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A NOTE ON THE COMPUTATION OF THE CUMULATIVE DISTRIBUTION FUNCTION OF THE NONCENTRAL F DISTRIBUTION

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SUMMARY

In this paper a computationally simple method is proposed for approximating the cumulative distribution function (c.d.f.) of the noncentral F distribution. A closed form expression for evaluating an incomplete beta integral is derived. Recursive formulas are obtained for computing products of the Poisson probabilities and incomplete beta functions involved in an infinite series representing the c.d.f. of the noncentral F distribution. The proposed technique can easily be implemented on personal computers.

KEY WORDS : F integral, Infinite series, Error bound, Complete and incomplete beta integrals, Complete gamma function, Poisson distribution, Binomial probabilities.

1. INTRODUCTION

The noncentral F distribution is required for evaluating the power of of a test of the general linear hypothesis. The derivation and properties of its density function are given in Johnson and Kotz (1970). However, the distribution function does not have a closed form expression. Among various well known approximations of the noncentral F distribution are those due to Severo and Zelen (1960) and Laubscher (1966), which require only the use of the normal distribution. A more accurate approximation due to Tiku (1966) utilizes the F distribution. An Edgeworth–series approximation is also available (Mudholkar and Chaubey, 1976).

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It is well known (Abramowitz and Stegun, 1965) that the c.d.f. of a noncentral F distribution can be written in the form of an infinite series. This fact is used here. Each term of this infinite series is a product of a Poisson probability and a noncentral incomplete beta integral. There is a relation between a central F distribution and an incomplete beta distribution. Most of the available algorithms evaluate the incomplete beta integrals in order to evaluate the F integrals. However, evaluating an incomplete beta integral is a fairly difficult task. A summary of the algorithms which evaluate an incomplete beta integral is given in Posten (1986).

In this paper, a method is proposed for computing the c.d.f. of the noncentral F distribution by evaluating an incomplete beta integral, in a closed form, and using a recurrence formula to evaluate the Poisson and the incomplete beta probabilities. The proposed method is an alternative to that given in Guenther (1978) and appears to be simple, efficient and accurate.

2. THE NONCENTRAL F DISTRIBUTION

Let $F(m, n; \lambda)$ denote a random variable which has a noncentral F distribution with degrees of freedom m and n and noncentrality parameter $\lambda > 0$, and let $I_w(a, b; \lambda)$ be the cumulative distribution function of a noncentral beta (usually referred to as a noncentral incomplete beta integral); namely

$$I_w(a, b; \lambda) = \sum_{k=0}^{\infty} p(k; \lambda/2).I_w(a+k, b; 0), \tag{1}$$

where

$$p(k; \lambda/2) = e^{-\lambda/2} (\lambda/2)^k / k!, \tag{2}$$

and

$$I_w(a,b; 0) = [B(a,b)]^{-1} \int_0^w x^{a-1} (1-x)^{b-1} dx \tag{3}$$

is the incomplete beta function, where $B(a,b) = \Gamma(a)\,\Gamma(b) / \Gamma(a+b)$ is the complete beta integral. It is easily checked that $(a + b - 1)\,B(a,b) = (b-1)\,B(a,b-1)$.

The cumulative distribution functions of the noncentral F distribution, $P[F(m, n; \lambda)]$, and noncentral beta distribution, $I_w(a, b; \lambda)$, are related by

$$P[F(m, n; \lambda) \leq f] = I_w(a, b; \lambda) \tag{4}$$

where $a = m/2$, $b = n/2$, and $w = mf/(n+mf)$. The function $I_w(a, b; \lambda)$ can be expressed as an infinite series involving central beta integrals $I_w(a, b; 0)$, denoted by $I_w(a,b)$. Thus

$$P[F(m, n; \lambda) \leq f] = \sum_{k=0}^{\infty} \left[p(k; \lambda/2)\, I_w(a+k, b) \right] \tag{5}$$

As already indicated, the infinite series (5) is a sum of products of Poisson probabilities with corresponding incomplete beta integrals. Therefore, to evaluate (5), we require terms, typical one of which is a product of $P(k; \lambda/2)$ by $I_w(a+k, b)$. This infinite series may be truncated at N terms and the error E_N will be less than one minus the sum of the Poisson terms evaluated to the N^{th} term. An upper bound for E_N, due to Lenth (1987), is given by

$$E_N < I_w(a + N + 1, b)\,[1 - \sum_{k=0}^{N} P(k; \lambda/2)] \tag{6}$$

This upper bound is sharper than that given in Norton (1983). By making use of (6), one can choose N such that (5) is computed to any desired accuracy. In the next section, a closed form expression for $I_w(a, b)$ is derived. The recursive formulas for evaluating $P(k; \lambda/2)$ and $I_w(a + k, b)$ are also given. These formulas minimize the computations to a considerable extent.

3. THE PROPOSED METHOD

We consider two cases.

Case 1: at least one among m and n is an even integer.

(i) For the situation when m and n are both even integers, $I_w(a, b)$ in (3) can be evaluated as the sum of binomial probabilities; namely

$$I_w(a, b) = \sum_{k=a}^{a+b-1} \binom{a+b-1}{k} w^k (1-w)^{a+b-k-1} \tag{7}$$

(ii) For the situation when either m or n (but not both) is an even integer, we make use of a symmetry arising from the formula $I_w(a,b) = 1 - I_{1-w}(b,a)$, and evaluate the incomplete beta integral $I_w(a,b)$ as follows:

If m is an even integer, then

$$I_w(a,b) = 1 - (1-w)^b \sum_{j=0}^{a-1} w^j \left[\prod_{i=1}^{j} \frac{b+i-1}{i} \right], \tag{8}$$

while if n is an even integer, then

$$I_w(a,b) = w^a \sum_{j=0}^{b-1} (1-w)^j \left[\prod_{i=1}^{j} \frac{a+i-1}{i} \right]. \tag{9}$$

To evaluate $P(k; \lambda/2)$ and $I_w(a+k, b)$ appearing in (5), the following recursive formulas are used.

$$P(0; \lambda/2) = e^{-\lambda/2}, \tag{10}$$

$$P(k; \lambda/2) = (\lambda/2k)\, P(k-1; \lambda/2), \quad k \geq 1 \tag{11}$$

and

$$A_k = A_{k-1} - C_{k-1}\, w^{a+k-1} (1-w)^b, \;\; k \geq 1 \tag{12}$$

where

$$C_{k-1} = [(a+k-1)\, B(a+k-1,\, b)]^{-1} = \prod_{i=1}^{a+k-1} [\frac{b+i-1}{i}],\; k \geq 1, \tag{13}$$

if a is an integer, while

$$C_{k-1} = [(b-1)\, B(a+k,\, b-1)]^{-1} = \prod_{i=1}^{b-1} [\frac{a+i-1}{i}],\; k \geq 1 \tag{14}$$

if b is an integer.

Equation (12) is evaluated by setting $A_0 = I_w(a, b)$ and $A_k = I_w(a+k, b)$ for $k \geq 1$. Thus, using equations (6) through (14), the c.d.f. of the noncentral F distribution in (5) can be easily evaluated for each case discussed before. It should be pointed out that Sibuya (1967) suggested the following closed form expression for computing $P[F(m, n, \lambda) \leq f]$ in the case when n is an integer:

$$P[\, F(m, n; \lambda) \leq f\,] = \sum_{i=0}^{b-1} \left[\, P[i; \lambda \frac{1-w}{2}].I_w(a+i, b-i) \,\right] \tag{15}$$

For more details on (15), see Johnson and Kotz (1970, p. 192).

<u>Case 2:</u> m and n both odd integers

A recurrence formula for evaluating $I_w(0.5, b)$ was suggested by Lee and Singh (1988). From Selby (1975, p. 406), a generalized recurrence relation to evaluate $I_w(a, b)$ is given by

$$I_w(a, b) = \frac{w^a\, (1-w)^{b-1}}{(b-1)\, B(a,\, b-1)} + I_w(a, b-1) \tag{16}$$

The use of the symmetry formula of $I_w(a, b)$ is the key to derive the recurrence formula:

$$I_w(a, b) = [B(a-1, b-1)]^{-1} \left[\frac{w^a \ (1-w)^{b-1}}{b-1} - \frac{w^{a-1} \ (1-w)^b}{a-1}\right] + I_w(a-1, b-1) \tag{17}$$

For the situation where m and n are both odd integers, a closed form expression for $I_w(a,b)$ is easily obtained and it is

$$I_w(a, b) = \frac{1}{2} + \frac{2}{\pi}[w(1-w)]^{1/2} D_1(D_2 - D_3) - \frac{1}{\pi} \text{Sin}^{-1} (1-2w), \tag{18}$$

where

$$D_1 = \prod_{j=1}^{a-.5} \left[\frac{j}{j-.5} w\right],$$

$$D_2 = \sum_{k=1}^{b-.5} (1-w)^{k-1} \left[\prod_{j=1}^{k-1} \frac{a+j-.5}{j+.5}\right],$$

and

$$D_3 = \sum_{k=1}^{a-.5} w^{k-1} \left[\prod_{j=1}^{k-1} \frac{j}{j+.5}\right].$$

In this case, the c.d.f. of the noncentral F distribution can be easily computed by redefining C_{k-1} in the recursive formula (12) as

$$C_{k-1} = [(a+k-1) \ B(a+k-1, b)]^{-1}, \ k \geq 1 \tag{19}$$

The algorithm of Pike and Hill (1966), which computes the natural logarithm of the complete gamma function, $\Gamma(a)$, for a strictly positive values of a, may also be used to compute B(a, b). Now, using expression (5), (6), (10), (11), (12), (18), and (19), the c.d.f. of the noncentral F distribution can be computed for the situation where m and n are odd integers.

4. CONCLUSION

All of the computations required for our method can be performed on a computer which does not need to have the F or incomplete beta integrals. An example given in Guenther (1978) is employed to check the performance of our method. In this example, $(m, n, \lambda, f) = (2, 6, 3, 5.1433)$. After 13 terms, the result coincides with 0.789135932 and $E_{14} < 1.922\ (10)^{-11}$. In addition, to illustrate the accuracy and efficiency of our method for computing $P[F(m, n, \lambda) \leq f]$, with a prescribed upper error bound $E_N < 10^{-10}$, let $(m, n, \lambda, f) = (2, 5, 2, 2)$. In this case, the exact value is 0.543470 while the approximated value turned out to be 0.543474. Furthermore, for $(m, n, \lambda, f) = (3, 15, 4, 5)$ the exact value is 0.786180 while the approximated value is 0.786167. This demonstrates that the method proposed here is simple, efficient and accurate.

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