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Research Article **Mathematics**

New qualitative results to a kind of non-autonomous third-order neutral delay differential equations

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ABSTRACT

In this paper, the qualitative properties of solutions for a kind of non-autonomous third-order neutral delay differential equations (NDDEs) are discussed. New results on the stability, boundedness and square integrability of solutions and their derivatives are obtained by using the method of Lyapunov functional. Our results generalize and extend many related results on third-order neural differential equations with and without delay in the literature. Moreover, an example is given to show the correctness and feasibility of the main results.

Introduction

In this research, we examine the asymptotic stability of a zero solution for the following non-autonomous third-order NDDEs:

$$\left[\eta(t)\left(x(t)+\zeta x(t-\sigma)\right)'\right]''+\vartheta(t)x''(t)+\varphi(t)x'(t-\tau)+\psi(t)R\left(x(t-\tau)\right)=0, \qquad (1.1)$$

as well as the square integrability and boundedness of

$$\begin{aligned}&\left[\eta(t)\left(x(t)+\zeta x(t-\sigma)\right)'\right]''+\vartheta(t)x''(t)+\varphi(t)x'(t-\tau)+\psi(t)R\left(x(t-\tau)\right)\\&=Q\left(t,x(t),x(t-\tau),x'(t),x'(t-\tau),x''(t)\right),\end{aligned} \qquad (1.2)$$

for all $t\geq t_1=t_0+\rho$, where $\rho=\sup\{\sigma,\tau\}$,ζ is a constant with $0\leq\zeta\leq 1$ and $\sigma,\tau\geq 0$.

The functions $\eta(t)$, $\vartheta(t)$, $\varphi(t)$, $\psi(t)$, $R(x)$ and $Q(\cdot)$ are continuous depending only on the arguments shown. In addition, it is also supposed that the derivative $\eta''(t)$ exists and continuous.

By a solution of (1.2) we mean a continuous function $x\colon[t_x,\infty)\to\mathbb{R}$ such that $x(t)+\zeta x(t-\sigma)\in\mathbb{C}^3([t_x;\infty),\mathbb{R})$, and which satisfies (1.2) on $[t_x;\infty)$.

A family of differential equations known as neutral differential equations includes both the function and its advanced or delayed argument. Control theory, biology, economics, and engineering are among the fields that use these equations to understand how a system's prior states influence its current behavior.

The qualitative characteristics of neutral differential equations are the subject of several textbooks, frequently in relation to functional or delay differential equations, for instance **(Hale and Verduyn Lunel, 1993; Kolmanovski and Myshkis, 1992 &1999; Kuang, 1993; Lakshmikantham et al., 1994; Strogatz, 2018).**

NDDEs can exhibit rich dynamics, including fixed points, limit cycles, and even more complex attractors. The study of bifurcations in such systems is crucial to understanding how these dynamics evolve as parameters change.

Third-order NDDEs are a specific type of delay differential equation where the highest derivative is third-order, and the equation involves neutral terms meaning that the delayed arguments appear not only in the terms involving derivatives but also in the terms without derivatives. These equations are more complex than standard delay differential equations because of their structure.

NDDEs are applied in many domains, including population dynamics, control theory, and mechanical systems, where the current behavior of the system is influenced by its prior state and rate of change.

In the advanced field of differential equations with delays, Lyapunov functionals are used to analyze the stability, boundedness, and integrability of solutions to NDDEs. Here is an organized method for doing this study, along with several sources for more in-depth knowledge.

Lyapunov functionals provide a powerful tool for analyzing the qualitative properties of NDDEs, though they require careful construction and manipulation to handle the neutral term effectively. The key is to find a functional that appropriately captures the dynamics of the system and yields a negative time derivative, indicating stability.

Several authors have studied that stability and boundedness of solutions for certain differential equations with and without delay using Lyapunov's direct method. We can mention in this direction, the works of **(Fellous et al., 2022; Graef et al., 2015; Mahmoud, 2016; Remili and Beldjerd, 2015; Remili and Oudjedi, 2014& 2016(a,b,c); Remili et al., 2016; Santra et al., 2020)** and the references therein.

The problem of neutral differential equations with and without delay has received considerable attention in recent years, for example, **(Ademola et al., 2019; Graef et al., 2018, 2019 and 2021**; **Khatir, 2022; Khatir et al., 2020; Oudjedi et al., 2019; Philos and Purnaras, 2010; Rahmane and Abdellaoui, 2024; Remili and Oudjedi, 2020)** and some other papers in literature before proving our results.

Asymptotic Stability

Setting $x'(t)=y(t),\ x''(t)=z(t)$ and $Y(t)=\eta(t)\big(x'(t)+\zeta\, x'(t-\sigma)\big)$, then (1.1) is equivalent to the system of first-order differential equations.

$$\left.\begin{aligned} x'(t) &= y(t)\,,\\ y'(t) &= z(t)\,,\\ Z'(t) &= -\vartheta(t)z-\varphi(t)y-\psi(t)R(x)+\varphi(t)\int_{t-\tau}^{t} z(u)du\\ &\quad +\psi(t)\int_{t-\tau}^{t} R'(x(u))y(u)du. \end{aligned}\right\} \qquad (2.1)$$

For the brevity, we put

$$Y(t)=\eta(t)\big(y(t)+\zeta\, y(t-\sigma)\big). \qquad (2.2)$$

According (2.1), we get

$$\begin{aligned} Z(t)=Y'(t) &= \big[\eta(t)\big(y(t)+\zeta\, y(t-\sigma)\big)\big]'\\ &= \eta'(t)\big(y(t)+\zeta y(t-\sigma)\big)+\eta(t)\big(z(t)+\zeta z(t-\sigma)\big). \end{aligned} \qquad (2.3)$$

The following theorem is the stability result of this paper:

Theorem 2.1. Assume that there are positive constants $\vartheta_0,\vartheta_1,\psi_0,\varphi_1,\eta_1,\eta_2,L_1,\ L_2,\delta_1,\delta_2,\beta,A$ and B, such that the following conditions are satisfied, for all $t\geq\ t_1=t_0+\rho$:

$(H_1)\ \ 0<\vartheta_0\leq\vartheta(t)\leq\vartheta_1\,, 0\leq\psi_0\leq\psi(t)\leq\varphi(t)\leq\varphi_1;$

$(H_2)\ \ 0<\eta_1\leq\eta(t)\leq\eta_2\,,\ and\ \ -L_2\leq\eta'(t)\leq 0;$

$(H_3)\ -L_1\leq\varphi'(t)\leq\psi'(t)\leq 0;$

$$(H_4)\ \ R(0)=0\,,\frac{R(x)}{x}\geq\delta_2>0\,,\ \ x\neq 0,\ \ and\ \ |R'(x)|\leq\delta_1, for\ all\ x;$$

$$(H_5)\ \ 2\delta_1\eta_2<\beta<\vartheta_0.$$

Then the zero solution of (2.1) is asymptotically stable, provided that

$$\tau < \min\left\{\frac{A}{2\{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1 M\delta_1 + \varphi_1(1+\delta_1)(L_2+\beta)\}}, \frac{B}{2\{\varphi_1 L_2\zeta(1+\delta_1) + \varphi_1 M + \eta_2\zeta\varphi_1(1+\delta_1)\}}\right\},$$

where

$$M = \beta + (L_2+\eta_2)(1+\zeta),$$

$$\beta\vartheta'(t) + 2\psi_0\{\delta_1\eta_2(2+\zeta) - \beta\} + \zeta\{\varphi_1\eta_2(1+\zeta+\delta_1) + L_2(\vartheta_1-\beta) + \varphi_1 L_2(1+\zeta)\} = -A < 0,$$

$$\eta_1(\beta-\vartheta_0)(2-\zeta) + L_2(\vartheta_1-\beta)(1+\zeta) + \zeta\varphi_1\eta_2 + \zeta\eta_2(\theta_1 - \beta + \varphi_1\zeta) = -B < 0.$$

Proof. Define a Lyapunov functional $\Gamma(\cdot) = \Gamma(t, x_t, y_t, z_t)$ such that:

$$\Gamma(\cdot) = \exp\left(-\frac{1}{p}\int_{t_1}^{t}|\eta'(s)|ds\right)V(t, x_t, y_t, z_t), \tag{2.4}$$

where

$$\begin{aligned} V = V(t, x_t, y_t, z_t) = {}& \beta\,\psi(t)\int_0^x R(\xi)d\xi + \psi(t)R(x)Y + \frac{1}{2}\frac{\varphi(t)}{\eta(t)}Y^2 + \frac{1}{2}Z^2 \\ & +\beta yZ + \frac{1}{2}\beta\,\vartheta(t)y^2 + \mu_1\int_{t-\sigma}^{t} y^2(u)du + \mu_2\int_{t-\sigma}^{t} z^2(u)du \\ & +\mu_3\int_{-\tau}^{0}\int_{t+s}^{t} z^2(u)duds + \mu_4\int_{-\tau}^{0}\int_{t+s}^{t} y^2(u)duds, \end{aligned} \tag{2.5}$$

where μ_i are positive constants and to be selected below suitably, for all $i = 1,2,3,4$.

The functional V defined in (2.5) can be written in the following form

$$\begin{aligned} V = {}& \beta\psi(t)\int_0^x\{\beta - 2\eta(t)R'(\xi)\}R(\xi)d\xi + \frac{\psi(t)}{\eta(t)}\left(\frac{1}{2}Y^2 + \eta(t)R(x)\right)^2 \\ & +\frac{1}{2}\frac{\varphi(t)}{\eta(t)}\left(1 - \frac{1}{2}\frac{\psi(t)}{\varphi(t)}\right)Y^2 + \frac{1}{2}(Z+\beta y)^2 + \frac{1}{2}\beta(\vartheta(t)-\beta)y^2 \\ & +\mu_1\int_{t-\sigma}^{t} y^2(u)du + \mu_2\int_{t-\sigma}^{t} z^2(u)du + \mu_3\int_{-\tau}^{0}\int_{t+s}^{t} z^2(u)duds \\ & +\mu_4\int_{-\tau}^{0}\int_{t+s}^{t} y^2(u)duds. \end{aligned}$$

Since the following integrals

$$\mu_1\int_{t-\sigma}^{t} y^2(u)du\,, \mu_2\int_{t-\sigma}^{t} z^2(u)du,$$

$$\mu_3\int_{-\tau}^{0}\int_{t+s}^{t} z^2(u)duds \;\text{ and }\; \mu_4\int_{-\tau}^{0}\int_{t+s}^{t} y^2(u)duds,$$

are positive, it follows that

$$V=\beta\psi(t)\int_0^x\{\beta-2\eta(t)R'(\xi)\}R(\xi)d\xi+\frac{\psi(t)}{\eta(t)}\left(\frac{1}{2}Y^2+\eta(t)R(x)\right)^2$$
$$+\frac{1}{2}\frac{\varphi(t)}{\eta(t)}\left(1-\frac{1}{2}\frac{\psi(t)}{\varphi(t)}\right)Y^2+\frac{1}{2}(Z+\beta y)^2+\frac{1}{2}\beta(\vartheta(t)-\beta)y^2.$$

By conditions $(H_2)-(H_5)$, it tends to

$$V\geq\frac{1}{2}\psi_0(\beta-2\eta_2\delta_1)x^2+\frac{1}{4}\frac{\psi_0}{\eta_2}Y^2+\frac{1}{2}\beta\,(\vartheta_0-\beta)y^2+\frac{1}{2}(Z+\beta y)^2.$$

Then there exists a positive constant K_1 such that

$$V\geq K_1(x^2+y^2+Y^2+Z^2). \tag{2.6}$$

Based on condition (H_2) and (2.4), we deduce that

$$\Gamma(\cdot)\geq\exp\left\{-\frac{1}{p}(\eta(t)-\eta(t_1))\right\}K_1(x^2+y^2+Y^2+Z^2)$$
$$\geq\exp\left\{\frac{1}{p}(\eta_1-\eta_2)\right\}K_1(x^2+y^2+Y^2+Z^2).$$

Then, we obtain

$$\Gamma(\cdot)\geq K_2\,(x^2+y^2+Y^2+Z^2), \tag{2.7}$$

with

$$K_2=K_1\exp\left(\frac{1}{p}(\eta_1-\eta_2)\right).$$

The derivative of the functional V in (2.5) along the trajectories of the system (2.1) is given by

$$V'=\beta\psi'(t)\int_0^x R(\xi)d\xi+\psi'(t)R(x)Y+\frac{1}{2}\left(\frac{\varphi(t)}{\eta(t)}\right)'Y^2$$
$$+\beta\psi(t)R(x)y\;+\psi(t)R'(x)yY+\psi(t)R(x)Z+\frac{\varphi(t)}{\eta(t)}\,YZ$$
$$+(Z+\beta y)\left\{-\vartheta(t)z-\varphi(t)y-\psi(t)R(x)+\;\varphi(t)\int_{t-\tau}^t z(u)du\right.$$
$$\left.+\psi(t)\int_{t-\tau}^t R'\big(x(u)\big)y(u)du\right\}+\beta zZ+\frac{1}{2}\beta\vartheta'(t)y^2$$
$$+\beta\vartheta(t)yz+\mu_1y^2-\mu_1y^2(t-\sigma)+\mu_2z^2-\mu_2\,z^2(t-\sigma)$$
$$+\mu_3\tau z^2-\mu_3\int_{t-\tau}^t z^2(s)ds+\mu_4\tau y^2-\mu_4\int_{t-\tau}^t y^2(s)ds.$$

From (2.2) and (2.3), then we can re-write it as the following

$$V' = \Delta_1(t) + \Delta_2(t) + \Delta_3(t) + \left\{\frac{1}{2}\beta\vartheta'(t) - \beta\varphi(t) + \psi(t)\eta(t)R'(x) + \mu_1 + \mu_4\,\tau\right\}y^2$$
$$+\{\beta\eta(t) - \vartheta(t)\eta(t) + \mu_2 + \mu_3\tau\}z^2 - \mu_1 y^2(t-\sigma) - \mu_2\, z^2(t-\sigma)$$
$$-\mu_3 \int_{t-\tau}^{t} z^2(s)ds - \mu_4 \int_{t-\tau}^{t} y^2(s)ds,$$

where

$$\Delta_1(t) = \beta\psi'(t)\int_0^x R(\xi)d\xi + \psi'(t)R(x)Y + \frac{1}{2}\left(\frac{\varphi(t)}{\eta(t)}\right)' Y^2,$$
$$\Delta_2(t) = \zeta\psi(t)\eta(t)R'(x)yy(t-\sigma) + \zeta\varphi(t)y(t-\sigma)Z + \big(\beta - \vartheta(t)\big)zZ,$$
$$\Delta_3(t) = (Z + \beta y)\left\{\varphi(t)\int_{t-\tau}^{t} z(u)du + \psi(t)\int_{t-\tau}^{t} R'\big(x(u)\big)y(u)du\right\}.$$

We have two cases:

Case I: If $\psi'(t) < 0$, then $\Delta_1(t)$ can be written as

$$\Delta_1(t) = \psi'(t)\left[\beta\int_0^x R(\xi)d\xi + R(x)Y\right] + \frac{1}{2}\left(\frac{\varphi(t)}{\eta(t)}\right)' Y^2$$
$$= \psi'(t)\left[\beta\int_0^x R(\xi)d\xi + \frac{1}{2}\frac{\varphi'(t)}{\eta(t)\cdot\psi'(t)}\left\{Y + \frac{\psi'(t)\eta(t)}{\varphi'(t)}R(x)\right\}^2 - \frac{\psi'(t)\eta(t)}{2\varphi'(t)}R^2(x)\right]$$
$$-\frac{\varphi(t)\eta'(t)}{2\eta^2(t)}Y^2.$$

From (H_1) and (H_3), we observe that

$$0 \le \frac{\psi(t)}{\varphi(t)} \le 1, \quad \text{and} \quad 0 \le \frac{\psi'(t)}{\varphi'(t)} \le 1,$$

thus from (H_5)

$$\Delta_1(t) \le \psi'(t)\left[\int_0^x (\beta - \delta_1\eta_2)R(\xi)d\xi\right] - \frac{\varphi(t)\eta'(t)}{2\eta^2(t)}Y^2$$
$$\le \frac{\varphi_1}{2\eta_1^2}|\eta'(t)|Y^2.$$

Case II: If $\psi'(t) = 0$, and from $(H_1) - (H_3)$, it follows that

$$\Delta_1(t) = \frac{1}{2}\left(\frac{\varphi(t)}{\eta(t)}\right)' Y^2 = \frac{1}{2}\frac{\varphi'(t)}{\eta(t)}Y^2 - \frac{1}{2}\frac{\varphi(t)\eta'(t)}{\eta^2(t)}Y^2 \le \frac{\varphi_1}{2\eta_1^2}|\eta'(t)|Y^2. \tag{2.9}$$

Hence, on combining (2.8) and (2.9), we have

$$\Delta_1(t) \le \frac{\varphi_1}{2\eta_1^2}|\eta'(t)|Y^2, \ \ for\ all\ t \ge t_1 \text{ and } Y.$$

From (2.3), we can re-write $\Delta_2(t)$ as

$$\Delta_2(t) = \zeta\varphi(t)y(t-\sigma)\{\eta'(t)\big(y(t) + \zeta y(t-\sigma)\big) + \eta(t)\big(z(t) + \zeta z(t-\sigma)\big)\}$$
$$+\big(\beta - \vartheta(t)\big)z\{\eta'(t)\big(y(t) + \zeta y(t-\sigma)\big) + \eta(t)\big(z(t) + \zeta z(t-\sigma)\big)\}$$
$$+\zeta\psi(t)\eta(t)R'(x)yy(t-\sigma).$$

From $(H_1) - (H_4)$ and applying the estimate $2ab \le a^2 + b^2$, we obtain

$$\Delta_2(t) \le \frac{1}{2}\,[\zeta\varphi(t)\eta_2\delta_1 + |\eta'(t)|\{(\vartheta_1-\beta)+\zeta\varphi_1\}]y^2$$
$$+\frac{1}{2}\,\{\zeta\varphi_1\eta_2 + L_2(\vartheta_1-\beta)(1+\zeta)+\zeta\eta(t)(\vartheta(t)-\beta)\}z^2$$
$$+\frac{1}{2}\{\zeta\varphi_1\delta_1\eta_2+(\vartheta_1-\beta)L_2\zeta+\varphi_1L_2\zeta(1+\zeta)+\zeta\varphi_1\eta_2+\zeta^2\varphi_1\eta_2\}y^2(t-\sigma)$$
$$+\frac{1}{2}\,\{\zeta\eta_2(\vartheta_1-\beta)+\zeta^2\varphi_1\eta_2\}z^2(t-\sigma).$$

Also, from the conditions $(H_1)-(H_4)$ and applying the inequality $2ab \le a^2+b^2$, we have

$$\Delta_3(t) = (Z+\beta y)\left\{\varphi(t)\int_{t-\tau}^{t} z(u)du + \psi(t)\int_{t-\tau}^{t} R'\big(x(u)\big)y(u)du\right\}$$
$$\le \frac{1}{2}\{\varphi_1(L_2+\beta)(1+\delta_1)\tau\}y^2 + \frac{1}{2}\{\varphi_1\eta_2(1+\delta_1)\tau\}z^2$$
$$+\frac{1}{2}\{\varphi_1\zeta L_2(1+\delta_1)\tau\}y^2(t-\sigma)+\frac{1}{2}\{\varphi_1\zeta\eta_2(1+\delta_1)\tau\}z^2(t-\sigma)$$
$$+\frac{1}{2}\,\varphi_1 M\int_{t-\tau}^{t} z^2(u)du + \frac{1}{2}\varphi_1\delta_1 M\int_{t-\tau}^{t} y^2(u)du,$$

where

$$M = \beta + (L_2+\eta_2)(1+\zeta).$$

Using $(H_1)-(H_4)$, we observe that V' can be replaced by

$$V' \le \left\{\frac{1}{2}\beta\vartheta'(t)+\varphi(t)(\delta_1\eta_2-\beta)+\frac{1}{2}\zeta\varphi(t)\delta_1\eta_2+\frac{1}{2}\varphi_1(L_2+\beta)(1+\delta_1)\tau+\mu_1+\mu_4\tau\right\}y^2$$
$$+\left\{\eta_1(\beta-\vartheta_0)\left(1-\frac{\zeta}{2}\right)+\frac{1}{2}L_2(\vartheta_1-\beta)(1+\zeta)+\frac{1}{2}\zeta\varphi_1\eta_2+\frac{1}{2}\varphi_1\eta_2(1+\delta_1)\tau\right.$$
$$\left.+\mu_2+\mu_3\tau\right\}z^2$$
$$+\frac{1}{2}\{\zeta\varphi_1\delta_1\eta_2+(\vartheta_1-\beta)L_2\zeta+\varphi_1L_2\zeta(1+\zeta)+\zeta\varphi_1\eta_2+\zeta^2\varphi_1\eta_2-2\mu_1$$
$$+\varphi_1\zeta L_2(1+\delta_1)\tau\}y^2(t-\sigma)$$
$$+\frac{1}{2}\{\zeta\eta_2(\vartheta_1-\beta)+\zeta^2\varphi_1\eta_2-2\mu_2+\varphi_1\zeta\eta_2(1+\delta_1)\tau\}z^2(t-\sigma)+\Delta_4(t)$$
$$+\frac{1}{2}(\varphi_1M-2\mu_3)\int_{t-\tau}^{t} z^2(u)du+\frac{1}{2}(\varphi_1\delta_1M-2\mu_4)\int_{t-\tau}^{t} y^2(u)du,$$

where

$$\Delta_4(t) = \frac{1}{2}|\eta'(t)|\left|\frac{\varphi_1}{2\eta_1^2}Y^2+\{(\vartheta_1-\beta)+\zeta\varphi_1\}y^2\right| \le K_3|\eta'(t)|(y^2+Y^2),$$

such that

$$K_3 = \max\left\{\frac{\varphi_1}{2\eta_1^2}\,,\ \ (\vartheta_1-\beta)+\zeta\varphi_1\right\}.$$

Let

$$\mu_1 = \frac{1}{2}\zeta\{\varphi_1\eta_2(1+\zeta+\delta_1) + (\vartheta_1-\beta)L_2 + \varphi_1 L_2(1+\zeta) + \varphi_1 L_2(1+\delta_1)\tau\},$$
$$\mu_2 = \frac{1}{2}\zeta\eta_2\{\vartheta_1-\beta+\varphi_1\zeta+\varphi_1(1+\delta_1)\tau\},$$
$$\mu_3 = \frac{1}{2}\varphi_1 M,$$
$$\mu_4 = \frac{1}{2}\varphi_1\delta_1 M.$$

Now, in view of estimates of A and B in Theorem 2.1, the inequality V' becomes

$$V' \le \frac{1}{2}[-A + \{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1\delta_1 M + \varphi_1(1+\delta_1)(L_2+\beta)\}\tau]y^2$$
$$+\frac{1}{2}[-B + \{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1 M + \varphi_1\eta_2\zeta(1+\delta_1)\}\tau]z^2 + K_3|\eta'(t)|(y^2+Y^2).$$

Then, from (2.6), we obtain

$$V' \le \frac{1}{2}[-A + \{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1\delta_1 M + \varphi_1(1+\delta_1)(L_2+\beta)\}\tau]y^2$$
$$+\frac{1}{2}[-B + \{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1 M + \varphi_1\eta_2\zeta(1+\delta_1)\}\tau]z^2 + \frac{K_3}{K_1}|\eta'(t)|V.$$

We take $p = \frac{K_3}{K_1}$ and from (2.4), we get

$$\Gamma'(\cdot) \le \frac{1}{2}\exp\left(-\frac{K_3}{K_1}\int_0^t |\eta'(s)|ds\right)[(-A$$
$$+\{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1(1+\delta_1)(L_2+\beta) + \varphi_1\delta_1 M\}\tau)y^2$$
$$+(-B + \{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1 M + \varphi_1\eta_2\zeta(1+\delta_1)\}\tau)z^2].$$

If

$$\tau < \min\left\{\frac{A}{2\{\varphi_1 L_2(1+\delta_1)\zeta + \varphi_1 M\delta_1 + \varphi_1(1+\delta_1)(L_2+\beta)\}}, \frac{B}{2\{\varphi_1 L_2\zeta(1+\delta_1) + \varphi_1 M + \eta_2\zeta\varphi_1(1+\delta_1)\}}\right\},$$

Then

$$\Gamma'(\cdot) \le -K_4(y^2+z^2) \quad \text{for some } K_4 > 0. \tag{2.10}$$

Therefore, all the conditions of Theorem 2.1, are satisfied.

Thus, the zero solution of system (2.1) is asymptotically stable.

Boundedness

To prove the boundedness of solutions of (1.2), we need to write (2.1) as the following form

$$x' = y,$$
$$y' = z,$$
$$Z' = -\vartheta(t)z - \varphi(t)y - \psi(t)R(x) + \varphi(t)\int_{t-\tau}^{t} z(u)du$$
$$+\psi(t)\int_{t-\tau}^{t} R'(x(u))y(u)du + Q(t, x(t), x(t-\tau), y(t), y(t-\tau), z(t)). \tag{3.1}$$

For the next theorem, we impose the following conditions:

$$|Q(t,x(t),x(t-\tau),y(t),y(t-\tau),z(t))| \le \Omega(t), \qquad (3.2)$$

and

$$\int_{t_1}^{t} |\Omega(s)|ds \le N. \qquad (3.3)$$

For the case $Q(t,x(t),x(t-\tau),y(t),y(t-\tau),z(t)) = Q(\cdot)$, our second main result of our paper is the following theorem.

Theorem 3.1. If the conditions of Theorem (2.1) and (3.2) – (3.3) hold, then there exists a positive constant D such that any solution of (3.1) satisfies

$$|x(t)| \le D\,,\ \ |y(t)| \le D\,, |Y(t)| \le D\,, \quad |Z(t)| \le D,\ \ for\ all\ \ t \ge t_1 \ge 0. \qquad (3.4)$$

Proof. Along any solution $(x(t),y(t),Z(t))$ of (3.1), we have

$$\Gamma'_{(3.1)}(\cdot) = \Gamma'_{(2.1)}(\cdot) + \exp\left(-\frac{1}{p}\int_0^t |\eta'(s)|ds\right)(Z+\beta y)Q(\cdot).$$

From (2.10), we obtain

$$\Gamma'_{(3.1)}(\cdot) \le K_5|\Omega(t)|(|y|+|Z|),$$

where

$$K_5 = \exp\left(\frac{2K_3}{K_1}\right)\max\{1,\beta\}.$$

From the relations $|y| \le 1+y^2$ and $|Z| \le 1+Z^2$, we get

$$\Gamma'_{(3.1)}(\cdot) \le K_5|\Omega(t)|(y^2+Z^2+2).$$

From the inequality (2.7), it follows that

$$\Gamma'_{(3.1)}(\cdot) \le K_6|\Omega(t)|\,\Gamma(\cdot) + K_6|\Omega(t)|, \qquad (3.5)$$

where

$$K_6 = \max\left\{2K_5\,,\ \frac{K_5}{K_2}\right\}.$$

Integrating the above estimate from t_1 to t, $t \ge t_1 = t_0 + \rho$, we can obtain

$$\Gamma(t) - \Gamma(t_1) \le K_6\int_{t_1}^{t} |\Omega(s)|\Gamma(s)ds + K_6\int_{t_1}^{t}|\Omega(s)|ds.$$

Thus, from (3.3)

$$\Gamma(t) \le \Gamma(t_1) + K_6\int_{t_1}^{t}|\Omega(s)|\Gamma(s)ds + K_6N.$$

Using Gronwell-inequality, it follows that

$$\Gamma(t) \le (\Gamma(t_1) + K_6N)\exp\left(K_6\int_{t_1}^{t}|\Omega(s)|ds\right) \le N_1, \qquad (3.6)$$

where

$$N_1 = (\Gamma(t_1) + K_6N)\exp(K_6N).$$

This result implies that there exists a constant D such that

$$|x(t)| \le D\,, |y(t)| \le D\,, |Y(t)| \le D\,, |Z(t)| \le D\,, for\ all\ \ t \ge t_1.$$

This completes the proof of Theorem 3.1.

Square integrability

Our next result concerns the square integrability of solutions for equation (1.2).

Theorem 4.1. In addition to the assumptions of Theorem 3.1, if we assume that

(H_6) $\psi_0\delta_2 - \frac{1}{2}\varphi_1 > 0,$

(H_7) $\int_{t_1}^{\infty} |\vartheta'(s)|ds < \alpha.$

Then all solution of (3.1) are elements in $L^2[t_1, +\infty)$.

Proof. Define $W(t)$ as

$$W(t) = \Gamma(t) + \gamma \int_{t_1}^{t} \left(y^2(s) + z^2(s)\right)ds\,, \qquad for\ all\ t \geq t_1, \tag{4.1}$$

where γ is a positive constant to be determined later.

By differentiating $W(t)$ along the solution of system (3.1) and using the inequalities (2.10) and (3.5), we obtain

$$\begin{aligned} W'_{(3.1)}(t) &= \Gamma'(t) + \gamma\left(y^2(t) + z^2(t)\right) \\ &\leq -K_4\left(y^2(t) + z^2(t)\right) + \gamma\left(y^2(t) + z^2(t)\right) + K_6(\Gamma(t) + 1)|\Omega(t)| \\ &= (\gamma - K_4)\left(y^2(t) + z^2(t)\right) + K_6(\Gamma(t) + 1)|\Omega(t)|. \end{aligned}$$

If we choose $\gamma - K_4 < 0$, then

$$W'_{(3.1)}(t) \leq K_6(\Gamma(t) + 1)|\Omega(t)|.$$

From (3.6), it follows that

$$W'_{(3.1)}(t) \leq K_6(N_1 + 1)|\Omega(t)| = K_7|\Omega(t)|. \tag{4.2}$$

where $K_7 = K_6(N_1 + 1)$.

Integrating (4.2) from t_1 to t and using the condition (3.3), we obtain

$$W(t) - W(t_1) \leq K_7 N.$$

Using equality $W(t_1) = \Gamma(t_1)$, we get

$$W(t) \leq K_7 N + \Gamma(t_1).$$

We can conclude by (4.1) that

$$\int_{t_1}^{t} \left(y^2(s) + z^2(s)\right)ds < \frac{K_7 N + \Gamma(t_1)}{\gamma}\,,$$

which imply the existence of positive constants κ_1 and κ_2 such that

$$\int_{t_1}^{t} x'^2(s)ds = \int_{t_1}^{t} y^2(s)ds \leq \kappa_1\,,$$

$$\int_{t_1}^{t} x''^2(s)ds = \int_{t_1}^{t} z^2(s)ds \leq \kappa_2.$$

Given the data above, we obtain

$$\int_{t_1}^{t} x'^2(s-\sigma)ds = \int_{t_0+\rho-\sigma}^{t-\sigma} x'^2(v)dv \le \int_{t_0+\rho-\sigma}^{t_1} x'^2(v)dv + \kappa_1 \le \lambda_1 + \kappa_1,$$

and

$$\int_{t_1}^{t} x''^2(s-\sigma)ds = \int_{t_0+\rho-\sigma}^{t-\sigma} x''^2(v)dv \le \int_{t_0+\rho-\sigma}^{t_1} x''^2(v)dv + \kappa_2 \le \lambda_2 + \kappa_2.$$

To prove that $\int_{t_1}^{t} x^2(s)ds < \infty$. We multiply (1.2) by $x(t-\tau)$, we obtain

$$\begin{aligned} &x(t-\tau)\left[\eta(t)\big(x(t)+\zeta x(t-\sigma)\big)'\right]'' + \vartheta(t)x(t-\tau)x''(t) \\ &+\varphi(t)x(t-\tau)x'(t-\tau) + \psi(t)x(t-\tau)R\big(x(t-\tau)\big) \\ &\quad = x(t-\tau)Q\big(t, x(t), x(t-\tau), x'(t), x'(t-\tau), x''(t)\big). \end{aligned} \tag{4.3}$$

Integrating (4.3) from t_1 to t, we have

$$\int_{t_1}^{t} \psi(v)x(v-\tau)R\big(x(v-\tau)\big)dv = \Omega_1(t) + \Omega_2(t) + \Omega_3(t), \tag{4.4}$$

where

$$\Omega_1(t) = -\int_{t_1}^{t} x(v-\tau)\left[\eta(v)\big(x(v)+\zeta x(v-\sigma)\big)'\right]'' dv,$$

$$\Omega_2(t) = -\int_{t_1}^{t} \vartheta(v)x(v-\tau)x''(v)dv - \int_{t_1}^{t} \varphi(v)x(v-\tau)x'(v-\tau)dv,$$

$$\Omega_3(t) = \int_{t_1}^{t} x(v-\tau)Q\big(v, x(v), x(v-\tau), x'(v), x'(v-\tau), x''(v)\big)dv.$$

Then, we have from **(3.3)** and **(3.4)**

$$\begin{aligned} \Omega_3(t) &\le \int_{t_1}^{t} |x(v-\tau)|\left|Q\big(v, x(v), x(v-\tau), x'(v), x'(v-\tau), x''(v)\big)\right|dv \\ &\le D\int_{t_1}^{t} |\Omega(v)|dv \\ &\le ND. \end{aligned} \tag{4.5}$$

From (2.3), it follows that

$$\begin{aligned} \Omega_1(t) &= -\int_{t_1}^{t} x(v-\tau)Z'(v)dv \\ &= -\int_{t_1}^{t} x(v-\tau)[\eta'(v)\{x'(v)+\zeta x'(v-\sigma)\} + \eta(v)\{x''(v)+\zeta x''(v-\sigma)\}]'dv \end{aligned}$$

By integrating the above equality by parts to get

$$\Omega_1(t) = \int_{t_1}^{t} x'(v-\tau)[\eta'(v)\{x'(v)+\zeta x'(v-\sigma)\} + \eta(v)\{x''(v)+\zeta x''(v-\sigma)\}]dv$$
$$+M_1(t) - M_1(t_1)$$
$$\leq |M_1(t) - M_1(t_1)| + \int_{t_1}^{t} |\eta'(v)x'(v-\tau)\{x'(v)+\zeta x'(v-\sigma)\}|dv$$
$$+ \int_{t_1}^{t} |\eta(v)x'(v-\sigma)\{x''(v)+\zeta x''(v-\sigma)\}|dv.$$

Next from the condition (H_2) and by use of inequality $2ab \leq a^2 + b^2$, we obtain

$$\Omega_1(t) \leq |M_1(t) - M_1(t_1)| + \frac{L_2}{2}\int_{t_1}^{t} [(1+\zeta)x'^2(v-\tau) + x'^2(v) + \zeta x'^2(v-\sigma)]dv$$
$$+\frac{\eta_2}{2}\int_{t_1}^{t} [(1+\zeta)x'^2(v-\tau) + x''^2(v) + \zeta x''^2(v-\sigma)]dv,$$

where

$$M_1(t) = -x(t-\tau)Z(t).$$

We remark by the inequalities (3.4) that

$$|M_1(t) - M_1(t_1)| \leq D^2 + |M_1(t_1)|\,, for\ all\ t \geq t_1,$$

and

$$\int_{t_1}^{t} x'^2(s-\tau)ds = \int_{t_0+\rho-\tau}^{t-\tau} x'^2(v)dv \leq \int_{t_0+\rho-\tau}^{t_1} x'^2(v)dv + \kappa_1$$
$$\leq \lambda_3 + \kappa_1,$$

thus

$$\Omega_1(t) \leq \frac{1}{2}[(1+\zeta)\{(L_2+\eta_2)(\lambda_3+\kappa_1) + L_2\kappa_1 + \eta_2\kappa_2\} + \zeta(L_2\lambda_1 + \eta_2\lambda_2)]$$
$$+D^2 + |M_1(t_1)| = \hbar_1. \qquad \textbf{(4.6)}$$

Similarly, we have

$$\Omega_2(t) = -\int_{t_1}^{t} \{\vartheta(v)x(v-\tau)x''(v) + \varphi(v)x(v-\tau)x'(v-\tau)\}dv$$
$$\leq \frac{\vartheta_1}{2}\int_{t_1}^{t} \{x'^2(v-\tau) + x'^2(v)\}dv + \frac{\varphi_1}{2}\int_{t_1}^{t} \{x^2(v-\tau) + x'^2(v-\tau)\}dv$$
$$+\vartheta_1 D^2 + |M_2(t_1)| + \int_{t_1}^{t} |\vartheta'(v)||x(v-\tau)||x'(v)|dv$$
$$+ \int_{t_1}^{t} |\vartheta'(v)| \left\{ \int_{t_1}^{v} |x'(u-\tau)||x'(u)|du \right\} dv,$$

where

$$M_2(t_1) = \vartheta(t_1)x(t_1-\tau)x'(t_1).$$

Then, we obtain by using the inequality $2ab \leq a^2 + b^2$

$$\Omega_2(t) \le \frac{\vartheta_1}{2}(\lambda_3 + 2\kappa_1) + \frac{\varphi_1}{2}(\lambda_3 + \kappa_1) + \frac{\varphi_1}{2}\int_{t_1}^{t} x^2(v-\tau)dv$$

$$+\vartheta_1 D^2 + |M_2(t_1)| + \left(D^2 + \frac{1}{2}\lambda_3 + \kappa_1\right)\int_{t_1}^{t} |\vartheta'(v)|dv.$$

It follows that

$$\Omega_2(t) \le \left(\vartheta_1 + \int_{t_1}^{t} |\vartheta'(v)|dv\right)\left(D^2 + \frac{1}{2}\lambda_3 + \kappa_1\right) + |M_2(t_1)|$$
$$+\frac{\varphi_1}{2}(\lambda_3 + \kappa_1) + \frac{\varphi_1}{2}\int_{t_1}^{t} x^2(v-\tau)dv. \tag{4.7}$$

On applying conditions (H_1) and (H_4), we conclude that

$$\int_{t_1}^{t} \psi(v)x(v-\tau)R\big(x(v-\tau)\big)dv \ge \psi_0\delta_2 \int_{t_1}^{t} x^2(v-\tau)dv. \tag{4.8}$$

Substituting estimates (4.5) – (4.8) and condition (H_7) in equation (4.4), we obtain

$$\psi_0\delta_2 \int_{t_1}^{t} x^2(v-\tau)dv \le \hbar_2 + \frac{\varphi_1}{2}\int_{t_1}^{t} x^2(v-\tau)dv,$$

where

$$\hbar_2 = ND + \hbar_1 + (\vartheta_1 + \alpha)\left(D^2 + \frac{1}{2}\lambda_3 + \kappa_1\right) + |M_2(t_1)| + \frac{\varphi_1}{2}(\lambda_3 + \kappa_1).$$

Therefore

$$\left(\psi_0\delta_2 - \frac{1}{2}\varphi_1\right)\int_{t_1}^{t} x^2(v-\tau)dv \le \hbar_2.$$

Then from condition (H_6), we observe that

$$\int_{t_1}^{t} x^2(v-\tau)dv \le \infty,$$

hence

$$\int_{t_1}^{\infty} x^2(v-\tau)dv \le \infty.$$

This fact completes the proof of Theorem 4.1.

Example

We consider the following non-autonomous third-order NDDE

$$\left[\left(\frac{1}{2} + \frac{1}{10+t^2}\right)\left(x(t) + \frac{1}{10}x(t-\sigma)\right)'\right]'' + \left(\frac{1}{\pi}\tan^{-1}t + \frac{13}{2}\right)x''(t)$$
$$+\left(3 + \frac{1}{2+4t^2}\right)x'(t-\sigma) + \left(3 + \frac{1}{4+4t^2}\right)\left[\frac{7}{10}\left(x(t-\tau) + \frac{x(t-\tau)}{1+x^2(t-\tau)}\right)\right] \tag{5.1}$$
$$= \frac{1}{1+t^2+x^2(t)+x^2(t-\tau)+x'^2(t)+x'^2(t-\tau)+x''^2(t)}.$$

The above equation is equivalent to a system of the first-order differential equations as the following

$$x' = y,\ y' = z,$$

$$z' = -\left(\frac{1}{\pi}\tan^{-1}t + \frac{13}{2}\right)z - \left(3 + \frac{1}{2+4t^2}\right)y - \left(3 + \frac{1}{4+4t^2}\right)\left[\frac{7}{10}\left(x + \frac{x}{1+x^2}\right)\right]$$
$$+\left(3 + \frac{1}{2+4t^2}\right)\int_{t-\tau}^{t} z(u)du + \frac{7}{10}\left(3 + \frac{1}{4+4t^2}\right)\int_{t-\tau}^{t}\left(1 + \frac{1-x^2}{(1+x^2)^2}\right)y(u)du$$
$$+\frac{1}{1+t^2+x^2(t)+x^2(t-\tau)+y^2(t)+y^2(t-\tau)+z^2(t)}. \tag{5.2}$$

Comparing the two systems (3.1) and (5.2), we find the following functions

- The function

$$\eta_1 = \frac{1}{2} \le \eta(t) = \frac{1}{2} + \frac{1}{10+t^2} \le \frac{3}{5} = \eta_2,$$

it follows that

$$-L_2 = -0.02 \le \eta'(t) = \frac{-2t}{(10+t^2)^2}.$$

Figure (1) shows the path of $\eta(t)$ on the interval $[0,10]$.

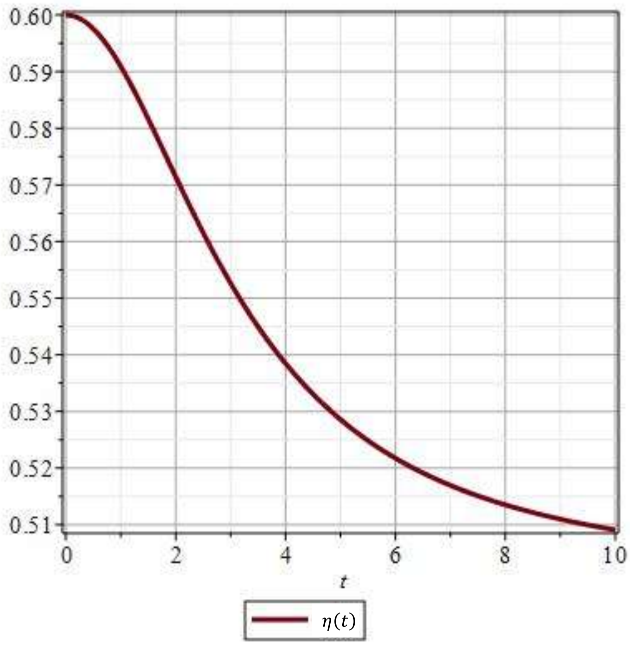

Fig. (2): The path of the function $\eta(t),\ t \in [0,10]$.

- The function

$$R(x) = \frac{7}{10}\left(x + \frac{x}{1+x^2}\right),$$

fulfills $R(0) = 0$ and

$$\delta_2 = \frac{7}{10} \le \frac{R(x)}{x} = \frac{7}{10}\left(1 + \frac{1}{1+x^2}\right),\ \text{with } x \ne 0.$$

The derivative of $R(x)$ is defined as

$$R'(x) = \frac{7}{10}\left(1 + \frac{1-x^2}{(1+x^2)^2}\right), |R'(x)| \le \frac{7}{5} = \delta_1,\ for\ all\ x.$$

The path of the functions $\frac{R(x)}{x}$ and $R'(x)$ appear in Fig. (3) through the interval $x \in [0,10]$.

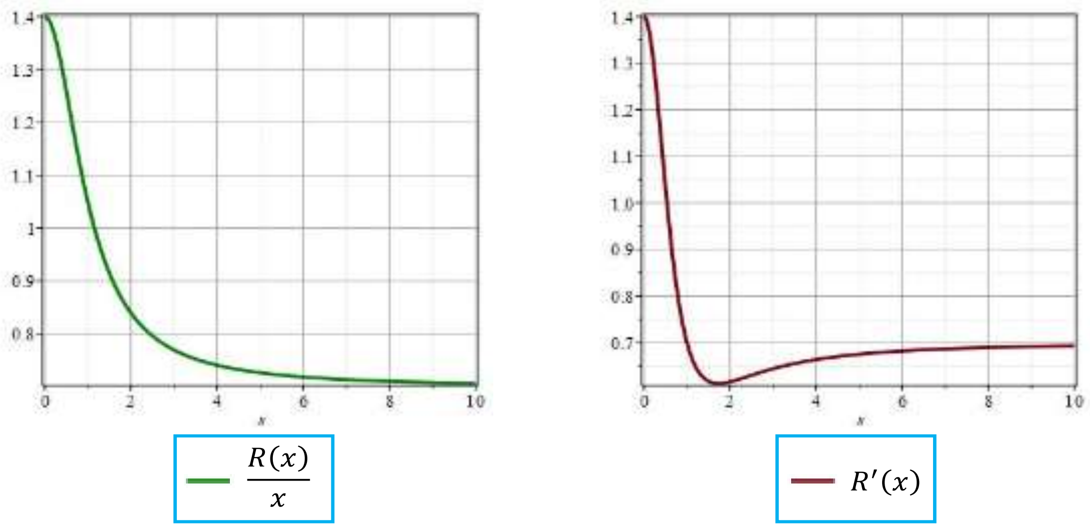

Fig. (4): The behavior of the functions $\frac{R(x)}{x}$ and $R'(x)$ for $x \in [0,10]$.

- The function

$$\vartheta(t) = \frac{1}{\pi}\tan^{-1} t + \frac{13}{2},$$

we note that

$$6 = \vartheta_0 \leq \vartheta(t) \leq \vartheta_1 = 7\,, \; for\ all\ t \in \mathbb{R}\,, \; \text{and}$$

$$\int_{t_1}^{t} |\vartheta'(s)| ds = \frac{1}{\pi}\int_{t_1}^{t} \left|\frac{1}{1+s^2}\right| ds < \frac{1}{2} = \alpha.$$

The shape and path of $\vartheta(t)$ is shown in Fig. (5).

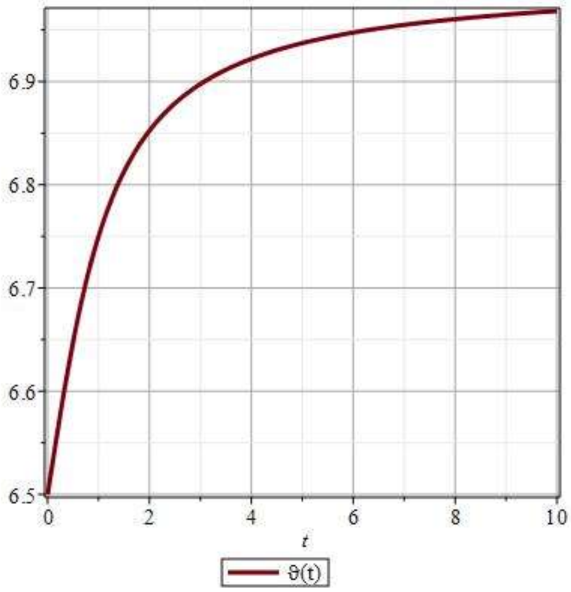

Fig. (6): The path of the function $\vartheta(t)$ for $t \in [0,10]$.

- The functions

$$\varphi(t) = 3 + \frac{1}{2+4t^2} \quad \text{and} \quad \psi(t) = 3 + \frac{1}{4+4t^2},$$

since $4 + 4t^2 \geq 2 + 4t^2$, for all $t \in \mathbb{R}$, then it follows that

$$3 = \psi_0 \le \psi(t) \le \varphi\,(t) \le \varphi_1 = 3.5,\ \ for\ all\ \ t \in \mathbb{R}.$$

The coinciding paths of $\varphi(t)$ and $\psi(t)$ on the interval $t \in [0,10]$ are presented in Fig. (4).

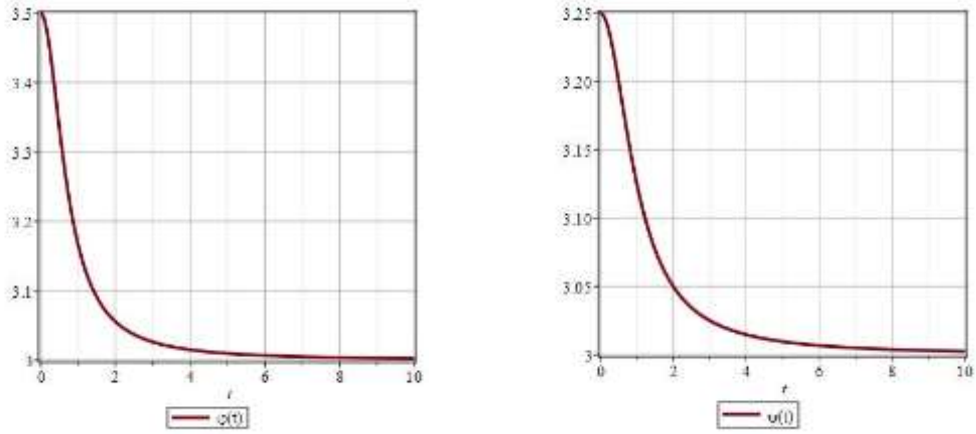

Fig. (7): The behaviors of the functions $\varphi(t)$ and $\psi(t)$ for $t \in [0,10]$.

Moreover, the derivatives of the functions $\psi(t)$ and $\varphi(t)$ with respect to the independent variable t

$$\varphi'(t) = \frac{-8t}{(2+4t^2)^2} \quad \text{and} \quad \psi'(t) = \frac{-8t}{(4+4t^2)^2}.$$

Noting that

$$\varphi'(t) < \psi'(t),\ \ for\ all\ t \in \mathbb{R},\quad \text{also}$$

$$\lim_{t\to\infty} \varphi'(t) = 0 = \lim_{t\to\infty} \psi'(t).$$

Thus, the inequality

$$\varphi'(t) \le \psi'(t) \le 0,\ \ for\ all\ t \in \mathbb{R}^+.$$

We can see that Fig. (8), illustrate the behaviour of $\varphi'(t)$ and $\psi'(t)$, through the interval $t \in [0,10]$.

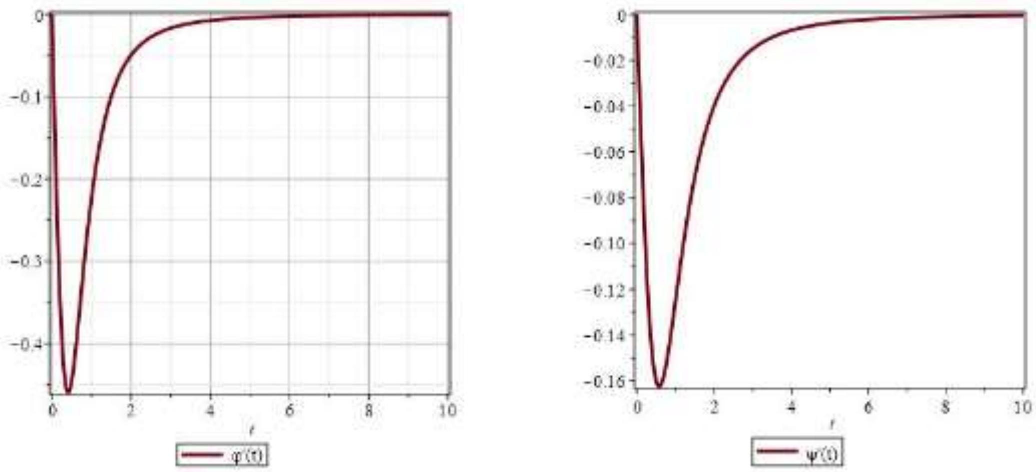

Fig. (9): Paths of $\varphi'(t)$ and $\psi'(t)$ for $t \in [0,10]$.

- The function

$$Q(\cdot) = \frac{1}{1+t^2+x^2(t)+x^2(t-\tau)+y^2(t)+y^2(t-\tau)+z^2(t)} \le \frac{1}{1+t^2} = \Omega(t),$$

Therefore, we conclude

$$\int_{t_1}^{t} |\Omega(s)| ds = \int_{t_1}^{t} \left|\frac{1}{1+s^2}\right| ds < \frac{\pi}{2} = N.$$

It is straightforward to verify that for all $t \geq t_1$, that

$$2\delta_1\eta_2 = 1.68 < \beta < 6 = \vartheta_0.$$

If we let $\beta = 2.6$, then we have the following estimates

$$M = \beta + (L_2 + \eta_2)(1+\zeta) = 3.282\,,$$

$$\beta\vartheta'(t) + 2\psi_0\{\delta_1\eta_2(2+\zeta) - \beta\} + \zeta\{\varphi_1\eta_2(1+\zeta+\delta_1) + L_2(\vartheta_1 - \beta) + \varphi_1 L_2(1+\zeta)\}$$
$$= -3.64828 \leq -A < 0,$$

$$\eta_1(\beta - \vartheta_0)(2-\zeta) + L_2(\vartheta_1 - \beta)(1+\zeta) + \zeta\varphi_1\eta_2 + \zeta\eta_2(\theta_1 - \beta + \varphi_1\zeta)$$
$$= -2.4282 \leq -B < 0\,,$$

$$\tau < \min\{0.046869\,, 0.101109\} \cong 0.046869\,,$$

$$\psi_0\delta_2 - \frac{1}{2}\varphi_1 = \frac{7}{20} > 0.$$

The behaviour of the solutions of (5.1) is depicted in Fig. (10).

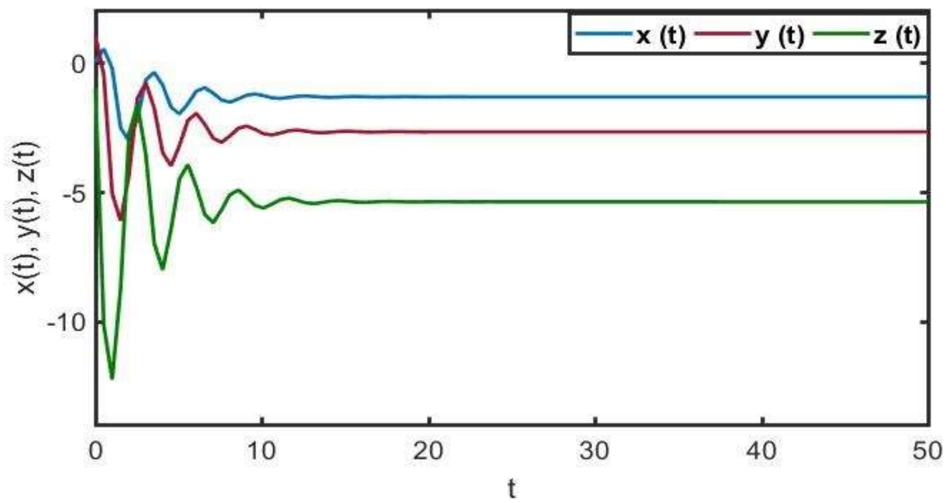

Fig. (11): The solution's behaviour for (5.1).

Then all the assumptions of Theorem 4.1 are satisfied, we can conclude that every solution of (5.1) are bounded and elements in $L^2[t_1, +\infty)$.

Conclusion

In this paper, the Lyapunov functional approach is applied to investigate a third-order non-autonomous neutral delay differential equation. To ensure the stability, bounderness, and square integrability of solutions, the standard Lyapunov functional is designed to fulfill the necessary requirements. Our findings add to the body of excellent existing research in the literature and make several new discoveries. To exemplify the efficacy of the achieved results, an example was suggested that performed the supplied results in all functions.

References

Ademola A. T., Mahmoud A. M., Arawomo P. O. (2019). On the behaviour of solutions for a class of third-order neutral delay differential equations, *Analele Universitatii Oradea*

Fasc. Matematica, Tom XXVI(2): 85–103.

Fellous A., Khatir A. M. and Remili M. (2022). On stability and square integrability of solutions to some third-order neutral differential equations, *Afr. Mat.*, 33(31): 1-12.

Graef J. R., Beldjerd D. and Remili M. (2019). Some new stability, boundedness, and square integrability conditions for third-order neutral delay differential equations, *Commun. Math. Anal.*, 22(1): 76-89.

Graef J. R., Beldjerd D. and Remili M. (2021). On Stability, boundedness, and square integrability of solutions of certain third-order neutral differential equations, *Math. Bohem.*, 147(3): 1-15.

Graef J. R., Oudjedi L. D. and Remili M. (2015). Stability and square integrability of solutions of nonlinear third-order differential equations, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, 22: 313-324.

Graef J. R., L. Oudjedi D. and Remili M. (2018). Stability and square integrability of solutions to third-order neutral delay differential equations, *Tatra Mt. Math. Publ.*, 71: 81-97.

Hale J. K. and Verduyn Lunel S. M. (1993). *Introduction to Functional Differential Equations*, Springer, Berlin, Heidelberg, New York.

Khatir A. M. (2022). On asymptotic properties of certain neutral differential equations, *Filomat*, 36(6): 2001-2009.

Khatir A. M., Graef J. R. and Remili M. (2020). Stability, boundedness, and square integrability of solutions to third-order neutral differential equations with delay, *Rend. Circ. Mat. Palermo II Ser.*, 69: 823-836.

Khatir A. M., Remili M. and Beldjerd D. (2020). Stability, boundedness and square integrability of solutions to certain third-order neutral delay differential equations, *Palestine J. Math.*, 9(2): 880-890.

Kolmanovski V. and Myshkis A. (1992). *Applied Theory of Functional Differential Equations*, Kluver Academic, Dordrecht.

Kolmanovskii V. B. and Myshkis A. (1999). *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer, Dordrecht.

Kuang K. (1993). *Delay Differential Equations: with Applications in Population Dynamics*, Academic Press.

Lakshmikantham V., Wen L. and Zhang B. (1994). *Theory of Differential Equations with Unbounded Delay*, Kluwer Academic Publishers, London.

Mahmoud A. M. (2016). On the asymptotic stability of solutions for a certain non-autonomous third-order delay differential equation, *British Journal of Mathematics and Computer Science*, 16(3): 1-12.

Oudjedi L. D., Lekhmissi B. and Remili M. (2019). Asymptotic properties of solutions to third-order neutral differential equations with delay, *Proyecciones Journal of Mathematics*, 38(1): 111-127.

Philos Ch. G. and Purnaras I. K. (2010). An asymptotic result for second-order linear nonautonomous neutral delay differential equations, *Hiroshima Math. J.*, 40: 47-63.

Rahmane M. and Abdellaoui F. (2024). Uniform stability, boundedness and square integrability for non-autonomous third-order neutral differential equations with delay, *Proyecciones Journal of Mathematics*, 43(4): 827-847.

Remili M. and Beldjerd D. (2015). A boundedness and stability results for a kind of third-order delay differential equations, *Appl. Appl. Math.*, 10(2): 772-782.

Remili M. and Oudjedi L. D. (2014). Uniform stability and boundedness of a kind of third-order delay differential equations, *Bull. Comput. Appl. Math.*, 2(1): 25-35.

Remili M. and Oudjedi L. D. (2016a). Boundedness and stability in third-order nonlinear differential equations with bounded delay, *An. Univ. Oradea, Fasc. Mat.*, XXIII (1): 135-143.

Remili M. and Oudjedi L. D. (2016b). On asymptotic stability of solutions to third-order nonlinear delay differential equation, *Filomat*, 30(12): 3217-3226.

Remili M. and Oudjedi L. D. (2016c). Uniform ultimate boundedness and asymptotic behaviour of third-order nonlinear delay differential equation, *Afr. Mat.*, 27(7): 1227-1237.

Remili M. and Oudjedi L. D. (2020). Stability and boundedness of non-autonomous neutral differential equation with delay, *Math. Morav.* , 24(1): 1-16.

Remili M., Oudjedi L. D. and Beldjerd D. (2016). On the qualitative behaviors of solutions to a kind of nonlinear third-order differential equation with delay, *Commun. Appl. Anal.*, 20: 53-64.

Santra S. S., Alotaibi H. and Bazighifan O. (2020). On the qualitative behavior of the solutions to second-order neutral delay differential equations, *J. Inequal. Appl.*, 256 (2020): 1-12.

Strogatz S. H. (2018). *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering (2nd ed.)*, Westview Press.

نتائج كيفية جديدة لنوع من المعادلات التفاضلية التأخيرية المحايدة غير الذاتية من الرتبة الثالثة

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يناقش البحث الحالي الخواص الكيفية لحلول نوع من المعادلات التفاضلية التأخيرية المحايدة غير الذاتية من الرتبة الثالثة. وقد تم التوصل إلي نتائج جديدة عن الإستقرار والمحدودية وقابلية التكامل التربيعية للحلول ومشتقاتها باستخدام طريقة دالية ليابونوف. نتائجنا تعمم وتوسع نطاق العديد من النتائج الموجودة في المقالات العلمية ذات الصلة بالمعادلات التفاضلية المحايدة من الرتبة الثالثة ذات التأخير أو بدونه. أكثر من ذلك، نقدم مثال لإثبات صحة النتائج الرئيسية وإمكانية تطبيقيها.