THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CAIRO UNIV., VOL.38 No.1, 1994

MAXIMUM FLOWS IN FLOW NETWORKS SUBJECT TO RANDOM ARC FAILURES

IBRAHIM MOUSA ABDEL-FATTAH*

The reliability of directed capacitated networks subject to random arc failures is evaluated by the expected value of maximum flow. It is known that calculating the expected value of maximum flow is very difficult and complicated. An upper bound on the expected value of maximum flow in directed networks is given by Onaga, while a lower bound is found by Carey and Hendrickson. The lower bound sometimes gives the exact value, e.g., if networks are bipartite. The purpose of this paper is to give necessary and sufficient conditions for a directed network to have the lower bound that is equal to the exact value. Finally, we develop a simple and efficient test that decides whether a given network satisfies the necessary and sufficient conditions and then an illustrative example is introduced.

1. INTRODUCTION

The reliability of capacitated directed networks subject to random arc failures, such as communication, transportation, power, and water networks, is often evaluated by such measures as probabilistic connectedness and expected value of maximum flow [8]. The problem of calculating probabilistic connectedness is very hard and complicated process [4], though efficient algorithms are known for series-parallel graphs. Since the expected maximum flow problem contains as its special case the probabilistic connectedness problem,

* Assistant Professor, Faculty of Commerce, Zagazig University.

it is also quite difficult to be computed. Efficient algorithms are known if at most two arcs are subject to failure in a directed network [7]. An upper bound on the expected value of maximum flow in general directed networks is given by Onaga [6], while a lower bound in general directed networks is found by Carey and Hendrickson [2]. In computational experiments for some directed networks, it was reported by Wallace [11] that the lower bound often provides more accurate approximate values than does the upper bound. However, the condition as to when the derived lower bound on the expected value of maximum flow gives the exact value is not known except for the case of bipartite directed networks [10].

In this paper, the main objective is to give necessary and sufficient conditions for a general directed network to have the lower bound that is equal to the exact value of the expected maximum flow. Additionally, a simple and efficient test is developed to check whether a given network satisfies the necessary and sufficient conditions or not.

2. DEFINITIONS AND NOTATION

Consider the single commodity flow problem in the following directed network $N=(G,s,t,c)$:

$G=(V,A)$: A connected directed graph, where V is a set of nodes and A is a set of arcs.

$s \in V$: A source node.

$t \in V$: A sink node ($s \neq t$).

$c: A \to R^+$ is a capacity function, where R^+ is the set of positive real numbers.

For a set $U \subseteq A$, the graph $G' = (V, A-U)$ obtained by removing U from G is denoted by G-U, and similarly the network (G-U, s, t, c) by N-U. Let

i(a): the initial node of arc a,
t(a): the terminal node of arc a.

For a set $X \subseteq V$, let

$$OUT(X) = \{ a \in A \mid i(a) \in X,\ t(a) \in V-X \},$$

$$IN(X) = \{ a \in A \mid i(a) \in V-X,\ t(a) \in X \}.$$

For two nodes x and y, the following sequence π alternating between nodes x_i and arcs $a_i \in A$ with $i(a_i) = x_i$ and $t(a_i) = x_{i+1}$ is called an x-y path:

$$\pi:\ x_1(=x),\ a_1, x_2, a_2, \ldots, a_k,\ x_{k+1}(=y) \qquad (k \geq 1).$$

Let us denote the set of all x-y paths in network N by P(x,y). The set of arcs in a path π is denoted by $A(\pi)$. An x-y path with x=y is called a cycle. A network without cycle is called acyclic. We define the capacity of a path π by

$$c(\pi) = \min \{ c(a) \mid a \in A(\pi) \}. \qquad (1)$$

For a subset $B \subseteq A$ and a constant α satisfying $0 \leq \alpha \leq \min\{c(a) \mid a \in A\}$, let

$$N - \alpha \mid B$$

denote the network (G', s, t, c') with G'=(V, A') obtained by decreasing the capacity of each arc in B by α, i.e.,

$$c'(a) = \begin{cases} c(a) - \alpha & \text{if } a \in B \\ c(a) & \text{if } a \in A - B, \end{cases}$$

and

$$A' = \{a \in A \mid c'(a) > 0\}.$$

Let f(a) denote the amount of flow in arc a, which is feasible if it satisfies the following capacity constraint:

$$f(a) \leq c(a) \qquad \text{for} \quad a \in A, \qquad (2)$$

where

$$f(a) = \sum_{\pi \in P(s,t)} \delta(a,\pi)\, f(\pi),$$

and

$$\delta(a,\pi) = \begin{cases} 1 & \text{if} \quad a \in A(\pi) \\ 0 & \text{if} \quad a \notin A(\pi). \end{cases}$$

For each $U \subseteq A$, define

$$v_f(N-U) = \sum_{\pi \in P(s,t) \text{ with } A(\pi) \subseteq A-U} f(\pi). \qquad (3)$$

Clearly, $v_f(N)$ is called the flow value of f in N. The following flow conservation rule is immediate from (2) and (3):

$$\sum_{a \in OUT(x)} f(a) - \sum_{a \in IN(x)} f(a) = \begin{cases} v_f(N) & \text{if } x=s \\ 0 & \text{if } x \neq s,t \\ -v_f(N) & \text{if } x=t. \end{cases}$$

Thus, any network has a feasible flow $f(\pi) = 0$, $\pi \in P(s,t)$. A feasible flow with maximum flow value is called a maximum flow in N, and the maximum flow value is denoted by v(N).

3. THE EXPECTED MAXIMUM FLOW IN PROBABILISTIC NETWORKS

We consider the expected value of v(N) in a network N that is subject to random arc failures. Arc failures are assumed to be mutually independent, and the failure (survival) probability of an arc $a \in A$ is denoted by p(a) [q(a) = 1 - p(a)]. We will assume that $0<p(a)<1$ for every $a \in A$. Let p denote the vector $(p(a),\ a \in A)$, and let (N,p) denote the resulting probabilistic network. The expected value of maximum flow in (N,p) is defined by

$$v(N,p) = \sum_{U \subseteq A} v(N-U)\ p(U), \tag{4}$$

where p(U) is the probability that all arcs in U fail while the rest of arcs survive, i.e.,

$$p(U) = \prod_{a \in U} p(a). \prod_{a \in A-U} q(a). \tag{5}$$

Clearly, the straightforward computation of (4) is intractable especially for dense and large size networks. In general, it is known that the problem of calculating the expected value of maximum flow is a very hard and complicated process [5].

Before proceeding, we will introduce the following definitions and lemmas which are essential for subsequent analysis.

Definition 3.1. The maximum real number $\alpha \geq 0$ such that

$c(a) - \alpha \geq f(a)$ for any acyclic feasible flow f, is called the residual capacity r(a) of arc a where $r(a) \geq 0$ for any a.

If a network N has an arc a' such that no s-t path passes through a', then r(a') = c(a').

Lemma 3.1. For an arc a' in network N=(G,s,t,c) with G= (V,A), let N' = N-r(a') | {a'}. Then, r'(a')=0 and r'(a) = r(a) for a $\in$ A - {a'}, where r'(a) denotes the residual capacity of arc a in N'.

Proof. Clearly, r'(a) $\geq$ r(a) for a$\neq$a'. By definition of r(a), each arc a has an acyclic feasible flow f(a) with f(a) = c(a) - r(a). Since each of such f(a) is obviously feasible in N', r'(a') = 0 and r'(a) $\leq$ r(a) for a$\neq$a'. So, r'(a) = r(a).

Q.E.D.

Definition 3.2. Lemma 3.1 asserts that a network obtained by removing all residual capacities is unique [2]. Such network is given by

$$R(N) = (G',s,t,c') \text{ with } G' = (V,A'),$$

where A' = {a$\in$A | c(a) - r(a) > 0} and c'(a) = c(a) - r(a) (a$\in$A').

Evidently,

$$P(s,t;N) = P(s,t;R(N)),$$

where P(s,t; R(N)) denotes the set of paths from s to t in network R(N).

Lemma 3.2. Networks N and R(N) satisfy v(N,p) = v(R(N),p) for any probability vector p.

Proof. Let N= (G=(v,A),c). Consider the network N'=(G,c') defined by c'(a) = c(a) - r(a) (a$\in$A), where c'(a) is allowed to be zero. It is easy to see that [3] v(N',p) = v(R(N),p).

Then, it suffices to show $v(N,p) = v(N',p)$, i.e., $v(N-U) = v(N'-U)$ for any $U \subseteq A$. $v(N'-U) \leq v(N-U)$ is obvious. Let f be an acyclic maximum flow in N-U. By definition of r(a), $f(a) \leq c(a) - r(a)$ for any $a \in A-U$. This implies that f is feasible in N'-U, proving $v(N'-U) \geq v(N-U)$, so, $v(N-U) = v(N'-U)$, and consequently, $v(N,p) = v(R(N), p)$.

Q.E.D.

4. BOUNDS ON THE EXPECTED MAXIMUM FLOW

To derive the lower bound due to Carey and Hendrickson [2], recall that, for each failure set U, it is possible to make rerouting of flows so as to maximize the flow value. If we prohibit this rerouting, a feasible flow f gives the following lower bound:

$$v_f(N,p) = \sum_{\pi \in P(s,t)} \{f(\pi) . \prod_{a \in A(\pi)} q(a)\} \leq v(N,p). \qquad (6)$$

I.e., $v_f(N,p)$ represents a lower bound on $v(N,p)$.

For an upper bound, Onaga [6] has shown that

$$v(N,p) \leq \sum_{\pi \in P(s,t)} \sum_{U \subseteq A-A(\pi)} c(\pi)\ p(U) = v^f(N,p), \qquad (7)$$

where $v^f(N,p)$ is the maximum flow value in the network $N = (G,s,t,c')$ defined by $c'(a) = q(a)\ c(a)$ for $a \in A$; that is $v^f(N,p)$ provides an upper bound on $v(N,p)$.

Definition 4.1. Let f be a feasible flow in N. f is called critical in (N,P) if it satisfies (6) by equality.

Clearly, if there is a critical flow, then it maximizes the expected flow $v_f(N,p)$ over all feasible flows f [9]. However, some networks do not have critical flows. In the subsequent parts, we derive necessary and sufficient conditions for a directed network to have a critical flow.

Properties of the Lower Bound

Let f be a feasible flow in N. For a path $\pi \in P(s,t)$,

$$\prod_{a \in A(\pi)} q(a) = \sum_{U \subseteq A-A(\pi)} \prod_{a \in U} p(a) . \prod_{a \in A-U} q(a)$$

$$= \sum_{U \subseteq A-A(\pi)} p(U)$$

holds, where p(U) is defined in (5). Then, we have

$$v_f(N,p) = \sum_{\pi \in P(s,t)} \{ f(\pi) \prod_{a \in A(\pi)} q(a) \}$$

$$= \sum_{\pi \in P(s,t)} \sum_{U \subseteq A-A(\pi)} f(\pi)\ p(U)$$

$$= \sum_{U \subseteq A} \sum_{A(\pi) \subseteq A-U} f(\pi)\ p(U)$$

$$= \sum_{U \subseteq A} v_f(N-U)\ p(U), \tag{8}$$

where $v_f(N-U)$ is shown in (3).

Lemma 4.1. Let f be a feasible flow in N.

(i) $v(N-U) \geq v_f(N-U)$ for all $U \subseteq A$.

(ii) If there exists a set $U \subseteq A$ such that $v(N-U) > v_f(N-U)$, then $v(N,p) > v_f(N,p)$ for any $0 < p < 1$.

Proof. (i) Immediate from definitions of v(N-U) and $v_f(N-U)$. (ii) Obvious from (i) and (8). Q.E.D.

Lemma 4.2. A feasible flow f is critical to (N,p) iff

$$f(\pi) = c(\pi) \qquad \forall\ \pi \in P(s,t) \qquad (9)$$

Proof. Sufficiency is obvious from $v_f(N,p)$ $= \sum_{\pi \in P(s,t)} c(\pi) . \prod_{a \in A(\pi)} q(a) = \sum_{U \subseteq A} \sum_{A(\pi) \subseteq A-U} c(\pi)\, p(U)$ and Lemma 4.1 (ii). To show necessity, assume that a maximum flow f in N has an s-t path π' with $f(\pi') < c(\pi')$. For $U' = A-A(\pi')$, $v(N-U') = c(\pi') > f(\pi') = v_f(N-U')$, and hence f is not critical by Lemma 4.1 (ii). Q.E.D.

Lemma 4.3. A feasible flow f which is critical to (N,p) has the following properties [11]:

(i) f is the unique maximum flow in N.

(ii) f is acyclic.

(iii) $R(N) = N_f$.

Proof. (i) The maximality follows from Lemma 4.1 (ii) with $U=\phi$. If there is another maximum flow f', there exist two s-t paths π_1 and π_2 such that $f(\pi_1) > f'(\pi_1)$ and $f(\pi_2) < f'(\pi_2)$, indicating that N has no f satisfying (9). (ii) N always has an acyclic maximum flow f, which is unique by (i). (iii) If an acyclic flow f' in N satisfies $f'(a') > f(a)$ for some $a' \in A$, there must exist an s-t path π with $f'(\pi) > f(\pi)$, which contradict $f(\pi) = c(\pi)$. Q.E.D.

5. NECESSARY AND SUFFICIENT CONDITIONS FOR A DIRECTED NETWORK TO HAVE CRITICAL FLOW

In this section, we characterize those networks that have critical flows.

Definition 5.1. A network $N = (G,s,t,c)$ with $G = (V,A)$ is called balanced if the capacity function satisfies

$$\sum_{a\in OUT(x)} c(a) - \sum_{a\in IN(x)} c(a) \begin{cases} \geq 0 & \text{for } x=s \\ = 0 & \text{for all } x\in V-\{s,t\} \\ \leq 0 & \text{for } x=t \end{cases} \quad (10)$$

The next Lemma follows directly from the definition.

Lemma 5.1. Let $N = (G,s,t,c)$ with $G = (V,A)$ be balanced. For an s-t path π, network $N-\alpha|A(\pi)$ $(0\leq \alpha \leq c(\pi))$ is also balanced and satisfies $v(N-\alpha|A(\pi)) = v(N) - \alpha$.

Definition 5.2. A node $x\in V - \{s,t\}$ in $N = (G,s,t,c)$ is called a junction if $|P(s,x)| \geq 2$ and $|P(x,t)| \geq 2$.

For example, network N in Figure 1 has junctions x_2, x_3, and x_4.

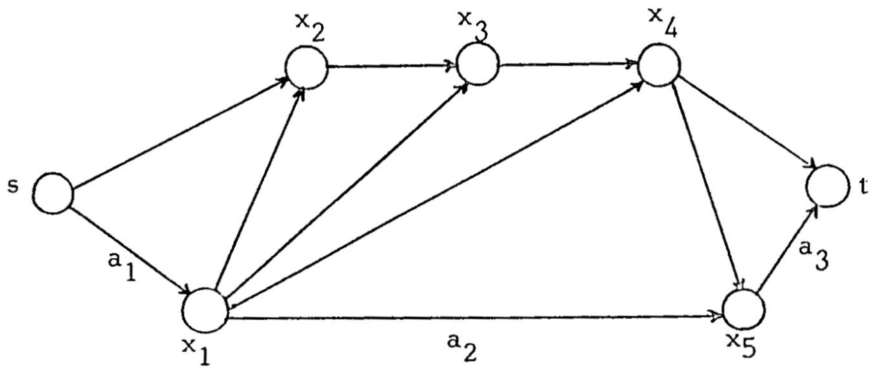

FIGURE 1. Network N.

Also, an arc a in N is called monofil if there is a unique s-t path that passes through a. Such an s-t path is also called monofil and is denoted by $\pi(a)$ [7].

For example, network N shown in Figure 1 has a monofil arc a_2 with $\pi(a_2)$: $s \to x_1 \to x_5 \to t$.

Lemma 5.2. [3]. Let N = (G,s,t,c) be a balanced acyclic network.

(i) If a maximum flow f in N is unique, N is junction free.

(ii) N is junction free iff the following (a) and (b) are satisfied:

(a) N has no node x with $|IN(x)| \geq 2$ and $|OUT(x)| \geq 2$.

(b) $P(y,z) = \phi$ for any y with $|IN(y)| \geq 2$ and any z with $|OUT(z)| \geq 2$.

(iii) Any monofil arc a in N, satisfies $c(a) = c(\pi(a))$.

(iv) For any arc $a \in A$, $P(s,v) \neq \phi$ and $P(v,t) \neq \phi$ for $v \in \{i(a), t(a)\}$.

Proof. The proof is given in the Appendix.

Definition 5.3. A network N is called monofil if N is balanced, acyclic and has no junction.

This definition is based on the following observation [2]: Assume without loss of generality that N = (G,s,t,c) with G = (V,A) is a monofil network in which no node x satisfies $|IN(x)| = |OUT(x)| = 1$ (remove such x by replacing $\{a,a'\} = IN(x) \cup OUT(x)$ by an arc a" with c(a") = c(a)). Let $\bar{X}$ be the set of all nodes x with $|IN(x)| \leq 1$ (i.e., x=s, where $|IN(x)| = 0$, or $|IN(x)| = 1$ by Lemma 5.2 (iv)). $IN(\bar{X}) = \phi$ holds since $t(a) \in \bar{X}$ implies $i(a) \in \bar{X}$ by $|OUT(t(a))| \geq 2$ and (b) of Lemma 5.2(ii). Thus, the subgraph G_s induced by $\bar{X}$ is a directed out-tree from source s. Similarly, $V-\bar{X}$ gives the set of all nodes x with $|OUT(x)| \leq 1$(i.e., x=t or x with $|OUT(x)|=1$), and the subgraph G_t induced by $V-\bar{X}$ is a directed in-tree with sink t. This means that P(s,t) is given by $\{\pi(a_1),\pi(a_2),\ldots,\pi(a_m)\}$, where $OUT(\bar{X}) = \{a_1,a_2,\ldots,a_m\}$, and every $\pi(a_i)$, i=1,2,...,m, is monofil. This is illustrated in Figure 2.

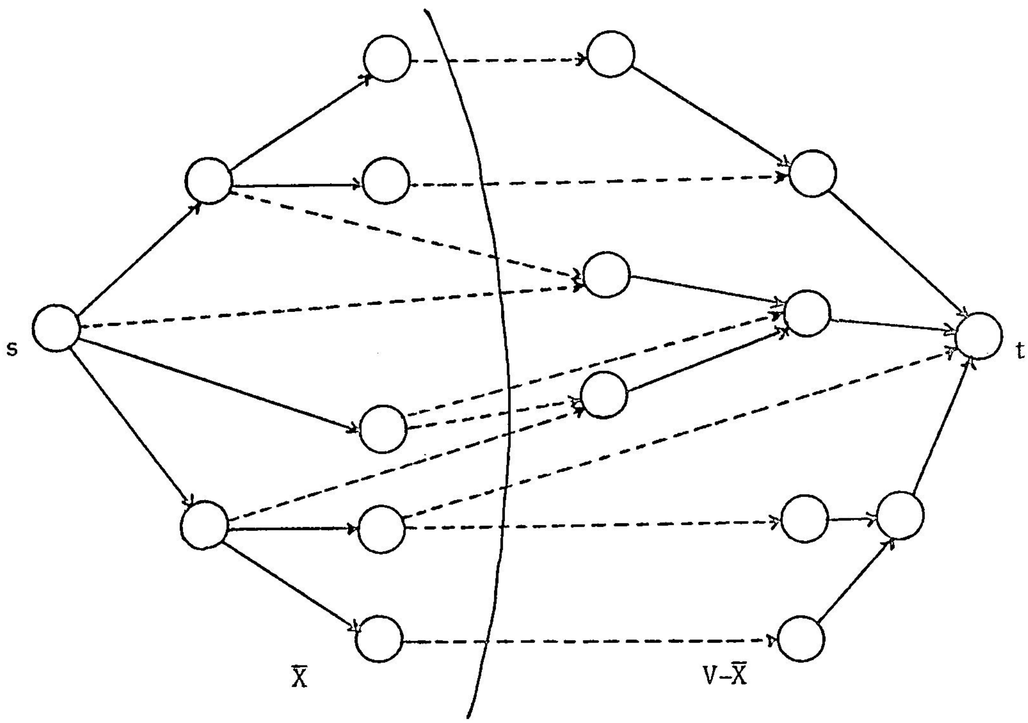

FIGURE 2. A monofil network (broken arcs are monofil).

From the previous concepts, definitions, and Lemmas discussed so far, it is possible to derive two main subsquent results.

I- A probabilistic network (N,p), where $0 < p < 1$, has a critical flow if and only if R(N) is monofil.

Proof. Necessity, By Lemma 4.3 (i), let f be the unique maximum flow in N. By Lemma 4.3 (iii), R(N) = N_f, which is balanced. By Lemma 4.3 (ii), N_f is acyclic. From Lemma 5.2 (i), N_f is junction free. Consequently, N_f = R(N) is monofil.

Sufficiency. Without loss of generality, assume that a monofil R(N) has node x with $|IN(x)| = |OUT(x)| = 1$ (see definition 5.3). Let $\bar{X}$ be the set of all nodes x with $|IN(x)| \leq 1$. Then, P(s,t) of R(N) is given by $\{\pi(a_1), \pi(a_2), \ldots, \pi(a_m)\}$, where $OUT(\bar{X}) = \{a_1, a_2, \ldots, a_m\}$, and every s-t path $\pi(a_i)$ is monofil. By Lemma 5.2 (iii), each monofil path $\pi(a_i)$ satisfies $c(\pi(a_i)) = c(a_i)$; i=2,3,...,m, hold. Repeating this, we see that there is a flow f satisfying $f(\pi(a_i)) = c(\pi(a_i))$, i=1,2,...,m. This f is critical by Lemma 4.2, the result follows.

As an example, consider the network N shown in Figure 3. When the capacity function is balanced, N is monofil.

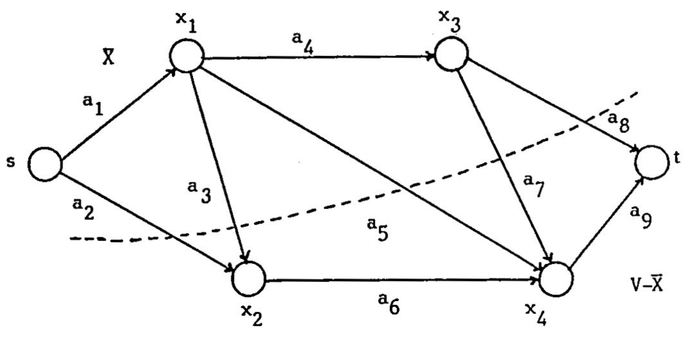

FIGURE 3. Directed Network.

In this case, its obvious that

$$\bar{X} = \{s, x_1, x_3\},$$

$$OUT(\bar{X}) = \{a_2, a_3, a_5, a_7, a_8\},$$

$$P(s,t) = \{s \to a_2 \to a_6 \to a_9 \to t,\ s \to a_1 \to a_3 \to a_6 \to a_9 \to t,\ s \to a_1 \to a_5 \to a_9 \to t,\ s \to a_1 \to a_4 \to a_7 \to a_9 \to t,\ s \to a_1 \to a_4 \to a_8 \to t\}.$$

and by (6) and the previous obtained result (I), we get

$$v(N,p) = v_f(N,p) = c(a_2)\ q(a_2)\ q(a_6)\ q(a_9) + c(a_3)\ q(a_1)\ q(a_3)\ q(a_6)\ q(a_9) + c(a_5)\ q(a_1)\ q(a_5)\ q(a_9) + c(a_7)\ q(a_1)\ q(a_4)\ q(a_7)\ q(a_9) + c(a_8)\ q(a_1)\ q(a_4)\ q(a_8).$$

Now, we characterize a second result concerning a network N whose R(N) is monofil.

II- A network $N = (G,s,t,c)$ with $G=(V,A)$ has a monofil R(N) iff the following (a), (b), and (c) hold for a maximum flow f in N:

(a) N_f is acyclic and junction free.

(b) $P(s,t;N) = P(s,t;N_f)$.

(c) $c(\pi) = f(\pi)$ for all $\pi \in P(s,t;N_f)$.

Necessity is immediate from result I and Lemma 4.3 (i), (iii) [(b) comes from that $P(s,t;N) - P(s,t;N_f) \neq \phi$ implies that $R(N) \neq N_f$].

Sufficiency is obvious since (a), (b), and (c) imply (9)

However, whether R(N) of a general network N is monofil or not can be easily checked by the following developed test which is denoted by MF-Test:

Step 1: Find an arbitrary maximum flow f in N, and let $A_f = \{ a \in A | f(a) > 0 \}$ and $N_f = ((V, A_f), s, t, f)$.

Step 2: If N_f contains a cycle or a junction, then terminate by concluding that R(N) is not monofil.

Step 3: For each path $\pi \in P(s,t;N_f)$, check whether $P(s,t;N-B_f(\pi))$ contains a path $\pi'(\neq \pi)$ or not, where

$$B_f(\pi) = \{a \in A | f(a) > 0, \ i(a) \in V(\pi), \ a \notin A(\pi)\},$$

in which arcs not in $A(\pi)$ have higher priority than those in $A(\pi)$. If there is such a π' for some $\pi \in P(s,t;N_f)$, then terminate since R(N) is not monofil.

Step 4. For each $\pi \in P(s,t;N_f)(= P(s,t;N))$, check whether $f(\pi) = c(\pi)$ holds or not. If there is a π with $f(\pi) < c(\pi)$, then terminate since R(N) is not monofil. Otherwise, R(N) is monofil and $R(N) = N_f$.

Illustrative Example: To illustrate the previous steps of MF-Test consider the network N shown in Figure 4. Arc capacities are given in Table 1.

TABLE 1

Arc Number : i	1	2	3	4	5	6	7	8	9	10	11	12	13
Arc Capacity : $c(a_i)$	6	4	7	3	5	5	4	3	4	6	5	4	4
Arc Number : i	14	15	16	17	18	19	20	21	22	23			
Arc Capacity : $c(a_i)$	3	3	5	3	8	4	5	4	6	7			

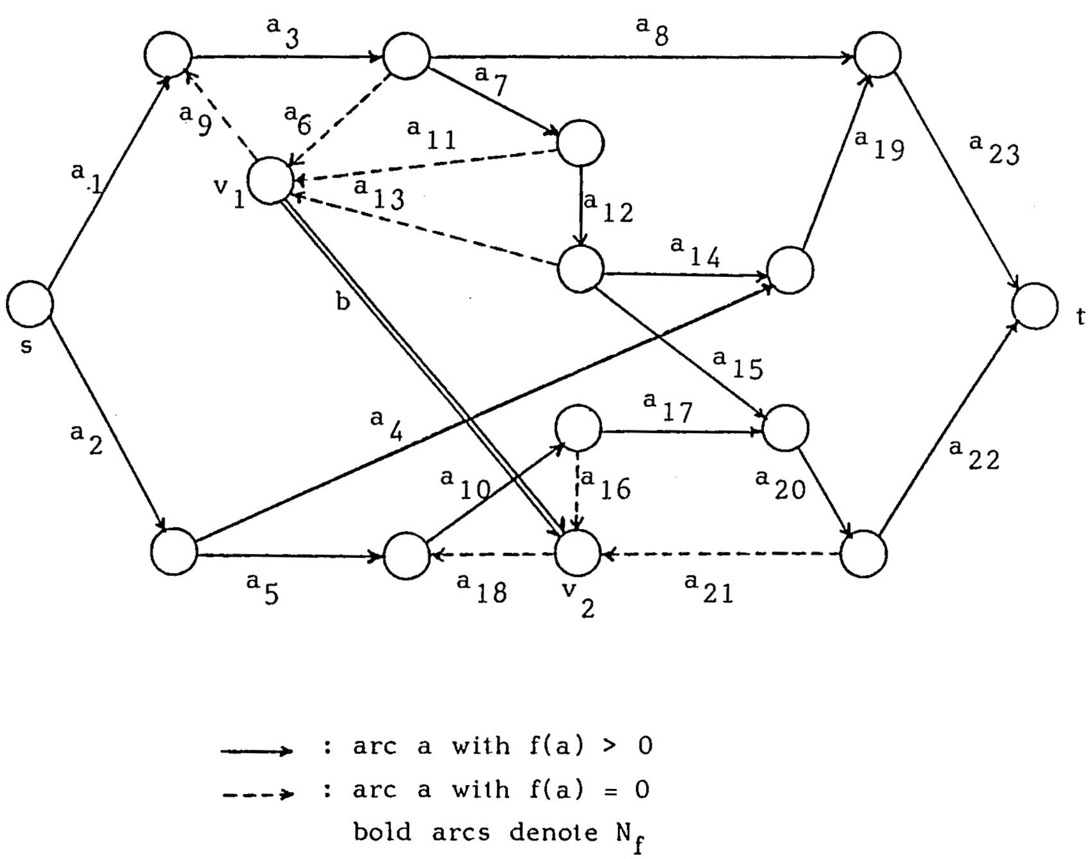

FIGURE 4. Example Network.

Step 1. Let f be a maximum flow, where

$f(\pi_1: s \to a_1 \to a_3 \to a_8 \to a_{23} \to t) = 3$

$f(\pi_2: s \to a_1 \to a_3 \to a_7 \to a_{12} \to a_{14} \to a_{19} \to a_{23} \to t) = 3$

$f(\pi_3: s \to a_1 \to a_3 \to a_7 \to a_{12} \to a_{15} \to a_{20} \to a_{22} \to t) = 3$

$f(\pi_4: s \to a_2 \to a_4 \to a_{19} \to a_{23} \to t) = 3$

$f(\pi_5: s \to a_2 \to a_5 \to a_{10} \to a_{17} \to a_{20} \to a_{22} \to t) = 3$

Step 2. It is shown that N_f is acyclic and junction free. Thus N_f is monofil and $\pi_i(i=1,2,\ldots,5)$ give all paths in $P(s,t;N_f)$.

Step 3. Test $\pi_1 \in P(s,t;N_f)$ as an example. Delecting arc set $B_f(\pi_1) = \{a_2,a_7\}$ from N, we conclude that $P(s,t;N-B_f(\pi_1)= \{\pi_1\}$ [if N has an arc b (indicated by a double arc) from v_1 to v_2, then path π': $s \to a_1 \to a_3 \to a_6 \to b \to a_{18} \to a_{10} \to a_{17} \to a_{20} \to a_{22} \to t$ is found at this point and the MF-test terminates by concluding that R(N) is not monofil]. Similarly, other paths $\pi_i(i=2,3,4,5)$ pass Step 3 successfully.

Step 4. We see that $f(\pi_i) = c(\pi_i) = 3$ for $i=1,2,\ldots,5$. So, R(N) is monofil and $R(N) = N_f$ holds.

6. SUMMARY AND CONCLUSIONS

This paper has been concerned with evaluation of the expected value of maximum flow in capacitated and directed networks subject to random arc failures. It is known that calculating the expected value of maximum flow in probabilistic networks is very hard and complicated process. An upper bound on the expected value of maximum flow in directed networks is given by Onaga, while a lower bound on it is found by Carey and Hendrickson. The lower bound sometimes gives the exact value, e.g., if networks are bipartite. We derive necessary and sufficient conditions for a directed networks to have the lower bound that is equal to the exact value. The conclusion is that it holds if and only if the network R(N) obtained by removing all residual capacities is monofil (i.e., balanced, acyclic, and junction free). Additionally, a simple and efficient test (MF-Test) is developed to check whether a given network R(N) is monofil or not. Furthermore, two points are still open for research and investigation. Firstly, extension the obtained results to undirected networks and multisource-multiterminal networks. Secondly, necessary and sufficient conditions for Onaga's upper bound to provide the exact value of the expected maximum flow in probabilistic networks.

APPENDIX

Proof of Lemma 5.2.

(i) Assume that N has a junction u, i.e., there are four distinct paths $\pi_1, \pi_2 \in P(s,u)$ and $\pi_3, \pi_4 \in P(u,t)$ (see Figure 5). Let π_{ij} denote the s-t path consisting of $\pi_i (i=1,2)$ and $\pi_j (j=3,4)$. Let

$$\alpha = \min \{ c(\pi_1),\ c(\pi_2),\ c(\pi_3),\ c(\pi_4) \} / 2,$$

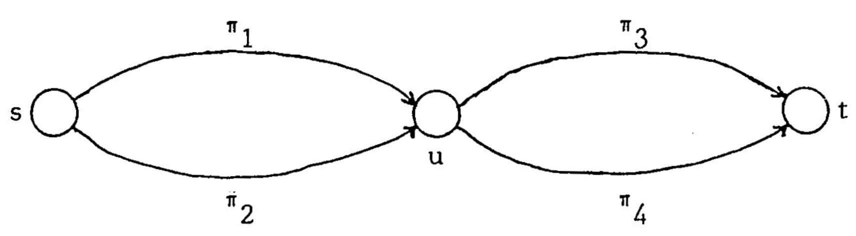

FIGURE 5. Network with a junction u.

and construct network $N' = (N - \alpha|\pi_{13}) - \alpha|\pi_{24}$. Since N is balanced, $v(N') = v(N) - 2\alpha$. Also, construct network $N'' = (G''(V,A''),s,t,c'')$ with $v(N'') = 2\alpha$ consisting only of paths π_{13} and π_{24}, i.e.,

$$A'' = A(\pi_{13}) \cup A(\pi_{24})$$

$$c''(a) = \begin{cases} 2\alpha & \text{if } a \in A(\pi_{13}) \cap A(\pi_{24}) \\ \alpha & \text{if } a \in \{A(\pi_{13}) - A(\pi_{24})\} \cup \{A(\pi_{24}) - A(\pi_{13})\}. \end{cases}$$

For any real number θ such that $0 \le \theta \le \alpha$, flow f_θ :

$$f_\theta(\pi) = \begin{cases} \alpha - \theta & \text{if } \pi = \pi_{13} \text{ or } \pi_{24} \\ \theta & \text{if } \pi = \pi_{14} \text{ or } \pi_{23} \\ 0 & \text{otherwise,} \end{cases}$$

is maximum in N''. Then, $f = f' + f_\theta$, where f' is a maximum flow in N', is maximum in N for any θ, indicating that a maximum flow in N is not unique.

(ii) Node x of (a), and nodes y,z of (b) with $P(y,z) \neq \phi$ are junctions as obvious from the acyclicity and capacity balance. This proves the necessity. Conversely, if N has a junction, then there exists x of (a) or y,z of (b) from definition of junction. This proves sufficiency.

(iii) Assume that $c(a) > c(\pi(a))$. Let f be a maximum flow in N. Since no s-t path except $\pi(a)$ contains a, this arc a is not saturated by f. By the capacity balance, a and other unsaturated arcs must form an s-t path or a cycle

But this contradicts the maximality of f and the acyclicity of N, respectively.

(iv) It suffices to show that $P(s,i(a)) \neq \phi$ and $P(t(a),t) \neq \phi$ for any $a \in A$. Let X be the set of nodes x (containing i(a)) such that $P(x,i(a)) \neq \phi$. From $IN(X)=\phi$ and the acyclicity, X contains a node x' satisfying $IN(x') = \phi$. Thus, x' must be s, that is, $P(s,i(a)) \neq \phi$. Similarly for $P(t(a),t) \neq \phi$. Q.E.D.

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ملخص البحـــــث

أقصى تدفق فى الشبكات الاحتمالية المعرضه لفشل بعض أنشطتها عشوائيـــا

تعتبر شبكات التدفق الاحتمالية - وهى تلك المعرضه لفشل أو تحطم بعـــض أنشطتها بشكل عشوائى - من النماذج الهامه للكثير من التطبيقات العمليه مثـــل شبكات المياه والصرف الصحى والكهرباء والنقل والاتصالات .... الخ .

وتقاس مدي صلاحية شبكة التدفق الاحتماليه بالقيمه المتوقعه لأقصى كميـــة تدفق خلال الشبكه ، هذه القيمه المتوقعه يصعب حسابها بشكل مباشر من الشبكه خصوصا فى الشبكات كبيرة الحجم كثيفه الأنشطه . لذلك قام كلا من كـــاري وهندريكسون بايجاد الحد الأدنى للقيمة المتوقعه لأقصى كمية تدفق بالشـــبكه الأحتمالية بينما قام أوناجا بايجاد الحد الأعلى لها ، وقد أثبتت الدراســات أن الحد الأدنى يعد أكثر قربا للقيمه المتوقعه من الحد الأعلـــى .

ويهدف هذا البحث الى أشتقاق الشروط الضروريه والكافيه والـــــتى اذا ماتوافرت فى الشبكه الأحتماليه (بعد حذف جميع السعات المتبقيـــــه للأنشطه) فان الحد الأدنى يساوي القيمه الفعليه لأقصى كمية تدفق متوقعــــه خلال تلك الشبكه ، وهذه الشروط هى :

١- أن تكون أنشطه الشبكه فى حالة اتزان (balanced) بمعنى أن تكـــون كمية التدفق الداخله فى النشاط تعادل كمية التدفق الخارجه منه .

٢- ألا تحتوي الشبكه على مسارات دورانيه من حدث البدايه الى حدث النهايه (acyclic) .

٣- ألا تحتوي الشبكه على أحداث يمثل أي منها نقطه اتصال وحيده بـــين أي مقطعـــين فى الشبكه بحيث يشتمل المقطع الأول على حدث البدايه والمقطـع الثانى على حدث النهايه (no junction) .

وقدقـدم الباحث اختبار يمكن بواسطته معرفة ما اذا كانت الشبكه موضـع البحث تحقق هذه الشروط أم لا . وتم تقديم مثال توضيحى باستخدام شـــبكه مكونه من ٢٣ نشاط بسعات افتراضيه لبيان كيفية استخدام هذا الأختبار للحكم على مدي توافر هذه الشروط بالشبكه من عدمـــه .