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# RECURRENCE RELATIONS BETWEEN MOMENT GENERATING FUNCTION OF ORDER STATISTICS FOR DOUBLY TRUNCATED CONTINUOUS DISTRIBUTIONS

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## ABSTRACT

we derive some results for obtaining recurrence relations between moment generating functions of order statistics and recurrence relations between moment generating functions of the sum of two order statistics. There are some applications of these results to doubly truncated distributions..

## 1. INTRODUCTION

khan et al (1983a,1983b,1987) obtaind general results for obtaining recurrence relations for moments of order statistics from doubly truncated continuous distribution and they found the single and product moments of order statistics from doubly truncated Burr distributions.

Mohie El-Din et al (1991)obtained the single and product moments of order statistics from doubly truncated parapolic and skewed distributions .In this paper, we find some results for obtaining recurrence relations between single moment generating functions and between moment generating functions of the sum of two order statistics from doubly truncated continuous distributions. These results are applied to exponential, logistic, extrem-value and weibull distributions.

Let $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$ be the order statistics obtained from a continuous distributions function (c.d.f.) $F(x)$ with probability density function(p.d.f) $f(x)$. The p.d.f of $x_{r:n}$ $(1 \le r \le n)$ is

$$\frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \quad , -\infty \le x \le \infty \tag{1-1}$$

Let $\quad \alpha^{(K)}_{r:n} = E(x^k_{r:n})$ ,

$M_{x_{r:n}}(t) = E(e^{tx_{r:n}})$ ,

then, we have

$$\alpha^{(k)}_{r:n} = \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^k [F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx \tag{1-2}$$

and

$$M_{x_{r:n}}(t) = \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} e^{tX}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx \tag{1-3}$$

The joint p.d.f of $x_{r:n}$ and $x_{s:n}$ is

$$\frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y) \ ,$$

$$-\infty \le x \le y \le \infty \tag{1-4}$$

Let $\quad \alpha^{(j,k)}_{r,s:n} = E(x^j_{r:n}\, y^k_{s:n})$

and $\quad M_{x_{r:n},x_{s:n}}(t_1,t_2) = E(e^{t_1 x_{r:n} + t_2 x_{s:n}})$ ,

$M_{U_{r,s:n}}(t) = E(e^{t(x_{r:n}+y_{s:n})})$, $U_{r,s:n} = x_{r:n}+y_{s:n}$ , $t_1 = t_2 = t$

then, we have

$$\alpha^{(j,k)}_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^j y^k [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$[1-F(y)]^{n-s}f(x)f(y)dydx \ , \quad x \le y \tag{1-5}$$

and

$$M_{U_{r,s:n}}(t) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\int_{-\infty}^{\infty}\int_{x}^{\infty} e^{t(x+y)}[F(x)]^{r-1} \times$$

$$[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dydx\ , \quad -\infty\le x\le y\le \infty \qquad (1\text{-}6)$$

Truncating a p.d.f. $f(x)$ from both sides at $Q_1$ and $P_1$, where $Q_1\le P_1$, the new p.d.f $f_1(x)$ is

$$f_1(x)=\frac{f(x)}{P-Q}\ , \qquad Q_1\le x< P_1 \qquad (1\text{-}7)$$

where $F(Q_1)=Q$ and $F(P_1)=P$

## 2. RECURRENCE RELATIONS FOR SINGLE MOMENT GENERATING FUNCTIONS OF ORDER STATISTICS

For doubly truncated continuous distribution, (1-2) and (1-3) take the form

$$\alpha^{(k)}_{r:n}=\frac{n!}{(r-1)!(n-r)!}\int_{Q_1}^{P_1} x^k[F_1(x)]^{r-1}[1-F_1(x)]^{n-r}f_1(x)dx\ , \qquad (2\text{-}1)$$

$$M_{X_{r:n}}(t)=\frac{n!}{(r-1)!(n-r)!}\int_{Q_1}^{P_1} e^{tx}\,[F_1(x)]^{r-1}\times$$
$$[1-F_1(x)]^{n-r}f_1(x)dx\ , \qquad Q_1\le x\le P_1 \qquad (2\text{-}2)$$

where $f_1(x)$ is given by (1-7) and $F_1(x)$ is the corresponding c.d.f.

An easy consequence of the definition is that

$$r\,M_{X_{r+1:n}}(t)=n\,M_{X_{r:n-1}}(t)-(n-r)\,M_{X_{r:n}}(t) \qquad (2\text{-}3)$$

Theorem(2-1) For $2\le r\le n$ and for any continuous disribution

$$M_{X_{r:n}}(t)-M_{X_{r-1:n-1}}(t)=\binom{n-1}{r-1}\,t\int_{Q_1}^{P_1} e^{tx}[F_1(x)]^{r-1}[1-F_1(x)]^{n-r+1}dx \qquad (2\text{-}4)$$

Proof From (2-2), we have

$$M_{X_{r:n}}(t)-M_{X_{r-1:n-1}}(t)=\frac{(n-1)!}{(r-1)!(n-r)!}\int_{Q_1}^{P_1} e^{tx}\,[F_1(x)]^{r-2}[1-F_1(x)]^{n-r}f_1(x)$$
$$\times[n\,F_1(x)-(r-1)]\,dx\ . \qquad (2\text{-}5)$$

Let $h(x)=-[F_1(x)]^{r-1}[1-F_1(x)]^{n-r+1}$ ,

then

$$\frac{dh(X)}{dX} = [F_1(x)]^{r-2}[1-F_1(x)]^{n-r}[nF_1(x)-(r-1)]f_1(x) \qquad (2\text{-}6)$$

From(2-5) and (2-6) we have

$$M_{X_{r:n}}(t)-M_{X_{r-1:n-1}}(t) = \binom{n-1}{r-1} \int_{Q_1}^{P_1} e^{tx} \left(\frac{dh(x)}{dx}\right) dx . \qquad (2\text{-}7)$$

Integration by parts yields

$$\int_{Q_1}^{P_1} e^{tx} \frac{d}{dx} h(x)dx = [e^{tx} h(x)]_{Q_1}^{P_1} - t \int_{Q_1}^{P_1} e^{tx} h(x)dx$$

$$= t \int_{Q_1}^{P_1} e^{tx} [F_1(x)]^{r-1}[1-F_1(x)]^{n-r+1}dx \quad , \qquad (2\text{-}8)$$

since $h(x)$ vanishes at both $Q_1$ and $P_1$. Finally substituting from (2-8) in (2-7), the required result is obtained.

<u>Proposition(2-2)</u>

$$M_{X_{r:n}}(t)-M_{X_{r-1:n-1}}(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} [\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)}], \; 2 \le r \le n-1 \qquad (2\text{-}9)$$

<u>Proof</u> By definition, we have

$$M_{X_{r:n}}(t)- M_{X_{r-1:n-1}}(t) = 1+ \sum_{k=1}^{\infty} \frac{t^k}{k!} \alpha_{r:n}^{(k)} - [1+ \sum_{k=1}^{\infty} \frac{t^k}{k!} \alpha_{r-1:n-1}^{(k)}]$$

$$= \sum_{k=1}^{\infty} \frac{t^k}{k!} [\alpha_{r:n}^{(k)} - \alpha_{r-1:n-1}^{(k)}]$$

## 3 RECURRENCE RELATIONS BETWEEN THE MOMENT GENERATING FUNCTIONS OF THE SUM OF TWO ORDER STATISTICS

For doubly truncated continuous distribution function, the product moments of $x_r$ and $x_s$ and the moment generating functions of $U_{r,s:n} = x_{r:n} + x_{s:n}$ are

$$\alpha_{r,s:n}^{(j,k)} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^k [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}$$

$$[1-F_1(y)]^{n-s} f_1(x)f_1(y)\, dy\, dx \quad , \qquad (3\text{-}1)$$

$$M_{X_{r:n},X_{s:n}}(t_1,t_2) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{Q_1}^{P_1} \int_{x}^{P_1} e^{(t_1x+t_2y)} [F_1(x)]^{r-1} \times$$

$$[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s}f_1(x)f_1(y)dydx \qquad (3-2)$$

and , when $t_1 = t_2 = t$ , we have

$$M_{U_{r,s:n}}(t) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{Q_1}^{P_1}\int_{x}^{P_1} e^{t(x+y)}[F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}$$

$$[1-F_1(y)]^{n-s}f_1(x)f_1(y)\,dy\,dx \quad , Q_1 \le x \le y \le P_1 \qquad (3-3)$$

**Theorem(3-1)** For any continuous distribution function , the moment generating function $M_{U_{r,s:n}}(t)$ satisfies,

$$M_{U_{r,s:n}}(t)-M_{U_{r,s-1:n}}(t) = \frac{n!\,t}{(r-1)!(s-r-1)!(n-s+1)!} \int_{Q_1}^{P_1}\int_{x}^{P_1} e^{t(x+y)}[F_1(x)]^{r-1} \times$$

$$[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s+1}f_1(x)dydx, 1\le r\le s\le n-1. \qquad (3-4)$$

**Proof** From (3-3), we have

$$M_{U_{r,s:n}}(t)-M_{U_{r,s-1:n}}(t) = \frac{n!}{(r-1)!(s-r-1)!(n-s+1)!} \int_{Q_1}^{P_1}\int_{x}^{P_1} e^{t(x+y)}[F_1(x)]^{r-1}$$

$$\times [F_1(y)-F_1(x)]^{s-r-2}[1-F_1(y)]^{n-s} \{(n-r)F_1(y)-(n-s+1)F_1(x)-(s-r-1)\} \times$$

$$f_1(x)f_1(y)\,dy\,dx \quad . \qquad (3-5)$$

Let

$$h(x,y) = -[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s+1} \quad ,$$

then,

$$\frac{\partial h(x,y)}{\partial Y} = [F_1(y)-F_1(x)]^{s-r-2}[1-F_1(y)]^{n-s} \times$$

$$\times \{(n-r)F_1(y)-(n-s+1)F_1(x)-(s-r-1)\}f_1(x) \quad . \qquad (3-6)$$

Now from (3-5) and (3-6), we get

$$M_{U_{r,s:n}}(t) - M_{U_{r,s-1:n}}(t) = \frac{n!}{(r-1)!(s-r-1)!(n-s+1)!} \int_{Q_1}^{P_1} e^{tx}[F_1(x)]^{r-1}f_1(x)$$

$$\times \{\int_{x}^{P_1} e^{tY} \partial/\partial y\, h(x,y)\,dy\}\,dx \qquad (3-7)$$

integration by parts yields

$$\int_x^{P_1} e^{ty}\,\partial/\partial y\, h(x,y)\,dy = [e^{ty} h(x,y)]_x^{P_1} - t\int_x^{P_1} e^{ty}\, h(x,y)\,dy$$

$$= t\int_x^{P_1} e^{ty}[\,F_1(y) - F_1(x)\,]^{s-r-1}\,[1-F_1(y)]^{n-s+1}\,dy, \qquad (3\text{-}8)$$

since h(x,y) vanishes at both the limits y=x and y $=P_1$. Substituting from (3-8) in (3-7), we get the required result.

Proposition (3.2) For any continuous distribution function , $1\le r< s\le n-1$ , the moment generating function $M_{U_{r,s:n}}(t)$ satisfies

$$M_{U_{r,s:n}}(t) - M_{U_{r,s-1:n}}(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{z=0}^{\infty}\binom{k}{z}\{\alpha_{r,s:n}^{(k-z,z)} - \alpha_{r,s-1:n}^{(k-z,z)}\} \qquad (3\text{-}7)$$

Proof The proof is clearly from the defination .

## 4.APPLICATIONS

### 1- Doubly truncated exponential distribution

The p.d.f. of a doubly truncated exponential distribution is

$$f_1(x)=\frac{e^{-x/\theta}}{\theta(P-Q)}, \quad Q_1 \le x_1 \le P_1 \quad ,\theta > 0 \qquad (4\text{-}1)$$

where $1-P=\exp(-P_1/\theta)$ and $1-Q=\exp(-Q_1/\theta)$

Let $P_2=\frac{1-P}{P-Q}$ and $Q_2=\frac{1-Q}{P-Q}$ , $1+P_2=Q_2$ . (4-2)

From (4-1) ,we have

$$1-F_1(x) = \int_x^{P_1} f_1(x)\,dx = -\frac{1-P}{P-Q} + \theta\, f_1(x) \qquad (4\text{-}3)$$

using (4-2) in (4-3) , we have

$$1-F_1(x) = -P_2 + \theta\, f_1(x) \qquad (4\text{-}4)$$

Frist we obtain a recurrence relation for $M_{X_{r:n}}(t)$

From theorem (2-1) ,using (4-4), we have

$$M_{X_{r:n}}(t) - M_{X_{r-1:n-1}}(t) = \binom{n-1}{r-1} t \int_{Q_1}^{P_1} e^{tx} [F_1(x)]^{r-1} [1-F_1(x)]^{n-r} \times$$

$$\times \ \{-P_2 + \theta f_1(x)\} dx \qquad (4-5)$$

$$= \frac{-(n-1)P_2}{(n-r)} [ M_{X_{r:n-1}}(t) - M_{X_{r-1:n-2}}(t) ] + \frac{\theta t}{n} M_{X_{r:n}}(t) , \qquad (4-5)$$

using (2-3) in (4-5),we have

$$[1- \frac{\theta t}{n} ] M_{X_{r:n}} = \frac{-(n-1)P_2}{(n-r)} [ M_{X_{r:n-1}}(t) - \frac{r-1}{n-r} M_{X_{r:n-1}}(t) - \frac{n-r}{n-1} M_{X_{r-1:n-1}}(t)]$$

$$+ M_{X_{r-1:n-1}}(t) \quad , n \neq \theta t \quad ,$$

and recalling the definition of $Q_2$ in (4-2) ,we have

$$[1- \frac{\theta t}{n} ] M_{X_{r:n}}(t) = -P_2 M_{X_{r:n-1}}(t) + Q_2 M_{X_{r-1:n-1}}(t) \quad , \quad n \neq \theta t \qquad (4-6)$$

Next ,we obtain a recurrence relation for $M_{U_{r,s:n}}(t)$ .

From (4-4), we have

$$1 - F_1(y) = - P_2 + \theta f_1(y),$$

and substituting with it in theorem (3- 1), we get

$$M_{U_{r,s:n}}(t) - M_{U_{r,s-1:n}}(t) = \frac{-nP_2}{n-s+1} [ M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t) ]$$

$$+ \frac{\theta t}{n-s+1} M_{U_{r,s:n}}$$

which yields

$$[1- \frac{\theta t}{n-s+1} ] M_{U_{r,s:n}}(t) = M_{U_{r,s-1:n}}(t) - \frac{nP_2}{n-s+1} [M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t) ]$$

$$, \theta t \neq n-s+1 \ , \ 1 \leq r \leq s \leq n-1 \qquad (4-7)$$

## 2 .Doubly truncated logistic distribution

The p.d.f and c.d.f of a logistic distribution are

$$f(x)=\frac{e^{-(x-\alpha)/\beta}}{\beta[1-e^{-(x-\alpha)/\beta}]^2},-\infty < x < \infty,\quad -\infty < \theta < \infty,\ \beta > 0 \quad \text{and}$$

$F(x)=[1+e^{-(x-\alpha)/\beta}]^2$ respectively .

Truncating f(x) at $Q_1$ and $P_1 (Q_1 < P_1)$ ,the new p.d.f.is

$$f_1(x)=\frac{e^{-(x-\alpha)/\beta}}{\beta\ (P-Q)\ [1+e^{-(x-\alpha)/\beta}]^2},\quad Q_1\leq x\leq P_1 \tag{4-8}$$

where $\frac{1}{P}-1=e^{-(P_1-\alpha)/\beta}$ and $\frac{1}{Q}-1=e^{-(Q_1-\alpha)/\beta}$.

Let $P_2=\frac{P}{P-Q}$ and $Q_2=\frac{Q}{P-Q}$ . (4-9)

From (4-8), we have

$$1-F_1(x)=\frac{1}{\beta(P-Q)}\int_x^{P_1}\frac{e^{-(x-\alpha)/\beta}}{[1+e^{-(x-\alpha)/B}]^2}\,dx$$

$$=\frac{[1+e^{-(P_1-\alpha)/B}]^{-1}}{P-Q}-\frac{[1+e^{-(x-\alpha)/\beta}]^{-1}}{P-Q}$$

$$=\frac{P}{P-Q}-\frac{1}{P-Q}\left\{\frac{[1+e^{-(x-\alpha)/\beta}]}{[1+e^{-(x-\alpha)/\beta}]^2}\right.$$

$$=\frac{P}{P-Q}-\frac{1}{P-Q}\left\{\frac{1}{[1+e^{-(x-\alpha)/\beta}]^2}+\frac{e^{-(x-\alpha)/\beta}}{[1+e^{-(x-\alpha)/\beta}]^2}\right\}, \tag{4-10}$$

using (4-8) and (4-9) in (4-10), we have

$$1-F_1(x)=P_2-\beta e^{(x-\alpha)/\beta}f_1(x)-\beta f_1(x)\quad . \tag{4-11}$$

Frist we obtain a recurrence relation for $M_{x_{r:n}}(t)$ .

Using (4-11), theorem (2-1) yields

$$M_{x_{r:n}}(t)-M_{x_{r-1:n-1}}(t)=\binom{n-1}{r-1}\,t\int_{Q_1}^{P_1}e^{tx}\,[F_1(x)]^{r-1}[1-F_1(x)]^{n-r}\times$$

$$\{P_2-\beta f_1(x)e^{(x-\alpha)/\beta}-\beta f_1(x)\}dx\ ,2\leq r\leq n-1$$

$$= \frac{(n-1)P_2}{(n-r)}[M_{X_{r:n-1}}(t)-M_{X_{r-1:n-2}}(t)] - \frac{\beta t}{n} e^{-\alpha/\beta} M_{X_{r:n}}(t+1/\beta) - \frac{\beta t}{n} M_{X_{r:n}}(t) \qquad (4\text{-}12)$$

Now using (2-3) in (4-12) , we have

$$M_{X_{r:n}}(t)= M_{X_{r-1:n-1}}(t) + \frac{(n-1)P_2}{(n-r)}[M_{X_{r:n-1}}(t) - \frac{r-1}{n-1}M_{X_{r:n-1}}(t) - \frac{n-r}{n-1}M_{X_{r-1:n-1}}(t)] - \frac{\beta t}{n} e^{-\alpha/\beta} M_{X_{r:n}}(t+1/\beta) - \frac{\beta t}{n} M_{X_{r:n}}(t)$$

Which gives

$$[1+\frac{\beta t}{n}] M_{X_{r:n}}(t) = -Q_2 M_{X_{r-1:n-1}}(t) + P_2 M_{X_{r:n-1}}(t) - \frac{\beta t}{n} e^{-\alpha/\beta} M_{X_{r:n}}(t+1/\beta), \quad 1 \le r \le n-1 \qquad (4\text{-}13)$$

Next, we obtain the recurrence relations for $M_{U_{r,s:n}}$.

From (4-11), we have ,

$$1- F_1(y) = P_2 - \beta e^{(y-\alpha)/\beta} f_1(y) - \beta f_1(y) \qquad (4\text{-}14)$$

using (4-14), theorem (3-1) yields

$$M_{U_{r,s:n}}(t) - M_{U_{r,s-1:n}} = \frac{n!\,t}{(r-1)!(s-r-1)!(n-s+1)!} \int_{Q_1}^{P_1}\int_{x}^{P_1} e^{t(x+y)} \times [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}[1-F_1(y)]^{n-s}[P_2 - \beta f_1(y)e^{(y-\alpha)/\beta} - \beta f_1(y)] f_1(x)\, dy\, dx$$

which gives

$$M_{U_{r,s:n}}(t) - M_{U_{r,s-1:n}}(t) = \frac{nP_2}{n-s+1}[M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t)] - \frac{n!\,\beta\, t\, e^{-\alpha/\beta}}{(r-1)!(s-r-1)!(n-s+1)!} \times \int_{Q_1}^{P_1}\int_{x}^{P_1} e^{tx+y(t+1/\beta)} [F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1} \times [1-F_1(y)]^{n-s} f_1(x)f_1(y)\, dx\, dy - \frac{\beta t}{n-s+1} M_{U_{r,s:n}}(t)$$

then,

$$[1+ \frac{\beta t}{n-s+1}]\, M_{U_{r,s:n}}(t) = M_{U_{r,s-1:n}}(t) + \frac{nP_2}{n-s+1}\,[M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t)] -$$

$$- \frac{\beta t}{n-s+1}\, e^{-\alpha/\beta}\, M_{X_{r:n},X_{s:n}}(t\,, t+1/\beta) \qquad (4\text{-}15)$$

## 3. Double truncated extreme-value distribution

The p.d.f.and c.d.f. of an extreme value distribution are

$$f(x)=\exp(-x - e^{-x})\;,\; -\infty \le x \le \infty$$

and $$F(x)=\exp(-e^{-x}) \text{ respectively}$$

Truncating f(x) at $Q_1$ and $P_1$ ($Q_1 < P_1$) the new p.d.f. is

$$f_1(x) = \frac{\exp(-x-e^{-x})}{P-Q} \quad , Q_1 \le x \le P_1 \qquad (4\text{-}16)$$

where , $P = \exp(-e^{-P_1})$ and $Q = \exp(-e^{-Q_1})$ (4-17)

Let $P_2 = \frac{P}{P-Q}$ and $Q_2 = \frac{Q}{P-Q}$ (4-18)

From (4-16),we have

$$1-F_1(x) = \frac{1}{P-Q}\int_x^{P_1} \exp(-x-e^{-x})\,dx \quad , \qquad (4\text{-}19)$$

using (4-18), ) in (4-19), we have

$$1-F_1(x) = P_2 - e^{x} f_1(x) \qquad (4\text{-}20)$$

using (4-20) in theorem (2-1) we obtain,

$$M_{X_{r:n}}(t) - M_{X_{r-1:n-1}}(t) = \frac{(n-1)P_2}{(n-r)}[M_{X_{r:n-1}}(t) - M_{X_{r-1:n-2}}(t)] - \frac{t}{n} M_{X_{r:n}}(t+1) \qquad (4\text{-}21)$$

from (2-3) , (4-18) and (4-21) , we get

$$M_{X_{r:n}}(t) + \frac{t}{n} M_{X_{r:n}}(t+1) = -Q_2 M_{X_{r-1:n-1}}(t) - P_2 M_{X_{r:n-1}}(t)\;,\; 1 < r \le n-1 \qquad (4\text{-}22)$$

Now we obtain a recurrence relation for $M_{U_{r,s:n}}(t)$

from (4-20) ,we have

$$1 - F_1(y) = P_2 - e^{y} f_1(y) \qquad (4\text{-}23)$$

using (4-23) in theorem (3-1), we obtain

$$M_{U_{r,s:n}}(t) - M_{U_{r,s-1:n}}(t) = \frac{nP_2}{n-s+1}[\, M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t)\,] - \frac{n!\ t}{(r-1)!(s-r-1)!(n-s+1)!}$$

$$\times \int_{Q_1}^{P_1}\int_{x}^{P_1} e^{tx+y(t+1)}\,[F_1(x)]^{r-1}[F_1(y)-F_1(x)]^{s-r-1}$$

$$[\,1-F_1(y)]^{n-s} f_1(x)f_1(y)dxdy\ ,1< r\le n-1$$

which yields

$$M_{U_{r,s:n}}(t) = M_{U_{r,s-1:n}}(t) + \frac{nP_2}{n-s+1}\,[M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t)\,] - \frac{t}{n-s+1} M_{X_{r:n},X_{s:n}}(t,t+1),$$

$$1 < r \le n-1 \qquad (4\text{-}24)$$

## 4. Doubly truncated Weibull distribuation

The p.d.f. and c.d.f. of a Weibull distribution are

$f(x)=(\,p\,/\,\theta)\,x^{p-1}\,\exp(-x^p/\,\theta),\quad x > 0,\quad \theta > 0$ and

$F(x)= 1-\exp(-x^p/\,\theta)$ respectively

Truncating $f(x)$ at $Q_1$ and $P_1$ $(Q_1<P_1)$, the new p.d.f $f_1(x)$ is

$$f_1(x)= (p\,/\,\theta(\alpha-\beta)\,)\,x^{p-1}\,\exp(-x^p/\,\theta),\ Q_1\le x\le P_1 \qquad (4\text{-}25)$$

where $1-\alpha=\exp(-P_1^{\,p}\,/\,\theta\,)$ and $1-\beta=\exp(-Q_1^{\,p}\,/\,\theta)$

let $P_2=\dfrac{1-\alpha}{\alpha-\beta}$ and $Q_2 = \dfrac{1-\beta}{\alpha-\beta}$ (4-26)

From (4-25), using (4-26 ), we have

$$1-F_1(x)= -P_2 + (\,\theta\,/\,p)\,x^{1-p}\,f_1(x) \qquad (4\text{-}27)$$

using (4-27) in theorem (2-1) , we get

$$M_{X_{r:n}}(t)- M_{X_{r-1:n-1}}(t) = \frac{-(n-1)P_2}{(n-r)}\,[M_{X_{r:n-1}}(t) - M_{X_{r-1:n-2}}(t)\,]+ \frac{\theta t}{p}\binom{n-1}{r-1}\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}$$

$$\int_{Q_1}^{P_1} x^{(k-p)}\,[F_1(x)]^{r-1}[1-F_1(x)]^{n-r} f_1(x)dx\ ,$$

$$2\le\ r \le n \qquad (4\text{-}28)$$

Now using (2-3) in (4-28), we have

$$M_{X_{r:n}}(t)= M_{X_{r-1:n-1}}(t)- \frac{(n-1)P_2}{n-r}\,[M_{X_{r:n-1}}(t)- \frac{r-1}{n-1} M_{X_{r:n-1}}(t)- \frac{n-r}{n-1} M_{X_{r-1:n-1}}(t)\,]$$

$$+ \frac{\theta t}{np} \sum_{k=1}^{\infty} \frac{t^{(k-1)}}{(k-1)!} \alpha_{r:n}^{(k-p)}$$

which yields by using (4-26)

$$M_{X_{r:n}}(t) = Q_2 M_{X_{r-1:n-1}}(t) - P_2 M_{X_{r:n-1}}(t) + \frac{\theta t}{np} \sum_{k=1}^{\infty} \frac{t^{(k-1)}}{(k-1)!} \alpha_{r:n}^{(k-p)}, \ 2 \le r \le n-1 \qquad (4\text{-}29)$$

since $\sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \alpha_{r:n}^{(k-1)} = M_{X_{r:n}}(t)$ ,(4-29) yields ,

$$\left[ 1 - \frac{\theta t}{n} \right] M_{X_{r:n}}(t) = -P_2 M_{X_{r:n-1}}(t) + Q_2 M_{X_{r-1:n-1}}(t) \quad , \theta t \ne n \quad ,$$

which is the recurrence relation for $M_{X_{r:n}}(t)$ of exponential distribution

Next we obtain a recurrence relation for $M_{U_{r,s:n}}$

From (4-27),we have

$$1 - F_1(y) = -P_2 + (\theta / p) y^{1-p} f_1(y) \qquad (4\text{-}30)$$

substituting from (4-30) in theorem (3-1), we get

$$M_{U_{r,s:n}}(t) - M_{U_{r,s-1:n}}(t) = \frac{-nP_2}{n-s+1} [M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t)] + \frac{\theta t \quad n!}{p(r-1)!(s-r-1)!(n-s+1)!}$$

$$\times \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{z=0}^{k} \binom{k}{z} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^{k-z} y^{z+1-P} [F_1(x)]^{r-1} \times$$

$$[F_1(y) - F_1(x)]^{s-r-1} [1 - F_1(y)]^{n-s} f_1(x) f_1(y) \, dy \, dx$$

$$= \frac{-nP_2}{n-s+1} [M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t)] + \frac{\theta t}{(n-s+1)p} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{z=0}^{k} \binom{k}{z} \alpha_{r,s:n}^{(k-z,z+1-p)}$$

which yields

$$M_{U_{r;s:n}}(t) - \frac{\theta t}{(n-s+1)p} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{z=0}^{k} \binom{k}{z} \alpha_{r,s:n}^{(k-z,z+1-p)} = M_{U_{r,s-1:n}}(t) + \frac{-nP_2}{n-s+1} \times$$

$$[ M_{U_{r,s:n-1}}(t) - M_{U_{r,s-1:n-1}}(t) ], \ 1 \le r < S \le n-1 \qquad (4\text{-}31)$$

Since $\sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{z=0}^{k} \binom{k}{z} \alpha_{r,s:n}^{(k-z,z)} = M_{U_{r,s:n}}(t)$, and at P= 1 in (4-31), we obtain

$$\left[1 - \frac{\theta t}{n-s+1}\right] M_{U_{r,s:n}}(t) = M_{u_{r,s-1:n}}(t) - \frac{nP_2}{n-s+1} [ M_{U_{r,s-1:n}}(t) - M_{U_{r,s-1:n-1}}(t) ],$$

$$\theta t \ne n-s+1, \ 1 \le r < s \le n-1$$

which is the recurrence relation for $M_{U_{r,s:n}}(t)$ of exponential distribution.

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