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APPROXIMATING THE CHARACTERISTIC FUNCTION OF THE STUDENT'S t DISTRIBUTION

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Abstract: The density function of Student's t random variable with n degrees of freedom, $f_n(t)$, can be approximated by a finite mixture density function from the same distribution. The criteria of approximation is to minimize the difference between the moments of $f_n(t)$ and that of its finite mixture approximation. The proposed approximation may be applied to obtain an explicit and simple form for the characteristic function of Student's t distribution of even number of n. It is evaluated for different values of n, and good results can be obtained even when n is as small as 5.

KEY WORDS: Distribution function; Characteristic function; Student's t variable; Finite mixture distribution.

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1-Introduction. The density function of Student's t-variable with n degrees of freedom, is given by

$$f_n(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{t^2}{n}\right)^{-(n+1)/2} , \qquad -\infty < t < \infty$$

where $\Gamma(.)$ is the gamma function.

The characteristic function of $f_n(t)$ has complicated form which can be simplified only if n is odd (see section 4). There has been an intense study of possible approximations to the t distribution. This has produced approximations of very high accuracy though sometimes rather complicated. Several approximations are based on finding either a direrct expansion or an asymptotic series approximation of the distribution function. Some other approximations built on considering formulas which may have a unit normal distribution. Johnson and Kotz (1970) gave an excellent review for some of these approximations (see also Kendall et.al.(1987)). However the characteristic function of this variable has little attention in spite of its important role in statistical theory.

The purpose of this article is to show that the density function of Student's t with n degrees of freedom, $f_n(t)$, may be approximated by a finite mixture from the same distribution, and to obtain an approximate simple form for characteristic function of t-variable with even degrees of freedom. In section 2, the finite mixture approximation of $f_n(t)$ is given, and calculated for different values of n in section 3. The proposed approximation is used in section 4 to obtain a simplified form for the characteristic function of Student's t distribution with even degrees of freedom.

Regarding notations; we used $< X , Y > = \sum_{i=1}^{p} x_i y_i$ to denote the inner product of X and $Y \in R^p$, while we use $|X|$ to denote the Euclidean norm of X where $|X|^2 = < X . X >$.

2. Mixtures Approximations of $f_n(t)$

Let $f_n(t)$ and $\mu_k(n)$ denote the density function and the kth moment of the t-distributions with n degrees of freedom respectivly. Then $\mu_k(n)$ exist only for k < n, and are then equal to zero by symmetry for odd-order moments, while for even moments, k = 2r, say,

$$\mu_{2r}(n) = n^r \frac{\Gamma(r + \frac{1}{2})\Gamma(\frac{n}{2} - r)}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}, \qquad 2r < n$$

2.1 Proposition

For arbitrary r, where $2r < n$, $\mu_{2r}(n)$ is a decreasing function of n.

Proof:

For arbitrary n, with $2r < n$, we have

$$\mu_{2r}(n) = n^r \frac{\Gamma(r+\frac{1}{2})\Gamma(\frac{n}{2}-r)}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}$$

$$= \frac{2^r\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \quad \frac{n^r}{(n-2)(n-4)\ldots(n-2r)}$$

$$= \frac{2^r\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})} \quad \frac{1}{(1-\frac{2}{n})(1-\frac{4}{n})\ldots(1-\frac{2r}{n})}$$

where $2r < n$.

Then

$$\frac{\mu_{2r}(n)}{\mu_{2r}(n+1)} = \frac{(1-\frac{2}{n+1})(1-\frac{4}{n+1})\ldots(1-\frac{2r}{n+1})}{(1-\frac{2}{n})(1-\frac{4}{n})\ldots(1-\frac{2r}{n})}$$

$$= \prod_{i=1}^{r} \frac{\left(1 - \frac{2i}{n+1}\right)}{\left(1 - \frac{2i}{n}\right)} > 1 \quad \forall r, \quad 2r < n$$

hence $\mu_{2r}(n)$ is a decreasing function of n. □

Furthermore, let us define M_n to be a vector whose ith component is the (2i)th moment of the t distribution with n degrees of freedom, i = 1,2, ... ,s (say) i.e.

$$M_n = (\mu_2(n), \mu_4(n), \ldots, \mu_{2s}(n))^t$$

2.2 Corollary

Let M_n be defined as above, then

(i) $\langle M_n - M_{n_1}, M_m - M_{m_1} \rangle > 0$ if

either $\{n < n_1$ and $m < m_1\}$

or $\{n > n_1$ and $m > m_1\}$

(ii) $\langle M_n - M_{n_1}, M_m - M_{m_1} \rangle < 0$ if

either $\{n < n_1$ and $m > m_1\}$

or $\{n > n_1$ and $m < m_1\}$

The proof is easily obtained by using the above proposition. □

Let $f_{m_1}(t)$ and $f_{m_2}(t)$ denote the density function of student's t random variables with m_1 and m_2 degrees of freedom respectivly. Define $\hat{f}_n(t) = \alpha f_{m_1}(t) + (1-\alpha) f_{m_2}(t)$, $0 < \alpha < 1$ We aim to determine the values of α, m_1 and m_2 that minimize the difference between the 2rth moments of $f_n(t)$ and $\hat{f}_n(t)$, for $r = 1, 2, \ldots, s$, where s is a positive integer arbitrary chosen. This will be given by the following theorem.

2.3 Theorem 1

Let $\mu_{2r}(n)$ be as defined above with $n > 3$, and suppose that

$\mu_{2r}(n) = \alpha\mu_{2r}(m_1) + (1-\alpha)\mu_{2r}(m_2) + \varepsilon_r$, $r = 1, 2, \ldots, s$

for some positive integers m_1 and m_2, then the least squares estimate of α that minimize the sum of squares of the errors ε_r, $r = 1, 2, \ldots, s$ is given by

$$\hat{\alpha} = \frac{\langle M_{m_1} - M_{m_2},\ M_n - M_{m_2} \rangle}{\| M_{m_1} - M_{m_2} \|}$$

$$= \frac{\sum_{r=1}^{s} [\mu_{2r}(m_1) - \mu_{2r}(m_2)]\,[\mu_{2r}(n) - \mu_{2r}(m_2)]}{\sum_{r=1}^{s} [\mu_{2r}(m_1) - \mu_{2r}(m_2)]^2}$$

where $0 < \hat{\alpha} < 1$, and $m_1 = n - 1$, $m_2 = n + 1$.

Proof:

Without loss of generality assume that $m_1 < m_2$.

We define

$$Q(\alpha, m_1, m_2) = \sum_{r=1}^{s} \varepsilon_r^2 = \| (M_n - M_{m_2}) - \alpha(M_{m_1} - M_{m_2}) \|$$

to be the function to be minimized. Differentiating the function Q with respect to α and setting $\partial Q / \partial \alpha = 0$, we get Q_{min} for given m_1 and m_2 at $\hat{\alpha}$, where

$$\hat{\alpha} = \frac{< M_{m_1} - M_{m_2} , M_n - M_{m_2} >}{| M_{m_1} - M_{m_2} |}$$

To find the conditions under which $\hat{\alpha} \in (0, 1)$, we note that:

(i) $\hat{\alpha} > 0$ if $< M_n - M_{m_2} , M_{m_1} - M_{m_2} > \; > 0$, by part (i) of the above Corollary, this occurs when $n < m_2$ (since $m_1 < m_2$).

(ii) $< M_n - M_{m_2} , M_{m_1} - M_{m_2} >$

$$= < M_n - M_{m_1} + M_{m_1} - M_{m_2} , M_{m_1} - M_{m_2} >$$

$$= < M_n - M_{m_1} , M_{m_1} - M_{m_2} > + | M_{m_1} - M_{m_2} |$$

hence, $\hat{\alpha} < 1$ if $< M_n - M_{m_1} , M_{m_1} - M_{m_2} > \; > 0$. By part (ii) of the above Corollary, this occurs when $m_1 < n$ (since $m_1 < m_2$).

Combining (i) and (ii), the condition $0 < \hat{\alpha} < 1$ is satisfied whenever $m_1 < n < m_2$.

It remains to show that $m_1 = n - 1$ and $m_2 = n + 1$. We proceed as follows:

$$Q(\hat{\alpha}, m_1, m_2) = |(M_n - M_{m_2}) - \hat{\alpha} (M_{m_1} - M_{m_1}) |^2$$

$$= \| (M_n - M_{m_1}) - \frac{<(M_n - M_{m_2}), (M_{m_1} - M_{m_2})>}{\|(M_{m_1} - M_{m_2})\|^2} (M_{m_1} - M_{m_2}) \|^2$$

$$= \| M_n - M_{m_2} \|^2 \left(1 - < \frac{M_n - M_{m_2}}{\| M_n - M_{m_2} \|^2}, \frac{M_{m_1} - M_{m_2}}{\| M_{m_1} \; M_{m_2} \|^2} >^2 \right)$$

But we have

$$\| M_n - M_{m_2} \|^2 = \| M_n - M_{m_{n+1}} + M_{m_{n+1}} + M_{m_2} \|^2 = < M_n - M_{m_{n+1}}, M_{m_{n+1}} + M_{m_2} >^2$$

$$= \| M_n - M_{m_{n+1}} \|^2 + \| M_{m_{n+1}} - M_{m_2} \|^2 + 2 < M_n - M_{m_{n+1}}, M_{m_{n+1}} + M_{m_2} >$$

which is minimum at $m_2 = n + 1$ (since both $\| M_{n+1} - M_{m_2} \|^2$ and $< M_m - M_{n+1}, M_{n+1} - M_{m_2} >$ are positive quantity). Then $Q(\hat{\alpha}, m_1, m_2)$ will be minimum for positive integer $m_2 > n$, *if* $m_2 = n + 1$.

Now $Q(\hat{\alpha}, m_1, m_2)$ at $m_2 = n + 1$ is given by

$$Q(\hat{\alpha}, m_1, n + 1) = \| M_n - M_{n+1} \|^2 \left(1 - < \frac{M_n - M_{n+1}}{\| M_n - M_{n+1} \|^2}, \frac{M_{m_1} - M_{n+1}}{\| M_{m_1} \; M_{n+1} \|^2} >^2 \right)$$

Then $Q(\hat{\alpha}, m_1, n + 1)$ is minimum for all values of the positive integer $m_1 < n$ if the quantity;

$$< \frac{M_n - M_{n+1}}{\| M_n - M_{n+1} \|^2}, \frac{M_{m_1} - M_{n+1}}{\| M_{m_1} - M_{n+1} \|^2} >^2$$

is maximum at this value. By Schwartz Inequality, this occurs if $m_1 = n$, which contradicts the condition that $m_1 < n$. Hence the

maximum of the above value is attainable whenever m_1 is as close as possible to n that is when $m_1 = n - 1$.
This complete the proof of the theorem.

Since $\mu_k(n)$ exists only for $k < n$, then "s" in the above theorem takes any integer value from one to the smallest integer less than or equal to (n - 1)/2. For instance, if n = 7 and s =2, using the given values of $\hat{\alpha}$, m_1 and m_2 to approximate $f_7(t)$, then the quantity $\sum_{i=1}^{2} (\mu_{2i}(7) - \hat{\mu}_{2i}(7))$ will be minimum .

When the number of degrees of freedom, n , is less than or equal to 3, the above theorem can't be applied. For instance if n = 2, then $m_1 = 1$ and $f_1(t)$ reduces to the Cauchy distribution. While for n = 3, $m_1 = 2$ and $f_2(t)$ has one moment which is zero. However, we may use Fisher's approximation of $f_n(t)$ to obtain a similar result as illustrated in the following.

Fisher (1925) gave a direct expansion of the probability density of t-variable. Much later, Fisher and Cornish(1960) inverted it to approximate $P(t_n \leq t)$ as a series in n^{-1} .
First define

$$R_1(t) = t^4 - 2t^2 - 1$$

$$R_2(t) = 3t^8 - 28t^6 + 30t^4 + 12t^2 + 3$$

$$R_3(t) = t^{12} - 22\, t^{10} + 113\, t^{8} - 92\, t^{6} - 33\, t^{4} - 6\, t^{2} + 15$$

and

$$R_4(t) = 15\, t^{16} - 600\, t^{14} + 7100\, t^{12} - 26616\, t^{10}$$
$$+ 18330\, t^{8} 3 + 6360\, t^{6} + 1980\, t^{4} - 1800\, t^{2} - 945$$

For the probability density function , Fisher gave the formula

$$f_n(t) = \phi(t)\, [\, 1 + \frac{1}{4n} R_1(t) + \frac{1}{96\, n^2} R_2(t)$$
$$+ \frac{1}{384\, n^3} R_3(t) + \frac{1}{92160\, n^4} R_4(t) + \ldots\,]$$

where

$$\phi(t) = \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}$$

Neglecting all terms which have n^{-4} or higer orders in its denominator give Fisher's approximation of $f_n(t)$, which can be rewritin as:

$$f_n(t) = \phi(t)\, [\, 1 + \frac{1}{4n} R_1(t) + \frac{1}{96\, n^2} R_2(t) + \frac{1}{384\, n^3} R_3(t)]$$

2.4 Theorem 2

For $n \geq 2$, $f_n(t)$ and $\hat{f}_n(t) = \frac{(n-1)}{2n} f_{n-1}(t) + \frac{(n+1)}{2n} f_{n+1}(t)$

are equal in the sense of Fisher's approximation of the probability density of t-variable.

Proof:

Using Fisher's approximation of the probability density of the t- variable, we have

$$\frac{n-1}{2n} f_{n-1}(t) + \frac{n+1}{2n} f_{n+1}(t)$$

$$\doteq \phi(t) \left[1 + \frac{1}{4} R_1(t) + \frac{1}{96n} \left(\frac{1}{2(n-1)} + \frac{1}{2(n+1)} \right) R_2(t) \right.$$

$$\left. + \frac{1}{384n} \left(\frac{1}{2(n-1)^2} + \frac{1}{2(n+1)^2} \right) R_3 \right]$$

$$= \phi(t) \left[1 + \frac{1}{4n} R_1(t) + \frac{1}{96} \left(\frac{1}{n^2} \quad \frac{1}{n^2(n^2-1)} \right) R_2(t) \right.$$

$$\left. + \frac{1}{384} \left(\frac{1}{n^3} + \frac{1}{n^3(n-1)^2(n+1)^2} \right) R_3(t) \right]$$

$$= \phi(t) \left[1 + \frac{1}{4n} R_1(t) + \frac{1}{96n^2} R_2(t) + \frac{1}{384n^3} R_3(t) \right.$$

$$\left. + \frac{1}{96n^2(n^2-1)} R_2(t) + \frac{1}{384n^3(n-1)^2(n+1)^2} R_3(t) \right]$$

As in Fisher's approximation of $f_n(t)$, if we neglect all terms which have n^{-4} or higher orders in its denominator we obtain

$$\hat{f}_n(t) = \frac{n-1}{2n} f_{n-1}(t) + \frac{n+1}{2n} f_{n+1}(t)$$
$$\doteq \varphi(t)\ [\,1 + \frac{1}{4n} R_1(t) + \frac{1}{96n^2} R_2(t) + \frac{1}{384n^3} R_3(t)]$$

which is exactly the same as Fisher's approximation of $f_n(t)$. This complete the proof. □

3.Results

We evaluate $f_n(t)$ and $\hat{f}_n(t)$ (the finite mixture approximation of $f_n(t)$) for different values of n, s and for t = 0, 0.1, 0.2, ..., 9.0.

Let $\mu_{2r}(n)$ be the 2rth moment of $\hat{f}_n(t)$. We define the maximum absolute error between the vector u and its approximation $\hat{u}$ by MAE($\hat{u}$) = $\max_i |u_i - \hat{u}_i| = e_r$ (say). Also the relative maximum absolute error is definde by RMAE($\hat{u}$) = e_r/u_r.

Table 1, shows the comparison between $f_n(t)$ and $\hat{f}_n(t)$ expressed in terms of MAE($\hat{f}$) and RMAE($\hat{f}$) for each selected pairs of n and s. The estimated value of α is given in column 3, while MAE($\hat{f}$) and RMAE($\hat{f}$) are shown in columns 4 and 5 respectively. Column 6 shows the value of t (say t^*) at which MAE($\hat{f}$) occurs.

Table 2, shows the comparison between $\mu_{2r}(n)$ and $\hat{\mu}_{2r}(n)$ expressed in terms of MAE($\hat{M}$) and RMAE($\hat{M}$) for each selected pair of n and s. These values are shown in columns 3 and 4 respectively, while column 5 refers to the order of the moments corresponding to MAE($\hat{M}$), (say s^*).

It is clear from Tables 1 and 2 that the maximum absolute error is closed to zero for n = 5. The approximation gets better as n increases. On the other hand, for a given n the approximation gets better as s decreases.
The MAE($\hat{f}$) occurs at t = 0.7 for all n > 4. For n = 2, it occurs at t = 0.8 while for n = 3, it occurs at t = 2.5.

Table 1 **

Comparison between $f_n(t)$ and $\hat{f}_n(t)$ in terms of

MAE($\hat{f}$) and RMAE($\hat{f}$)

n	s	α	MAE($\hat{f}$)	RMAE($\hat{f}$)	t*
2	-	.25	3×10^{-4}	1×10^{-4}	0.8
3	-	.666667	8×10^{-4}	2×10^{-4}	2.5
5	1	.333333	7×10^{-4}	3×10^{-3}	0.7
10	4	.224482	6×10^{-4}	2×10^{-3}	0.7
	2	.413376	9×10^{-5}	3×10^{-4}	0.7
15	6	.28978	2×10^{-4}	7×10^{-4}	0.7
	4	.41446	6×10^{-5}	2×10^{-4}	0.7
	2	.45307	2×10^{-5}	6×10^{-5}	0.7
20	8	.320037	1×10^{-4}	3×10^{-4}	0.7
	6	.413424	4×10^{-5}	1×10^{-4}	0.7
	4	.450813	2×10^{-5}	5×10^{-5}	0.7
25	8	.412149	3×10^{-5}	9×10^{-5}	0.7
	6	.44728	1×10^{-5}	5×10^{-5}	0.7
	4	.465955	6×10^{-6}	2×10^{-5}	0.7

** *For $n = 2$ and 3, we follow Theorem 2, while for $n > 3$, we follow Theorem 1.*

Table 2

Copmarison between $\mu_{2r}(n)$ and $\hat{\mu}_{2r}(n)$ in terms of

MAE($\hat{M}$) and RMAE($\hat{M}$)

n	s	MAE($\hat{M}$)	RMAE($\hat{M}$)	s*
5	1	0.0	0.0	2
10	4	5.59378	7×10^{-2}	6
	3	0.06088	9×10^{-3}	4
	2	2×10^{-3}	1×10^{-3}	2
15	6	811.40941 *	5×10^{-2}	10
	5	7.83219	1×10^{-2}	8
	4	0.15163	4×10^{-3}	6
	3	5×10^{-3}	1×10^{-3}	4
	2	2×10^{-4}	2×10^{-4}	2
20	6	34.79075	6×10^{-3}	10
	4	0.024187	8×10^{-4}	6
	2	5×10^{-5}	5×10^{-5}	2
25	6	5.91438	1×10^{-3}	10
	5	.177195	7×10^{-4}	8
	4	7×10^{-2}	3×10^{-4}	6
	3	3×10^{-4}	8×10^{-5}	4
	2	2×10^{-5}	2×10^{-5}	2

**For this particular case the approximation includes the first twenty moments.*

4.Approximation Form of The Characteristic Function

The characteristic function of Student's t-variable with n degrees of freedom, denoted by $\Phi_t(\theta; n)$, is given by (Fogiel, M.1991);

$$\phi_t(\theta; n) = E(e^{i\theta t}) = \frac{\left(\frac{|\theta|}{2\sqrt{n}}\right)^{n/2}}{\pi\,\Gamma(\frac{n}{2})} \; Y_{\frac{n}{2}}\left(\frac{|\theta|}{\sqrt{n}}\right)$$

where $Y_s(x)$ is Bessel function of the second kind of order s, for s = 0,1,2..., $Y_s(x)$ takes the following form

$$Y_s(x) = \frac{2}{\pi}\left(\ln\left(\frac{x}{2}\right) + \gamma\right) J_s(x) - \frac{1}{\pi}\sum_{k=0}^{s-1} \frac{(s-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-s}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty} (-1)^k \left(\psi(k) + \psi(s+k)\right) \frac{\left(\frac{x}{2}\right)^{2k+s}}{k!\,(s+k)!}$$

γ = 0.5772165... is Euler,s constant ,

$$\psi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p}, \quad \psi(0) = 0$$

and

$$J_s(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+s}}{(s+k)!}$$

is Bessel function of the first kind of order s .

To simplify slightly, put $x = t/\sqrt{n}$, then the denisty function of x is

$$h_n(x) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})}(1+x^2)^{-(\frac{n+1}{2})} \qquad , x > 0.$$

When n is odd , say n = 2 m + 1, it is known that (see,e.g., Chapman,1950) $\Phi_x(\theta;2m+1)$ is proportional to the modified Bessel function of the third kind and reduces to:

$$\phi_x(\theta;2m+1) = e^{-|\theta|}\frac{m!}{(2m)!}\sum_{k=0}^{m}\frac{(2m-k)!}{k!\,(m-k)!}(2|\theta|)^k .$$

When n is even say n = 2 m, m =1,2,..., no such simple form for $\phi_x(\theta;2m)$ exists (see e.g.,Chapman 1950).

Now, if F_1 , F_2 , ... be probability distributions with characteristic functions ϕ_1 , ϕ_2 , ... with $c_i \geq 0$ and $\Sigma c_i = 1$, the mixture $F = \Sigma c_i F_i$ is a probability distribution with characteristic function $\phi = \Sigma c_i \phi_i$ (see e.g. Feller,W. Vol.II pp. 477).

Hence a simple approximation of the charactiristic function of Student's t distribution with even number of degrees of freedom ,say n = 2 m , m = 1,2,..., can be put in either one of the following forms:

(I) If $m \geq 2$, then on following Theorem 1 of section 2, we set

$$\phi_x(\theta, 2m) \doteq \hat{\alpha}\, \phi_x(\theta, 2m-1) + (1-\hat{\alpha})\, \phi_x(\theta, 2m+1)$$

$$= e^{-|\theta|} \left[\hat{\alpha} \frac{(m-1)!}{(2(m-1))!} \sum_{k=0}^{m-1} \frac{(2(m-1)-k)!}{k!\,(m-1-k)!} (2|\theta|)^k \right.$$

$$\left. + (1-\hat{\alpha}) \frac{m!}{(2m)!} \sum_{k=0}^{m} \frac{(2m-k)!}{k!\,(m-k)!} (2|\theta|)^k \right].$$

where $\hat{\alpha}$ is as given in Theorem 1 .

(II) If $m = 1$, then on following Theorem 2 in section 2, we set

$$\phi_x(\theta, 2) \doteq \frac{1}{4} \phi_x(\theta, 1) + \frac{3}{4} \phi_x(\theta, 3)$$

$$= e^{-|\theta|} \left[1 + \frac{3}{4} |\theta| \right]$$

Conclusion.The proposed approximation developed here is useful in approximating the density function of the t-variable with n degrees of freedom.This approximation is given as a finite mixture of two t-variables having n-1 and n+1 degrees of freedom. It can be used to obtain a simplified and explicit form for the characteristic function of the t-variable with even degrees of freedom n.

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