DJS Vol. 50(1)(2025) –pp. 45-53 ISSN: 1012-5965

1969 ACCREDITED 2016

Delta Journal of Science

Available online at

https://djs.journals.ekb.eg/

Research Article **Mathematics**

A finite difference scheme for the two-dimensional sine-Gordon equation

A. A. Soliman[1], Manar M. Dahshan[*2] and Ahmed S. Elgazzar[3]

Mathematics Department, Faculty of Science, Arish University, 45516 Arish, Egypt

[1]asoliman 99@yahoo.com, [2] manar.mohamed@sci.aru.edu.eg, [3]aselgazzar@aru.edu.eg

[*]**Corresponding author:** Manar M. Dahshan **E-mail:** manar.mohamed@sci.aru.edu.eg

Received: 4/2/2025 **Accepted:** 27/4/2025

KEY WORDS

Sine-Gordon equation; Finite difference; Von Neumann stability; Local truncation error; Error analysis.

ABSTRACT

The sine-Gordon (SG) equation is a fundamental aspect of nonlinear physics. It models a wide range of phenomena in many scientific fields. While its mathematical structure allows analytical solutions under certain conditions, the complexity of real-world applications often requires numerical methods. Accurate and efficient numerical solutions enable a deeper understanding and advances applications in many fields. This study presents a finite difference scheme for the SG equation in two dimensions. Both the local truncation error and stability of the scheme are studied. We also present numerical simulations and error analysis to ensure the accuracy of the scheme.

Introduction

Two-dimensional sine-Gordon (2D SG) equation **(Ablowitz and Clarkson, 1991)** is a fundamental, nonlinear partial differential equation (NPDE). It describes the dynamics of a scalar field over time and two spatial dimensions. The 2D SG equation models a variety of physical phenomena involving wave propagation and nonlinear interactions. In superconductivity, it describes the dynamics of the phase difference across Josephson junction (**Perring and Skyrme, 1962; Barone and Paterno, 1982)**. The 2D SG equation also models the propagation of light pulses in nonlinear optical media (**Agrawal, 2013**). The 2D SG equation can model the behavior of various phenomena in many fields (**Cuevas-Maraver et al., 2014**). NPDEs arise in various mathematical models of biological systems (**Ahmed et al. 2002a; 2004; 2020)** and theoretical games (**Ahmed et al., 2002b; 2005)**.

In general, NPDEs (**Evans, 2010)** are difficult to solve analytically. The presence of nonlinear terms makes the traditional methods such as separation of variables and Fourier analysis ineffective. While there are certain analytical solutions for some special cases and certain boundary conditions **(Ablowitz and Clarkson, 1991)**, many real-world problems involve nonlinearity or complex boundary conditions that make analytical solutions intractable. In these cases, numerical methods **(Press et al., 2007)** provide robust tools to solve these challenges.

The finite difference (FD) method **(Thomas, 1995; Samarskii, 2001)** is widely used to solve NPDEs numerically. It involves the following steps. The spatial and temporal domains are divided into a grid of points. The derivatives are approximated using FD formulas, and the resulting algebraic equations are solved numerically. Advantages of the FD method include its simplicity and efficiency, especially for problems with regular grids. The FD method plays a crucial role in solving real-life problems **(Elgazzar, 2021; Soliman et al. 2025)**.

The stability of a numerical technique is an important concept in numerical analysis. The von Neumann stability analysis **(Hoffman, 2001)** is a method for examining the stability of linear FD schemes. It involves analyzing the amplification factor, which represents the growth of error modes over time. A numerical scheme is considered stable if the absolute value of the amplification factor is less than or equal to one for all possible error modes.

The local truncation error (LTE) **(Dahlquist and Bjorck, 2008)** of a FD scheme measures the error caused by the approximation of the derivatives. It is calculated using Taylor series expansions to express the function values at neighboring grid points, then comparing the discretized equation to the original equation.

Numerical convergence is an important aspect of numerical analysis. It is evaluated by error norms such as the L_2 and L_∞ norms **(Djidjeli et al., 1995)**. These error norms measure the inconsistency between the numerical and analytical solutions.

Various numerical methods were developed for solving the SG equation in one and two dimensions. Here, some examples are given, starting with the one-dimensional case. A meshfree numerical scheme was proposed demonstrating ease of implementation and high accuracy **(Jiang and Wang, 2012)**. Predictor-corrector schemes **(Bratsos, 2008a;**

2008b) were successfully applied. Dehghan and Mirzaei **(Dehghan and Mirzaei, 2008)** used the dual reciprocity technique. Raslan et al. **(Raslan et al., 2023)** used both FD and non-standard FD methods.

With respect to the 2D SG equation, the differential quadrature technique was used effectively **(Jiwari et al., 2012)**. Dehghan and Shokri **(Dehghan and Shokri, 2008)** provided accurate results by using the radial basis functions. A modified predictor-corrector technique was introduced to improve the accuracy of the numerical solutions **(Bratsos, 2007; Kaya, 2004)**.

Soliman et al. **(2007; 2008a; 2008b; 2009a; 2009b; 2012)** presented various methods for solving complex equations and simulations. These include numerical solutions based on similarity reductions, exact solutions using the exponential function method, decomposition methods for solving coupled modified equations, improved tanh-function methods for solving nonlinear physical problems, numerical simulations of equations modelling neuron interactions, and exact travelling wave solutions for NPDEs. The combined findings offer efficient and accurate solutions applicable to a range of mathematical and physical models.

The development of accurate and efficient numerical solutions for the 2D SG equation is necessary for the following reasons.

1. Accurate numerical solutions enable deeper understanding and more accurate predictions in areas where the 2D SG equation is applied.
2. Numerical methods can help in the development and optimization of systems subject to the 2D SG equation. This can lead to advances in technologies and materials.
3. Numerical methods allow researchers to study the complex dynamics of solutions, especially when analytical solutions are not available.

This work aims to extend the numerical techniques for solving the 2D SG equation. The FD method is used. The FD technique is known to be of second-order accuracy and unconditionally stable for linear problems, but its performance for nonlinear problems requires further investigation. We investigate the stability and LTE of the FD scheme of the 2D SG equation. We also present numerical solutions and error analysis to ensure the scheme's accuracy. The remaining content is structured as follows. Section 2 describes the 2D SG equation. Section 3 presents the FD method. In sections 4 and 5, stability and LTE of the scheme are studied. In section 6, we present numerical simulations and verify the accuracy of the numerical solution. The conclusion is presented in section 7.

The problem

The 2D SG equation is an important NPDE. It arises in various physical contexts. It is given by

$$\frac{\partial^2 v}{\partial t^2} - \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \sin(v) = 0, \qquad (1)$$

where $v(x, y, t)$ represents a physical quantity, and x, y and t refer to the spatial and temporal coordinates, respectively. Let $-1 \le x, y \le 1$, $t > 0$. Equation (1) arises in many physical topics. We assume the following initial conditions

$$v(x, y, 0) = 4\tan^{-1}(e^{x+y}), \qquad v_t(x, y, 0) = \frac{-4e^{x+y}}{1+e^{2(x+y)}}. \qquad (2)$$

The exact analytical solution **(Dehghan and Ghesmati, 2010)** is

$$v(x, y, t) = 4\tan^{-1}(e^{x+y-t}). \qquad (3)$$

The boundary conditions are obtained from Eq. (3). To better understand of the FD method, we use it to numerically solve the 2D SG equation. We perform a stability and error analysis and present numerical simulations that include comparing the numerical solution to the exact one. This is discussed in the following sections.

The finite difference method

The approximation of the derivatives using finite differences plays a crucial role in the FD technique. Suppose $v(x, y, t)$ is a C^4 continuous function. For a grid with spatial spacing $\Delta x = \Delta y = h$ and $\Delta t = k$ in time, where $h, k > 0$, the second-order partial derivatives of v are expressed in discretized form as follows

$$v_{tt}(i,j,n) = \frac{1}{k^2}[v(i,j,n+1) - 2v(i,j,n) + v(i,j,n-1)], \tag{4}$$

$$v_{xx}(i,j,n) = \frac{1}{h^2}[v(i+1,j,n) - 2v(i,j,n) + v(i-1,j,n)], \tag{5}$$

$$v_{yy}(i,j,n) = \frac{1}{h^2}[v(i,j+1,n) - 2v(i,j,n) + v(i,j-1,n)], \tag{6}$$

where $v(i, j, n)$ denotes the value of v at grid point (i, j) and time step n. By substituting from Eqs. (4)-(6) into Eq. (1), we get

$$\begin{aligned} v(i,j,n+1) &= 2v(i,j,n) - v(i,j,n-1) \\ &+ r^2[v(i+1,j,n) + v(i-1,j,n) \\ &+ v(i,j+1,n) - 4v(i,j,n)] \\ &- k^2 \sin(v(i,j,n)), \end{aligned} \tag{7}$$

where $r^2 = k^2/h^2$. By including this recurrence relation with both initial and boundary conditions, the corresponding value at time $n+1$ can be obtained. The stability and LTE of the scheme (Eq. (7)) are discussed in the next two sections.

Stability analysis

The von Neumann method **(Hoffman, 2001)** is a powerful tool for examining stability of linear FD scheme. It involves the analysis of the behavior of Fourier modes of the error over time. We can apply this method to a linearized form of the scheme of the 2D SG equation (Eq. (7)). This is explained in Theorem (1).

Theorem (1): *The numerical scheme* (*Eq.* (7)) *is conditionally stable. The stability condition is*

$$\frac{2k^2}{h^2} \le 1.$$

Proof. We begin by approximating the nonlinear term in Eq. (7) as $\sin(v(i, j, n)) \simeq v(i, j, n)$, and Eq. (7) becomes

$$\begin{aligned} v(i,j,n+1) &= 2v(i,j,n) - v(i,j,n-1) \\ &+ r^2[v(i+1,j,n) + v(i-1,j,n) \\ &+ v(i-1,j,n) + v(i,j+1,n) + \\ &v(i,j-1,n) - 4v(i,j,n)] - k^2 v(i,j,n). \end{aligned} \tag{8}$$

Then we represent $v(i, j, n)$ as a Fourier mode as follows

$$v(i,j,n) = U(n) e^{I h (k_x i + k_y j)}. \tag{9}$$

In Eq. (9), $U(n)$ refers to the amplitude of Fourier mode at time step n, k_x, and k_y refer to the wave numbers in the directions of the axes x and y, respectively, and I is the imaginary unit. Substituting from Eq. (9) into Eq. (8) yields

$$\begin{aligned} U(n+1) &= 2U(n) - U(n-1) + 4r^2 U(n) \\ &(\sin^2(\frac{k_x h}{2}) + \sin^2(\frac{k_y h}{2})) - k^2 U(n). \end{aligned} \tag{10}$$

Let the amplification factor be G, such that

$$U(n+1)=GU(n). \quad (11)$$

By substituting in Eq. (10), we obtain the characteristic equation of the amplification factor as follows

$$G^2-2[1-\frac{k^2}{2}-2r^2(\sin^2(\frac{k_x h}{2}) +\sin^2(\frac{k_y h}{2}))]G+1=0. \quad (12)$$

The stability of a numerical technique requires that the magnitude of the amplification factor must satisfy

$$|G|\le 1. \quad (13)$$

This requires

$$\left|1-\frac{k^2}{2}-2r^2(\sin^2(\frac{k_x h}{2})+\sin^2(\frac{k_y h}{2}))\right|\le 1. \quad (14)$$

Then

$$-2\le -\frac{k^2}{2}-2r^2(\sin^2(\frac{k_x h}{2})+\sin^2(\frac{k_y h}{2}))\le 0. \quad (15)$$

The term $(-k^2/2)-2r^2\ (\sin^2\ (k_xh/2)\ +\sin^2\ (k_yh/2))$ is always negative, then the condition (Eq. (13)) requires

$$r^2(\sin^2(\frac{k_x h}{2})+\sin^2(\frac{k_y h}{2}))\le 1-\frac{k^2}{4}\le 1. \quad (16)$$

To ensure the satisfaction of Eq. (16) for all possible wave numbers *kx* and *ky*, we take $\sin^2\ (k_xh/2)\ +\sin^2\ (k_yh/2)$ at its maximum value, then

$$2r^2\le 1. \quad (17)$$

Therefore, the numerical scheme (Eq. (7)) is conditionally stable, and the stability condition is

$$\frac{2k^2}{h^2}\le 1. \quad (18)$$

Local truncation error

The LTE **(Dahlquist and Bjorck, 2008)** is the error that occurs in a single time step due to approximating the derivatives. The LTE can be analyzed by comparing the analytical solution with the numerical scheme using Taylor expansions. The following theorem determines the LTE of the numerical scheme (Eq. (7)).

Theorem (2): *The LTE of the FD numerical scheme of the 2D SG equation (Eq. (7)) is* $O(k^2+h^2)$.

Proof. The Taylor expansions of $v(i, j, n)$ around n are

$$v(i,j,n+1)=v(i,j,n)+k\frac{\partial v}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 v}{\partial t^2} +\frac{k^3}{3!}\frac{\partial^3 v}{\partial t^3}+O(k^4), \quad (19)$$

$$v(i,j,n-1)=v(i,j,n)-k\frac{\partial v}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 v}{\partial t^2} -\frac{k^3}{3!}\frac{\partial^3 v}{\partial t^3}+O(k^4). \quad (20)$$

Then

$$\frac{1}{k^2}[v(i,j,n+1)-2v(i,j,n)+v(i,j,n-1)] =\frac{\partial^2 v}{\partial t^2}+O(k^2). \quad (21)$$

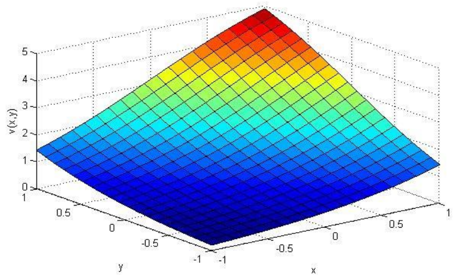
This means that the LTE of the time discretization is $O(k^2)$. Similarly, we derive that the LTE of the spatial discretization is $O(h^2)$. Therefore, the LTE is $O(k^2+h^2)$ for the entire scheme.

This result agrees with the corresponding result in the one-dimensional case **(Raslan et al., 2023)**.

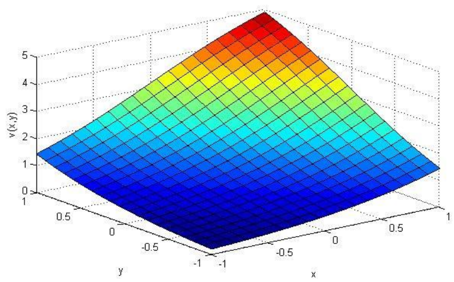
Numerical simulations

In this section, we present numerical simulations of Eq. (1). A FD scheme (Eq. (7)) is used. The boundary and initial conditions are given in section 2. We also perform an error analysis to evaluate the accuracy of the method. Based on the stability condition (Eq. (18)), a grid of size $\Delta x = \Delta y = h = 0.05$ and a time step of size $\Delta t = k = 0.01$ are used in the simulations.

Figure (1) compares the exact analytical solution (Fig. 1(a)) with the numerical solution (Fig. 1(b)) at $t = 1.0$. To illustrate the comparison, we graph the solution v for certain values of y as a function of x. In Fig. (2), both solutions are displayed as functions of x at $y = -0.7$, $y = 0.7$ and $t = 1.0$. Figures (1&2) show excellent agreement between both solutions. Absolute error of the numerical solution is shown in Fig. (3).



(a)



(b)

Fig. (1): (Color online) (a) Exact solution, (b) Numerical solution of the 2D SG equation at $t = 1.0$. In the simulations, we take $h = 0.05$ and $k = 0.01$.

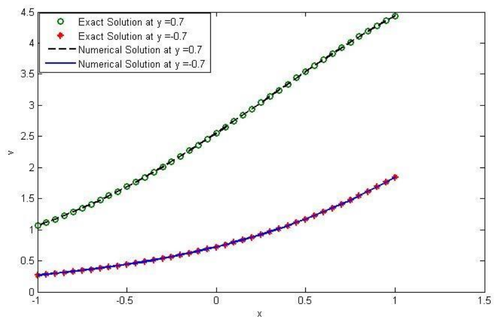


Fig. (2): (Color online) An illustrative comparison between the exact and numerical solutions of the 2D SG equation at certain values of y at $t = 1.0$. In the simulations, we take $h = 0.05$ and $k = 0.01$.

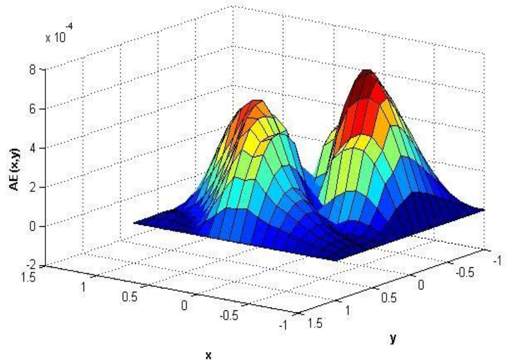


Fig. (3): (Color online) Absolute error of the numerical scheme of the 2D SG equation at $t = 1.0$. In the simulations, we take $h = 0.05$ and $k = 0.01$.

To examine the consistency and accuracy of the numerical solution, we calculate the error norms L_2 and L_∞ **(Djidjeli et al., 1995)**, where

$$L_2 := \sqrt{(\Delta x)(\Delta y)\sum_i \sum_j \left| v^{exact}(i,j) - v(i,j) \right|^2}, \tag{22}$$

$$L_\infty := \max_{i,j} \left| v^{exact}(i,j) - v(i,j) \right|. \tag{23}$$

Both L_2 and L_∞ norms are calculated at selected time points, and the results are given in Table 1. From Fig. 3 and Table 1, the overall magnitude of the error is very small. This indicates that the numerical scheme of the 2D SG equation using the FD technique is accurate.

t	$L_2 * 10^{-4}$	$L_\infty * 10^{-4}$
0.1	7.034	5.005
0.2	12.776	9.919
0.3	16.875	14.579
0.4	19.090	18.850
0.5	19.302	22.234
0.6	17.530	23.243
0.7	13.942	20.585
0.8	8.892	14.397
0.9	3.478	6.638
1.0	5.718	7.502

Table (1): The results of the error norms L_2 and L_∞ of the numerical scheme (Eq. (7)) of the 2D SG (Eq. (1)) at different time points.

Conclusion

The 2D SG equation has numerous applications in various fields. The FD technique is powerful in solving NPDEs. We investigate a FD numerical scheme for the 2D SG equation. It is shown that the scheme is stable under the condition, $(2k^2/h^2)\leq 1$. The LTE is $O(k^2+h^2)$. This ensures that the FD scheme of the 2D SG equation is accurate of the second-order in space and time.

Numerical simulations show good consistency between both the exact and numerical solutions. Our results are consistent with those of the one-dimensional case **(Raslan et al., 2023)**. Despite its simplicity, the presented scheme provides similarly accurate results as given in **(Jiwari et al., 2012; Dehghan and Shokri, 2008; Bratsos, 2007; Kaya, 2004)**. These results are important for understanding the properties of the FD method, the 2D SG equation and its numerous applications. This scheme could be used for modeling wave propagation in nonlinear media and superconducting Josephson junctions. Future work could focus on the extension to more complex boundary conditions and/or higher-dimensional problems.

Acknowledgements

The authors are very thankful to the reviewers for their valuable suggestions.

Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

References

Ablowitz, M. A.; Clarkson, P. A. (1991): Solitons, Nonlinear Evolution Equations and Inverse Scattering, *Cambridge University Press*.

Agrawal, G. (2013): Nonlinear Fiber Optics (Fifth Edition), *Academic Press*.

Ahmed, E.; Hegazi, A. S.; Elgazzar, A.S. (2002a): An Epidemic Model on Small-World Networks and Ring Vaccination, *International Journal of Modern Physics C* **13**: 189-198.

Ahmed, E.; Hegazi, A. S.; Elgazzar, A.S. (2002b): On Spatial Asymmetric Games, *Advances in Complex Systems* **5**: 433-443.

Ahmed, E.; Hegazi, A. S.; Elgazzar, A.S. (2004): On Persistence and Stability of Some Biological Systems with Cross Diffusion, *Advances in Complex Systems* **7**: 65-76.

Ahmed, E.; Hegazi, A. S.; Elgazzar, A.S. (2005): On Some Variants of Prisoner's Dilemma Dynamics, *Applied Mathematics and Computation* **163**: 163-168.

Ahmed, E.; Elgazzar, A. S.; Elsadany, A. A. (2020): Simple Mathematical Models of Antimicrobial Resistance, *Journal of Fractional Calculus and Applications* **11**: 22-25.

Barone, A.; Paterno, G. (1982): Physics and Applications of the Josephson Effect, *John Wiley & Sons, Inc.*

Bratsos, A. (2007): A Modified Predictor-Corrector Scheme for the Two-Dimensional Sine-Gordon Equation, *Numerical Algorithms* **43**: 295-308.

Bratsos, A. (2008a): A Numerical Method for the One-Dimensional Sine-Gordon Equation, *Numerical Methods for Partial Differential Equations* **24**: 833-844.

Bratsos, A. (2008b): A Fourth Order Numerical Scheme for the One-Dimensional Sine-Gordon Equation, *International Journal of Computer Mathematics* **85**: 1083-1095.

Cuevas-Maraver, J.; Keverkidis, P. G.; Williams, F. (Eds.) (2014): The Sine-Gordon Model and Its Applications, *Springer*.

Dahlquist, G.; Bjorck, A. (2008): Numerical Methods in Scientific Computing, *Society for Industrial and Applied Mathematics*.

Dehghan, M.; Mirzaei, D. (2008): The Boundary Integral Equation Approach for Numerical Solution of the One-Dimensional Sine-Gordon Equation, *Numerical Methods for Partial Differential Equations* **24**: 1405-1415.

Dehghan, M.; Shokri, A. (2008): A Numerical Method for Solution of the Two-Dimensional Sine-Gordon Equation Using the Radial Basis Functions, *Mathematics and Computers in Simulation* **79**: 700-715.

Dehghan, M.; Ghesmati, A. (2010): Numerical Simulation of Two-Dimensional Sine-Gordon Solitons via a Local Weak Meshless Technique Based on Radial Point Interpolation Method (RPIM), *Computer Physics Communications* **181**: 772-786.

Djidjeli, K.; Price, W. G.; Twizell, E.H. (1995): Numerical Solutions of a Damped Sine-Gordon Equation in Two Space Variables, *Journal of Engineering Mathematics* **29**: 347-369.

Elgazzar, A. S. (2021): Simple Mathematical Models for Controlling COVID-19 Transmission through Social Distancing and Community Awareness, *Zeitschrift für Naturforschung C* **67**: 393-400.

Evans, L. C. (2010): Partial Differential Equations, *American Mathematical Society*.

Hoffman, J. D. (2001): *Numerical* Methods for Engineers and Scientists, *CRC Press*.

Jiang, Z.; Wang, R. (2012): Numerical Solution of One-Dimensional Sine-Gordon Equation Using High Accuracy Multiquadric Quasi-Interpolation, *Applied Mathematics and Computation* **218**: 7711-7716.

Jiwari, R.; Pandit, S.; Mittal, R. (2012): Numerical Simulation of Two-Dimensional Sine-Gordon Solitons by Differential Quadrature Method, *Computer Physics Communications* **183**: 600-616.

Kaya, D. (2004): An Application of the Modified Decomposition Method for Two-Dimensional Sine-Gordon Equation, *Applied Mathematics and Computation* **159**: 1-9.

Perring, J. K.; Skyrme, T. H. R. (1962): A Model Unified Field Equation, *Nuclear Physics* **31**: 550-555.

Press, W. H.; Teukolsky, S. A.; Vetterling, W. T.; Flannery, B. P. (2007): Numerical Recipes: The Art of Scientific Computing, *Cambridge University Press*.

Raslan, K. R.; Soliman, A. A.; Ali, K. K.; Gaber, M.; Almahdy, S. R. (2023): Numerical Solution for the Sine-Gordon Equation Using the Finite Difference Method and Nonstandard Finite Difference Method, *Applied Mathematics & Information Sciences* **17**: 253-260.

Samarskii, A. A. (2001): The Theory of Difference Schemes, *CRC Press*.

Soliman, A. A.; Abdou, M. A. (2007): Exact Travelling Wave Solutions of Nonlinear Partial Differential Equations, *Chaos, Solitons & Fractals* **32**: 808-815.

Soliman, A. A.; Abdou, M. A. (2008a): The Decomposition Method for Solving the Coupled Modified KdV Equations, *Mathematical and Computer Modelling* **47**: 1035-1041.

Soliman, A. A. (2008b): Extended Improved Tanh-Function Method for Solving the Nonlinear Physical Problems, *Acta Applicandae Mathematicae* **104**: 367-383.

Soliman, A. A.; Ali, A. H. A.; Raslan, K. R. (2009a): Numerical Solution for the KdV Equation Based on Similarity Reductions, *Applied Mathematical Modelling* **33**: 1107-1115.

Soliman, A. A. (2009b): Exact Solutions of KdV-Burgers Equation by Exp-Function Method, *Chaos, Solitons & Fractals* **41**: 1034-1039.

Soliman, A. A. (2012): Numerical Simulation of the FitzHugh-Nagumo Equations, *Abstract and Applied Analysis* **2012**: 762516.

Soliman, A. A.; Dahshan, M. M.; Elgazzar, A. S. (2025): Stability Properties of an Epidemic Model with Cross-Diffusion, *Applied Mathematics & Information Sciences* **19**: 541-549.

Thomas, J. W. (1995): Numerical Partial Differential Equations: Finite Difference Methods, *Springer*.

"مخطط الفروق المحدودة لمعادلة ساين-غوردون ثنائية الأبعاد"

عبد المقصود عبد القادر سليمان، منار محمد دهشان، أحمد سعد الجزار

قسم الرياضيات، كلية العلوم، جامعة العريش، مصر

يهدف هذا البحث إلى تطوير طريقة عددية فعالة لحل معادلة ساين-غوردون ثنائية الأبعاد. تعتبر هذه المعادلة مهمة في العديد من التطبيقات الفيزيائية والهندسية. تم استخدام طريقة الفروق المحدودة نظراً لسهولة تطبيقها وكفاءتها في حل المشكلات الرياضية.
تركز الدراسة على اختبار أداء هذه الطريقة مع المعادلات غير الخطية. يتم تحليل استقرار الطريقة وحساب الأخطاء الناتجة عن التقريب العددي. كما يتم تقديم اختبارات عملية للتأكد من دقة النتائج.
ينقسم البحث إلى عدة أقسام. يبدأ بشرح معادلة ساين-غوردون الأساسية. ثم يتم عرض طريقة الفروق المحدودة المستخدمة في الحل. بعد ذلك يتم دراسة استقرار الطريقة وتحليل الأخطاء. أخيراً يتم تقديم النتائج العددية والتحقق من صحتها.
تساهم هذه الدراسة في توفير أداة فعالة للباحثين في مجالات الرياضيات التطبيقية والفيزياء النظرية. تساعد النتائج في فهم أفضل لسلوك معادلة ساين-غوردون في بعدين. كما تفتح المجال لتطبيقات عملية في مجالات متعددة.