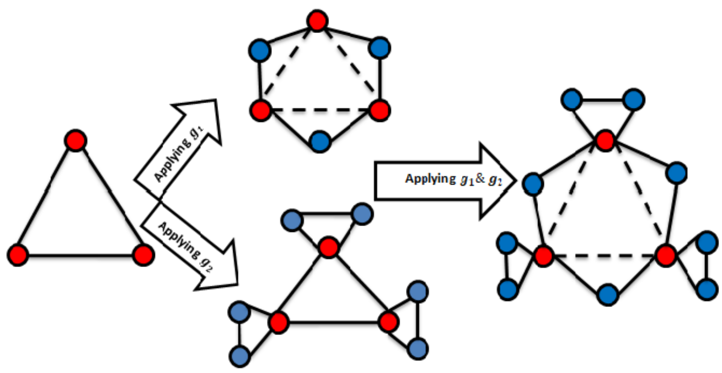

Fig. 2: The diagrams of operations g_1 and g_2

3 Special case: Model $F(t)$ and its topological properties

Model $F(t)$ can be constructed by a recursive process as follows:
At $t = 0$, $F(0)$ is a cycle of length 3. For $t \geq 1$, $F(t)$ is constructed with the aid of $F(t-1)$ by applying the operations g_1 and g_2 to $F(t-1)$. That is, each edge of $F(t-1)$ is changed out to a path of length 2 and each vertex of $F(t-1)$ is mapped to a cycle of length 3. Let n_t and e_t be the order and size of $F(t)$ respectively, then $n_0 = 3, e_0 = 3, n_1 = 12, e_1 = 15$,... and so on.The process of generating $F(t)$ can be repeated indefinitely, see Figure 3. After t time replications, the order and size of $F(t)$ will follow the following recurrence relations:

$$n_t = 3n_{t-1} + e_{t-1}, \quad t \geq 1 \tag{5}$$

$$e_t = 3n_{t-1} + 2e_{t-1}, \quad t \geq 1 \tag{6}$$

From (5) and (6), we can derive the following difference equations:

$$n_t - 5n_{t-1} + 3n_{t-2} = 0 \tag{7}$$

$$e_t - 5e_{t-1} + 3e_{t-2} = 0 \tag{8}$$

Solving (7) and (8) gives

$$n_t = c_1 \left(\frac{5+\sqrt{13}}{2}\right)^t + c_2 \left(\frac{5-\sqrt{13}}{2}\right)^t, \tag{9}$$

$$e_t = c_3 \left(\frac{5+\sqrt{13}}{2}\right)^t + c_4 \left(\frac{5-\sqrt{13}}{2}\right)^t, \tag{10}$$

where

$$c_1 = \frac{3\sqrt{13}+9}{2\sqrt{13}}, \text{ and } c_2 = \frac{3\sqrt{13}-9}{2\sqrt{13}}, \tag{11}$$

$$c_3 = \frac{3\sqrt{13}+15}{2\sqrt{13}}, \text{ and } c_4 = \frac{3\sqrt{13}-15}{2\sqrt{13}}. \tag{12}$$

Next, the topological characteristics of model $F(t)$ prescribed above are determined exactly. We emphasize the average degree, clustering coefficient, diameter, number of spanning trees and entropy.

3.1 Average degree of model $F(t)$

The average degree is an important topological invariant that determines the irregularity degrees of the network. It is defined as the arithmetic mean of the degrees of all vertices.

Lemma 1. *The average degree of $F(t)$ satisfies*

$$\langle d \rangle = \frac{2e_t}{n_t} \approx \frac{2c_3}{c_1} \approx 2.6056. \tag{13}$$

We note that model $F(t)$ is sparse because the average degree is small.

3.2 Clustering coefficient of model $F(t)$

The clustering coefficient is one of the most important parameters of network, since it provides a way to assess the network's local structure. Recall that the clustering coefficient $\mathscr{C}_v$ of a vertex v with degree d_v is the portion of the number of edges that exist between the d_v neighbors of v to the number of the whole edges between them. The clustering coefficient $\overline{\mathscr{C}}$ of the network is the average value of the clustering coefficients of all vertices in the network. Now, we calculate $\mathscr{C}_v, \forall v \in F(t)$ and $\overline{\mathscr{C}}$ of $F(t)$.

For $F(0)$, there are 3 vertices of degree 2 each of which has a clustering coefficient equals 1. For $F(1)$, there are 6 vertices of degree 2 whose clustering coefficient equals 1 and 3 vertices of degree 4 whose clustering coefficient is $\frac{1}{6}$. Generally speaking, the values of the clustering coefficient $\mathscr{C}_v$ of all vertices of degree d_v in model $F(t), t > 1$ are given in the following tables.

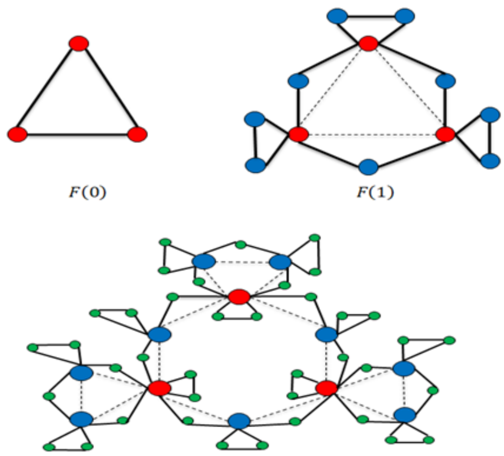

Fig. 3: The diagrams of model $F(t)$ at $t = 0, 1, 2$

d_v	2	2	4	6	...	2t	2t+2
c_v	0	$\frac{2}{2\times 1}$	$\frac{2}{4\times 3}$	$\frac{2}{6\times 5}$	...	$\frac{2}{2t(2t-1)}$	$\frac{2}{(2t+2)(2t+1)}$

The clustering coefficient $\overline{\mathscr{C}}$ of model $F(t)$ is obtained from the following lemma.

Lemma 2. *For sufficiently large t, the clustering coefficient $\overline{\mathscr{C}}$ of model $F(t)$ is given by*

$$\overline{\mathscr{C}} \approx 0.4973 \tag{14}$$

Proof: Clearly,

$$\overline{\mathscr{C}} = \begin{cases} 1 & t = 0, \\ 0.5417 & t = 1, \\ 0.5039 & t = 2, \end{cases}$$

To calculate $\overline{\mathscr{C}}$ for $F(t)$ with sufficiently large t, we need to compute the probability $p_v = \frac{n_v}{n_t}$, where n_v is the number of vertices whose degree is d_v. We mark down the clustering coefficient spectrum of model $F(t)$ in the given table.

d_v	2	2	4	6	...	2t	2t+2
$\mathscr{C}_v$	0	1	$\frac{1}{6}$	$\frac{1}{15}$	...	$\frac{1}{t(2t+1)}$	$\frac{1}{(t+1)(2t+2)}$
p_v	$\frac{e_{t-1}}{n_t}$	$\frac{2n_{t-1}}{n_t}$	$\frac{2n_{t-2}+e_{t-2}}{n_t}$	$\frac{2n_{t-3}+e_{t-3}}{n_t}$	...	$\frac{2n_0+e_0}{n_t}$	$\frac{e_0}{n_t}$

From the above table, we find that $\overline{\mathscr{C}}$ of $F(t)$ for sufficiently large t is given by

$$\overline{\mathscr{C}} = \sum p_v c_v \approx \frac{2n_{t-1}}{n_t} + \frac{2n_{t-2} + e_{t-2}}{6n_t} + \frac{2n_{t-3} + e_{t-3}}{15n_t} \approx 0.4973 \tag{15}$$

3.3 Diameter of model $F(t)$

Lemma 3. *For $t \geq 1$, the exact solution of diameter $D(t)$ of the model $F(t)$ is*

$$D(t) = \begin{cases} 1 & t = 0, \\ D(t-1) + 2^{t-1} \times 3 & t \geq 1. \end{cases} \tag{16}$$

Proof: For $t = 0$, the diameter of $F(0)$ equals 1, and we may say that $D(0) = 1$. For $t = 1$, the diameter of $F(1)$ equals 4, that is $D(1) = 4 = 1+3$. Also, for $t = 2$, the diameter of $F(2)$ is 10, $D(2) = 10 = 4+2\times 3$. It is clear that the structure of $F(t)$ can be decomposed to n_{t-1} cycles of length $2^0 \times 3$, n_{t-2} cycles of length $2\times 3, \ldots, n_0$ cycles of length $2^{t-1}\times 3$, and one cycle of length $2^t \times 3$. Therefore $D(t) = D(t-1) + 2^{t-1}\times 3$.

3.4 The number of spanning trees of model $F(t)$

The number of spanning trees of a network is a vital parameter relevant to the topological attributes and dynamic characteristics of the network. However, counting spanning trees of complex networks and investigating their properties is computationally demanding.

Recall that at any iteration $t = T$, the structure of $F(t)$ can be decomposed to n_{T-1} cycles of length $2^0 \times 3$, n_{T-2} cycles of length $2\times 3, \ldots, n_0$ cycles of length $2^{T-1}\times 3$, and one cycle of length $2^T \times 3$, we can state the following theorem.

Theorem 2. *The number of spanning trees of $F(t)$ model is given by*

$$\tau(F(t)) = 2^{t+\lambda} \times 3^{1+\mu}, \tag{17}$$

where $\lambda = \frac{4c_1}{(3+\sqrt{13})^2}\left(\frac{5+\sqrt{13}}{2}\right)^t + \frac{4c_2}{(3-\sqrt{13})^2}\left(\frac{5-\sqrt{13}}{2}\right)^t - 3$, $\mu = \frac{2c_1}{3+\sqrt{13}}\left(\frac{5+\sqrt{13}}{2}\right)^t + \frac{2c_2}{3-\sqrt{13}}\left(\frac{5-\sqrt{13}}{2}\right)^t$.

Proof: From the structure of model $F(t)$, the number of spanning trees of $F(t)$ is given by

$$\begin{aligned} \tau(F(t)) &= \left[\prod_{i=1}^{t} (2^{i-1}\times 3)^{n_{t-i}}\right] \times 2^t \times 3, \quad t \geq 1 \\ &= 2^{t+\lambda} \times 3^{1+\mu}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} \lambda &= n_{t-2} + 2n_{t-3} + 3n_{t-4} + \ldots + (t-1)n_0 \\ &= c_1 \sum_{i=0}^{t-2} (t-1-i)\left(\frac{5+\sqrt{13}}{2}\right)^i + c_2 \sum_{i=0}^{t-2} (t-1-i)\left(\frac{5-\sqrt{13}}{2}\right)^i \\ &= \frac{4c_1}{(3+\sqrt{13})^2}\left(\frac{5+\sqrt{13}}{2}\right)^t + \frac{4c_2}{(3-\sqrt{13})^2}\left(\frac{5-\sqrt{13}}{2}\right)^t - 3. \end{aligned} \tag{19}$$

$$\begin{aligned} \mu &= n_{t-1} + n_{t-2} + \ldots + n_0 \\ &= c_1 \sum_{i=0}^{t-1}\left(\frac{5+\sqrt{13}}{2}\right)^i + c_2 \sum_{i=0}^{t-1}\left(\frac{5-\sqrt{13}}{2}\right)^i \\ &= \frac{2c_1}{3+\sqrt{13}}\left(\frac{5+\sqrt{13}}{2}\right)^t + \frac{2c_2}{3-\sqrt{13}}\left(\frac{5-\sqrt{13}}{2}\right)^t. \end{aligned} \tag{20}$$

$\tau(F(t))$	Formula of Ma and Yao [27]	Formula of El Atik et al [19]	Our formula
$t=1$	0.0216	2×3^4	2×3^4
$t=2$	3.1453×10^{-5}	$2^5\times 3^{16}$	$2^5\times 3^{16}$
$t=3$	7.4714×10^{-17}	$2^{21}\times 3^{67}$	$2^{21}\times 3^{67}$
$t=4$	3.7595×10^{-68}	$2^{88}\times 3^{286}$	$2^{88}\times 3^{286}$
$t=5$	4.8852×10^{-291}	$2^{374}\times 3^{1228}$	$2^{374}\times 3^{1228}$

Table 1: A comparison between results of $\tau(F(t)), t=1,...,5$

It is obvious that our strategy is based on the divide and conquer procedure. We decompose model $F(t)$ into number of sub-graphs and analyse each sub-graph individually. Moreover, we combine these results together to determine the number of spanning trees of $F(t)$. We notice that El Atik et al. [19] as well as Ma and Yao [27] relied on an induction and iterative procedure. Table 1 shows a comparison between the results of $\tau(F(t)), t=1,\dots,5$ obtained from our formula and other formulas in [19, 27]. Table 1 depicts that our results coincide with the results in [19]. Moreover, it enhances that the results in [27] are incorrect.

3.5 The entropy of spanning trees of model $F(t)$

Theorem 3. *The entropy of spanning trees of $F(t)$ is*

$$\xi(F(t)) = \lim_{t\to\infty} \frac{\ln \tau(F(t))}{n_t} \approx 0.3962 \tag{21}$$

Proof: We have

$$\begin{aligned}\xi(F(t)) &= \lim_{t\to\infty} \frac{\ln \tau(F(t))}{n_t} \\ &= \lim_{t\to\infty} \frac{\ln 2^{t+\lambda} \times 3^{1+\mu}}{c_1 a^t + c_2 b^t},\end{aligned}$$

where $a = \frac{5+\sqrt{13}}{2} > 1$ and $b = \frac{5-\sqrt{13}}{2} < 1$.
From equations (19) and (20), the entropy of spanning trees of $F(t)$ can be estimated as follows:

$$\begin{aligned}\xi(F(t)) &= \lim_{t\to\infty} \frac{(t+\lambda)\ln 2 + (1+\mu)\ln 3}{c_1 a^t + c_2 b^t} \\ &\approx \frac{4}{(3+\sqrt{13})^2}\ln 2 + \frac{2}{3+\sqrt{13}}\ln 3 \approx 0.3962\end{aligned}$$

4 General case: Model $\mathbb{F}(t)$ and its structural properties

In section 3, model $F(t)$ is generated with the aid of a cycle of length 3 which is considered as $F(0)$. Now, we investigate the general case of a class of small world networks $\mathbb{F}(t)$ which is generated by a cycle of length k, where $k \geq 3$. We discuss its construction, analyze its topological properties and compute its number of spanning trees.
Let $\mathbb{P}$ and $\mathbb{C}$ be the set of paths of length $k-1$ and the set of cycles of length k respectively. Define two operations γ_1 and γ_2 to be applied on a graph $G=(V,E)$ as follows:

–In γ_1-operation, each edge $e \in E$ would be exchanged by a path $P \in \mathbb{P}$. Denote this operation by $\gamma_1 : E \to \mathbb{P}$.
–In γ_2-operation, each vertex $v \in V$ would be associated with a cycle $C \in \mathbb{C}$. Denote this operation by $\gamma_2 : V \to \mathbb{C}$.

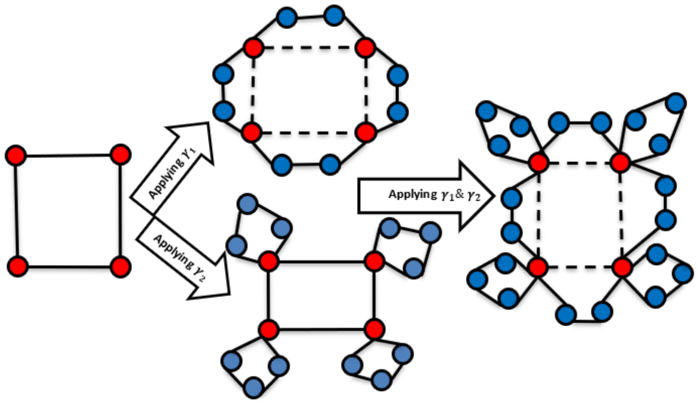

Fig. 4: The diagrams of occurrence of operations γ_1 and γ_2 to cycle of length 4.

Applying operations γ_1 and γ_2 to a cycle of length 4 is illustrated in Figure 4.
The model $\mathbb{F}(t)$ is established as stated below:
At $t = 0$, $\mathbb{F}(0)$ is a cycle of length $k \geq 3$. At $t \geq 1$, $\mathbb{F}(t)$ is generated from $\mathbb{F}(t-1)$ by applying γ_1-operation and γ_2-operation to $\mathbb{F}(t-1)$. Let $\mathbb{N}_t$ and $\mathbb{E}_t$ be the order and size of $\mathbb{F}(t)$ respectively. It is clear that $\mathbb{N}_0 = k, \mathbb{E}_0 = k, \mathbb{N}_1 = 2k^2 - 2k$ and $\mathbb{E}_1 = 2k^2 - k$. It is easy to get recursive relations for $\mathbb{N}_t$ and $\mathbb{E}_t$.

$$\mathbb{N}_t = k\mathbb{N}_{t-1} + (k-2)\mathbb{E}_{t-1} \tag{22}$$

$$\mathbb{E}_t = k\mathbb{N}_{t-1} + (k-1)\mathbb{E}_{t-1} \tag{23}$$

Equations (22) and (23) give us the following difference equations:

$$\mathbb{N}_t + (1-2k)\mathbb{N}_{t-1} + k\mathbb{N}_{t-2} = 0 \tag{24}$$

$$\mathbb{E}_t + (1-2k)\mathbb{E}_{t-1} + k\mathbb{N}_{t-2} = 0 \tag{25}$$

Solving (24) and (25) leads to

$$\mathbb{N}_t = \alpha_1 \beta^t + \alpha_2 \varepsilon^t \tag{26}$$

$$\mathbb{E}_t = \alpha_3 \beta^t + \alpha_3 \varepsilon^t, \tag{27}$$

where

$$\beta = \frac{2k-1+\sqrt{4k^2-8k+1}}{2}, \quad \varepsilon = \frac{2k-1-\sqrt{4k^2-8k+1}}{2},$$

$$\alpha_1 = \frac{2k^2-3k+k\sqrt{4k^2-8k+1}}{2\sqrt{4k^2-8k+1}}, \quad \alpha_2 = \frac{-2k^2+3k+k\sqrt{4k^2-8k+1}}{2\sqrt{4k^2-8k+1}},$$

$$\alpha_3 = \frac{2k^2-k+k\sqrt{4k^2-8k+1}}{2\sqrt{4k^2-8k+1}}, \quad \alpha_4 = \frac{-2k^2+k+k\sqrt{4k^2-8k+1}}{2\sqrt{4k^2-8k+1}}.$$

4.1 Average degree of model $\mathbb{F}(t)$

Lemma 4. *The average degree of model $\mathbb{F}(t)$ is bounded by*

$$2 \leq \langle d \rangle = \frac{2\mathbb{E}_t}{\mathbb{N}_t} \approx \frac{2\alpha_3}{\alpha_1} \leq 2.6056. \tag{28}$$

The sparsity of model $\mathbb{F}(t)$ is valid since the average degree value is small.

4.2 Clustering coefficient of model $\mathbb{F}(t)$

For model $\mathbb{F}(t)$ constructed with the aid of a cycle of length $k > 3$, it is clear that the clustering coefficient $\mathscr{C}_v, \forall v \in \mathbb{F}(t)$ is identically zero since there is no triangle existing between v and its neighbors.

Lemma 5. *The clustering coefficient $\overline{\mathscr{C}}$ of model $\mathbb{F}(t)$, $k \geq 4$ satisfies*

$$\overline{\mathscr{C}} = 0. \tag{29}$$

4.3 Diameter of model $\mathbb{F}(t)$

For model $\mathbb{F}(t)$ constructed by aid of a cycle of length $k \geq 3$, it is obvious that the diameter $D(0)$ of $\mathbb{F}(t)$ is given by

$$D(0) = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even,} \\ \lfloor \frac{k}{2} \rfloor & \text{if } k \text{ is odd.} \end{cases}$$

Lemma 6. *For $t \geq 1$, the exact solution of diameter $D(t)$ of model $\mathbb{F}(t)$ is*

$$D(t) = \begin{cases} 2D(t-1) + \frac{(k-1)^t \times k}{2} & \textit{if } k \textit{ is even,} \\ D(t-1) + \frac{(k-1)^t \times k}{2} & \textit{if } k \textit{ is odd.} \end{cases} \tag{30}$$

4.4 The number of spanning trees of model $\mathbb{F}(t)$

For $t = T$, the structure of model $\mathbb{F}(T)$ can be decomposed to $\mathbb{N}_{T-1}$ cycles of length $(k-1)^0 \times k$, $\mathbb{N}_{T-2}$ cycles of length $(k-1) \times k$, ..., $\mathbb{N}_0$ cycles of length $(k-1)^{t-1} \times k$, and only one cycle of length $(k-1)^t \times k$, the number of spanning trees of $\mathbb{F}(t)$ is estimated as follows:

Theorem 4. *The number of spanning trees of $\mathbb{F}(t)$ model is given by*

$$\tau(\mathbb{F}(t)) = (k-1)^{t+\zeta} \times k^{1+\eta}, \tag{31}$$

where $\zeta = \frac{\alpha_1}{(\beta-1)^2}\beta^t + \frac{\alpha_2}{(\varepsilon-1)^2}\varepsilon^t - \frac{k}{k-2}$ *and* $\eta = \frac{\alpha_1}{\beta-1}\beta^t + \frac{\alpha_2}{\varepsilon-1}\varepsilon^t$.

Proof: Similar to the proof of Theorem 2.

4.5 The entropy of spanning trees of model $\mathbb{F}(t)$

Theorem 5. *The entropy of spanning trees of model $\mathbb{F}(t)$ is given by*

$$\xi(\mathbb{F}(t)) = \lim_{t\to\infty} \frac{\ln \tau(\mathbb{F}(t))}{\mathbb{N}_t} \approx \frac{\ln(k-1)}{(\beta-1)^2} + \frac{k}{\beta-1}. \tag{32}$$

Proof: Similar to the proof of Theorem 3.

5 Conclusion

The examination of complex networks is inspired by empirical studies of many phenomena. It is relevant to many applications as stated above and nowadays it is used to investigate the spread of epidemics such as COVID-19 [25]. In this paper, we suggested a class of complex networks $\mathbb{F}(t)$ constructed in an iterative manner and discussed some of their topological properties. We found the analytic solutions for the entropy and number of spanning trees of $\mathbb{F}(t)$.

Availability of data and material

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

The authors declare that they have no competing interests.

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Authors' contributions

The authors contribute equally to the work.

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