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Exact Test for Diagonals-Parameter Symmetry Model in a 3×3 Table with Ordered Categories

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Summary

This short note gives an exact test for Goodman's (1979) diagonals-parameter symmetry model in a 3×3 table with ordered categories, which can be easily done applying Fisher's exact test of independence in a 2×2 table. An example is given.

Key words: Conditional distribution; Fisher's exact test.

1. Introduction

Fisher's exact test of independence in a 2×2 table is widely known (e.g., Plackett, 1981, p.47). For the analysis of square contingency tables with ordered categories, Goodman (1979) proposed the diagonals-parameter symmetry (DPS) model. The purpose of this short note is to give an exact test for the DPS model in a 3×3 table, which can be easily done applying Fisher's exact test.

2. Exact test for diagonals-parameter symmetry model

Consider a 3×3 table with cell probabilities $\{p_{ij}\}$. The DPS model is defined by

$$p_{ij} = \begin{cases} \tau_{j-i}\omega_{ij} & \text{for } i<j, \\ \omega_{ij} & \text{for } i\geq j, \end{cases}$$

where $\omega_{ij}=\omega_{ji}$. This model is equivalent to

$$\psi = 1,$$

where $\psi=(p_{12}p_{32})/(p_{21}p_{23})$. Suppose the cell counts (n_{ij}) have a multinomial distribution with $n=\Sigma\Sigma n_{ij}$ fixed. The probability function of (n_{ij}) is

$$P((n_{ij})|n) = \frac{n!}{\Pi\Pi n_{ij}!}\Pi\Pi p_{ij}^{n_{ij}}$$

$$= \frac{n!}{(\prod_{k=1}^{3} n_{kk}!)n_{13}!n_{31}!n^*_{1\cdot}!n^*_{2\cdot}!} \gamma^n (\prod_{k=1}^{3} \theta_k^{n_{kk}})\delta_1^{n_{13}}\delta_2^{n_{31}}$$

$$\times(\prod_{s=1}^{2} \phi_s^{n^*_{s\cdot}} \lambda_s^{n^*_{\cdot s}})h(n_{12};(n^*_{i\cdot}),(n^*_{\cdot j}),\psi),$$

where $n^*_{1\cdot}=n_{12}+n_{23}$, $n^*_{2\cdot}=n_{21}+n_{32}$, $n^*_{\cdot 1}=n_{12}+n_{21}$, $n^*_{\cdot 2}=n_{23}+n_{32}$ and

$\gamma=p_{32}$, $\theta_k=p_{kk}/p_{32}$, $\delta_1=p_{13}/p_{32}$, $\delta_2=p_{31}/p_{32}$,

$\phi_1=p_{23}/p_{32}$, $\phi_2=1$, $\lambda_1=p_{21}/p_{32}$, $\lambda_2=1$,

$$h(n_{12};\cdot,\psi) = \binom{n^*_{1\cdot}}{n_{12}}\binom{n^*_{2\cdot}}{n^*_{\cdot 1}-n_{12}}\psi^{n_{12}}.$$

[Note that γ is a function of (θ_k), (δ_s), (ϕ_s), (λ_s) and ψ.] The sufficient statistics for the nuisance parameters are (n_{kk}), n_{13}, n_{31}, $(n^*_{s\cdot})$ and $(n^*_{\cdot t})$. Given them (then n_{12} determines all other cell counts), the conditional probability function of n_{12} is

$$P(n_{12}|(n_{kk}), n_{13}, n_{31}, (n^*_{s\cdot}), (n^*_{\cdot t})) = h(n_{12};\bullet,\psi) / \sum_{u=A}^{B} h(u;\bullet,\psi), \tag{2.1}$$

where $n_{12}=A, A+1, \cdots, B$; $A=\max(0, n^*_{\cdot 1}-n^*_{2\cdot})$, $B=\min(n^*_{1\cdot}, n^*_{\cdot 1})$. The exact test for the DPS model could be done using this conditional distribution. When $\psi=1$, (2.1) may be expressed as

$$P(n_{12}|\bullet) = \binom{n^*_{1\cdot}}{n_{12}}\binom{n^*_{2\cdot}}{n^*_{\cdot 1}-n_{12}} \Big/ \binom{n^*_{1\cdot}+n^*_{2\cdot}}{n^*_{\cdot 1}}.$$

Note that this exact test for the DPS model is equivalent to Fisher's exact test of independence applied to the 2×2 table with the first row (n_{12}, n_{23}) and the second row (n_{21}, n_{32}).

3. An example

For the data in Table 1, we conduct the exact test of H_0: $\psi=1$ against H_1: $\psi\neq 1$. We get

$$h(u;\bullet,\psi) = \binom{55}{u}\binom{38}{78-u}\psi^u \quad \text{for } u=40,41,\cdots,55.$$

From Table 2, for the usual two-sided test of the 0.05 significance level (whose the critical region is determined such that each tail area is at most 0.025), the critical region consists of the values 40,41,42,51,52,53,54,55 of n_{12}. (Then the actual significance level is less than 0.05.) Since n_{12} for the observed table is 52, we now reject H_0 (twice the minimum of the exact upper and lower p-values is 0.0020384). Therefore, the odds that the number of lambs born to a ewe in 1953 is more, instead of less, than the number in 1952 when a pair of numbers in 1952 and 1953 is (0,1), is *not* equal to the odds when a pair of numbers in 1952 and 1953 is (1,2).

Table 1

Cross-Classification of Ewes According to Number of Lambs Born in Consecutive Years

Number of Lambs 1953	Number of Lambs 1952: 0	1	2	Totals
0	58	52	1	111
1	26	58	3	87
2	8	12	9	29
Totals	92	122	13	227

Taken directly from Bishop et al.(1975, p.288).

Table 2

Values of $P(n_{12}|(n_{kk}),n_{13},n_{31},(n^*_{s\cdot}),(n^*_{\cdot t}))$ with $\psi=1$,

given $(n_{kk}),n_{13},n_{31},(n^*_{s\cdot}),(n^*_{\cdot t})$ in Table 1

n_{12}	$P(n_{12}\|\bullet)$
40	0.0001523
41	0.0021178
42	0.0130599
43	0.0473801
44	0.1130661
45	0.1879411
46	0.2247121
47	0.1967085
48	0.1270409
49	0.0604957
50	0.0210525
51	0.0052538
52[a]	0.0009093
53	0.0001029
54	0.0000068
55	0.0000002

[a] Observed table

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