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MODIFIED MAXIMUM LIKELIHOOD PREDICTORS OF FUTURE ORDER STATISTICS FROM RAYLEIGH SAMPLES

by

Mohammad Z. Raqab
Department of Mathematics
University of Jordan
Amman 11942 JORDAN

ABSTRACT

In this paper we present some modified maximum likelihood predictors of the s-th order statistic based on r order statistics of random samples of size n from Rayleigh distribution, where $r < s \leq n$. We suggest four types of modifications to the predictive likelihood equations in order to find such predictors. We simulate the values of the bias, mean square prediction error, and likelihood function. On the basis of these criteria, we select the one obtained by the so called Type II modification to be the best predictor. Its efficiencies compared to those for the best linear unbiased predictors and alternative linear predictors are remarkably high.

Key Words: prediction, order statistics, Rayleigh distribution, maximum likelihood predictor, modified maximum likelihood predictor.

1991 Mathematics Subject Classifications: 62G30, 62F99, 62M20

1. Introduction

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics of a random sample from an absolutely continuous cumulative distribution function (cdf) $F(x;\underset{\sim}{\theta})$ having probability density function (pdf) f, where $\underset{\sim}{\theta}$ is possibly a vector valued parameter. Let $F_{k:n}$ and $f_{k:n}$ denote the cdf and pdf of $X_{k:n}$, respectively, for k = 1 to n. Suppose we observe only $\underset{\sim}{X} = (X_{1:n}, \ldots, X_{r:n})$ and the goal is to predict $X_{s:n}$ where $1 \leq r < s \leq n$. In the context of reliability theory, $X_{s:n}$ represents the life-length of a (n-s+1)-out-of-n system made up of n identical components with independent life-lengths. When s= n, it is better known as the parallel system. Thus our concern is the prediction of the life length of such a system based on the first r failure times of the components.

When F belongs to a location - scale family with location parameter μ and scale parameter σ (> 0), the best known predictor is the best linear unbiased predictor (BLUP) (see, for example, David, 1981, p. 156). This is based on the work by Kaminsky and Nelson (1975) who applied Goldberger's (1962) results in the context of a linear model for the prediction of $X_{s:n}$. Let $Z_{i:n} = (X_{i:n} - \mu)/\sigma$ (i=1, 2, ..., r), $\underset{\sim}{Z} = (Z_{1:n}, \ldots, Z_{r:n})$, and let $\underset{\sim}{\alpha}$ and V denote the mean vector and the covariance matrix of $\underset{\sim}{Z}$, respectively. Then, the BLUP of $X_{s:n}$ is given by

$$\delta_L(1) = (\hat{\mu}_L + \alpha_s \hat{\sigma}_L) + \underset{\sim}{w}' V^{-1}(\underset{\sim}{X} - \hat{\mu}_L \underset{\sim}{1} - \hat{\sigma}_L \underset{\sim}{\alpha}), \tag{1.1}$$

where $\hat{\mu}_L$ and $\hat{\sigma}_L$ are the least-squares estimators of μ and σ, respectively, $\alpha_s = E(Z_{s:n})$, and $\underset{\sim}{w}' = (w_1, \ldots, w_r)$ with $w_i = Cov(Z_{i:n}, Z_{s:n})$ (i=1, 2,..., r) and $\underset{\sim}{1}$ is a vector whose elements are all unity. Gupta (1952) has proposed alternative linear estimators of μ and σ by replacing the variance-covariance matrix of $\underset{\sim}{Z}$, V, by an identity matrix I. Raqab (1993) has also obtained an alternative linear unbiased predictor (ALUP) of $X_{s:n}$ by replacing V by D, where $D = diag(v_{11}, \ldots, v_{rr})$. Let us denote this ALUP by $\delta_L(2)$.

Kaminsky and Rhodin (1985) have extended the method of maximum likelihood to allow the joint prediction of a future random variable and estimation of an unknown parameter. The resulting predictor is being called the maximum likelihood predictor (MLP). They gave some sufficient conditions for the existence and uniqueness of the MLP and illustrated the method with several examples.

It is not possible to obtain a closed form expression for the MLP in most cases. In estimation problem, Mehrotra and Nanda (1974) obtained approximate maximum likelihood estimators (MLE's) for the normal and gamma distributions by replacing h(x) or xh(x) by their respective expected values, where h, the hazard rate function, is given by

$$h(x) = \frac{f(x)}{1-F(x)}. \tag{1.2}$$

Using numerical computations they showed this procedure produces estimators that are efficient when compared to the best linear unbiased estimators (BLUE's). Balakrishnan and Cohen (1991, Ch.6) used the Taylor series expansion of h(x) and f(x)/F(x) around the p_r-th quantile where $p_r = r/(n+1)$, to obtain modified MLE's of the parameters of the normal and Rayleigh distributions. The main point in their approach is that the likelihood equation involves messy terms and it is impossible to obtain an explicit form for the MLE. Similar is the case with prediction as well because of the nature of the predictive likelihood function(PLF).

Now, the PLF of $X_{s:n}$ and $\underset{\sim}{\theta}$ is given by

$$L(X_{s:n}, \underset{\sim}{\theta} ; \underset{\sim}{X}) = C_n(r,s)\prod_{j=1}^{r} f(X_{j:n})\, [F(X_{s:n})-F(X_{r:n})]^{s-r-1} f(X_{s:n}) \cdot [1-F(X_{s:n})]^{n-s}, \tag{1.3}$$

where $C_n(r,s) = n!/\{(s-r-1)!(n-s)!\}$. The PLF of $X_{s:n}$ and θ in (1.3) can be written as a product of two likelihood functions L_1 and L_2 where

$$L_1(\underset{\sim}{\theta}; \underset{\sim}{X}) = f(\underset{\sim}{\theta}; \underset{\sim}{X}) = \frac{n!}{(n-r)!}\prod_{j=1}^{r} f(X_{j:n})\, [1-F(X_{r:n})]^{n-r},$$

and

$$L_2(X_{s:n}; \underset{\sim}{\theta}, \underset{\sim}{X}) = \frac{(n-r)!}{(s-r-1)!(n-s)!}\,\frac{[F(X_{s:n})-F(X_{r:n})]^{s-r-1}}{[1-F(X_{r:n})]^{n-r}} \cdot [1-F(X_{s:n})]^{n-s} f(X_{s:n}).$$

The predictive likelihood equations (PLE's) corresponding to L, L_1, and L_2 are in the following forms:

$$\frac{\partial \log L}{\partial X_{s:n}} = 0, \text{ and } \frac{\partial \log L}{\partial \theta_i} = 0, \tag{1.4}$$

$$\frac{\partial \log L_1(\underset{\sim}{\theta})}{\partial \theta_i} = 0, \tag{1.5}$$

where θ_i is a component of $\underset{\sim}{\theta}$; and

$$\frac{\partial \log L_2(X_{s:n})}{\partial X_{s:n}} = 0. \tag{1.6}$$

So the PLE's involve $h(X_{i:n})$ (i=r, s), where h is defined in (1.2) and the extended hazard rate functions $h_1(X_{r:n}, X_{s:n})$, $h_2(X_{r:n}, X_{s:n})$, defined by

$$h_1(x,y) = \frac{f(x)}{F(y)-F(x)}, \quad h_2(x,y) = \frac{f(y)}{F(y)-F(x)}, \quad x < y. \tag{1.7}$$

The following section introduces four types of modifications (Type I-IV) to the PLE's to derive the modified maximum likelihood predictors (MMLP's). Two of these are based on replacing h, h_1 and h_2 by their expected values in the PLE's and the remaining two are based on Taylor series expansions. We apply these techniques in Section 3 to predict a future order statistic using a Type II censored sample from Rayleigh distribution when the scale parameter σ is unknown. We compare the predictors so obtained on the basis of bias, MSPE, and the value of the likelihood function at the predicted value. On the basis of these comparisons we select the one whose performance is the best most often and compare it with the BLUP's and ALUP's. All these numerical comparisons are discussed in Section 4.

2. Types of Modification

We consider four types of modification to approximate the terms involved hazard rate and extended hazard rate functions that appear in the likelihood equations (1.4-1.6). The modifications used are of the following types:

One-Stage Modified Maximum Likelihood Prediction:

In this technique, $h(X_{s:n})$, $h_1(X_{r:n}, X_{s:n})$, and $h_2(X_{r:n}, X_{s:n})$ are replaced by their respective expected values. This allows us to solve the PLE's in (1.4) and obtain simple MMLP of $X_{s:n}$ and PMLE of $\underset{\sim}{\theta}$. The resulting predictor is called Type I MLP.

Two-Stage Modified Maximum Likelihood Prediction:

Here, $h(X_{r:n})$ is replaced by its expected value in the likelihood equation (1.5). Therefore the PMLE of $\underset{\sim}{\theta}$ can be obtained in the first stage. In the second stage, this estimator will be substituted in the likelihood equation (1.6). Also $h(X_{s:n})$, and $h_2(X_{r:n}, X_{s:n})$ are replaced by their respective expected values. This technique leads us finally to modified MLP of $X_{s:n}$ and modified PMLE of $\underset{\sim}{\theta}$. The resulting predictor is called Type II MMLP.

One-Stage Approximate Maximum Likelihood Prediction:

The third modification involves the use of Taylor series approximation of $h(X_{s:n})$, $h_1(X_{r:n}, X_{s:n})$, and $h_2(X_{r:n}, X_{s:n})$ around the quantile functions at p_i, where $p_i = i/(n+1)$ $(i = r, s)$. Thus, the left sides of the equations in (1.4) are approximated by linear functions. Therefore this technique gives simultaneous approximate MLP of $X_{s:n}$ and approximate PMLE of $\underset{\sim}{\theta}$. The resulting predictor is called Type III MMLP.

Two-Stage Approximate Maximum Likelihood Prediction:

In this approach, the Taylor series expansion around the quantile function at $p_r=r/(n+1)$ is used to approximate $h(X_{r:n})$ in the likelihood equation (1.5). This yields the modified PMLE of $\underset{\sim}{\theta}$ in the first stage. In the second stage this estimator of $\underset{\sim}{\theta}$ is substituted in the likelihood equation (1.6). Further, the Taylor series expansion of $h(X_{s:n})$ and $h_2(X_{r:n}, X_{s:n})$ around $F^{-1}(p_s)$ and $(F^{-1}(p_r), F^{1}(p_s))$, respectively, is applied again. Hence an approximation to the MLP of $X_{s:n}$ may be obtained as an explicit solution. The resulting predictor is called Type IV MMLP.

3. Modified Predictors for the Rayleigh Samples

Consider the Rayleigh distribution with density function $f(x; \sigma) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}$, $x > 0$, and cumulative distribution function $F(x; \sigma) = 1 - e^{-x^2/2\sigma^2}$, $x > 0$. This distribution can be considered as a special case of a two-parameter Weibull distribution. It serves as a model for the failure time distribution with unknown (scale) parameter σ.

Denote $Z_{s:n} = X_{i:n}/\sigma$, and for notational convenience, let

$$\Delta_1(L) = \frac{\partial \log L}{\partial Z_{s:n}}, \quad \Delta_2(L) = \frac{\partial \log L}{\partial \sigma},$$

$$\Delta_{11}(L) = \frac{\partial^2 \log L}{\partial Z^2_{s:n}}, \text{ and } \Delta_{22} = \frac{\partial^2 \log L}{\partial \sigma^2}.$$

When the likelihood is either L_1 or L_2, the partial derivatives are similarly defined.

The PLE's corresponding to (1.4) can be written as

$$\Delta_1(L) = (s-r-1)h_2(Z_{r:n}, Z_{s:n}) - (n-s)h(Z_{s:n}) - \Psi(Z_{s:n}) = 0, \tag{3.1}$$

$$\Delta_2(L) = -\frac{1}{\sigma}\{(r+1) - \sum_{i=1}^{r} Z_{i:n}\,\Psi(Z_{i:n}) - Z_{s:n}\,\Psi(Z_{s:n}) - (s-r-1)$$
$$[Z_{r:n}\,h_1(Z_{r:n}, Z_{s:n}) - Z_{s:n}\,h_2(Z_{r:n}, Z_{s:n})] - (n-s)\,Z_{s:n}\,h(Z_{s:n})\} = 0. \tag{3.2}$$

where $\Psi(z) = -\{f'(z)/f(z)\}$. The equations in (3.1), (3.2) are the main equations for producing the PMLEs of σ and the MLP of $X_{s:n}$. If μ is unknown, we will have one more equation to deal with. However, in this paper we assume that the parameter involved in this prediction problem is the scale parameter σ.

Using fact that $h(z) = z$ and $\frac{\partial f(z)}{\partial z} = f(z)[-z + \frac{1}{z}]$, the equations (3.1) and (3.2) reduce respectively to

$$\Delta_1(L) = -(n-s+1)\,Z_{s:n} + \frac{1}{Z_{s:n}} + (s-r-1)h_2(Z_{r:n}, Z_{s:n}) = 0, \tag{3.3}$$

$$\Delta_2(L) = -\frac{1}{\sigma}\Big\{2(r+1) - (n-s+1)\,Z^2_{s:n} - \sum_{i=1}^{r} Z^2_{i:n}$$
$$- (s-r-1)\,[Z_{r:n}\,h_1(X_{r:n}, X_{s:n}) - Z_{s:n}\,h_2(Z_{r:n}, Z_{s:n})]\Big\} = 0. \tag{3.4}$$

The PLE's corresponding to L_1 and L_2 are given by

$$\frac{1}{\sigma}\{\sum_{i=1}^{r} Z_{i:n}^2 + (n-r)\, Z_{r:n}^2 - 2r\} = 0, \tag{3.5}$$

$$-(n-s+1)\, Z_{s:n} + \frac{1}{Z_{s:n}} + (s-r-1)\, h_2(Z_{r:n}, Z_{s:n}) = 0. \tag{3.6}$$

For the two stage procedure, the PMLE of σ can be evaluated explicitly from (3.5). Therefore the MMLP of $X_{s:n}$ can be obtained by modifying only the extended hazard function $h_2(Z_{r:n}, Z_{s:n})$ in (3.6).

When $s = r+1$, the PLF is reduced to

$$L(Z_{r+1:n}, \sigma; \underset{\sim}{X}) = C_n(r, r+1)\prod_{i=1}^{r} f(Z_{i:n})\, f(Z_{r+1:n})\, [1 - F(Z_{r+1:n})]^{n-r-1}.$$

So the PLE's are given by

$$\Delta_1(L) = -\Psi(Z_{r+1}) - (n-r-1)\, h(Z_{r+1:n}) = 0.$$

$$\Delta_2(L) = -\frac{1}{\sigma}\{(r+1) - \sum_{i=1}^{r} Z_{i:n}\Psi(Z_{i:n}) - Z_{r+1:n}\Psi(Z_{r+1:n}) - (n-r-1)\, Z_{r+1:n}^2\} = 0,$$

It is clear that the above equations need not be modified. Hence the exact PMLE of σ and MLP of $X_{r+1:n}$ are obtained by the following equations:

$$(n-r)\, X_{r+1:n}^2 - \sigma^2 = 0, \tag{3.7}$$

$$-(n-r)\, X_{r+1:n}^2 - \sum_{i=1}^{r} X_{i:n}^2 + 2(r+1)\,\sigma^2 = 0. \tag{3.8}$$

Upon solving (3.7) and (3.8) simultaneously, we obtain the solutions of σ and $X_{r+1:n}$ as the following:

$$\hat{\sigma}_a(1) = \frac{1}{\sqrt{2r+1}}\sqrt{\sum_{i=1}^{r} X_{i:n}^2}, \quad \text{and}$$

$$\delta_a(1) = \frac{1}{\sqrt{(2r+1)(n-r)}} \sqrt{\sum_{i=1}^{r} X_{i:n}^2} \, .$$

Although $\delta_a(1)$ is the exact solution, it is in inadmissible region since $\delta_a(1) \le X_{r:n}$. Thus we take $X_{r:n}$ to be $\delta_a(1)$, the MLP of $X_{r+1:n}$, and will try to maximize L with respect to σ. In that case

$$\log L(\sigma) \propto -2(r+1)\log \sigma - \frac{1}{2\sigma^2}\{\sum_{i=1}^{r} X_{i:n}^2 + (n-r)\,X_{r:n}^2\}.$$

Hence, the PMLE of σ is found to be

$$\hat{\sigma}_a(1) = \sqrt{\frac{\sum_{i=1}^{r} X_{i:n}^2 + (n-r)\,X_{r:n}^2}{2(r+1)}} \, .$$

At $\hat{\sigma}_a(1)$, the approximation to $\Delta_{22}(L)$ is less than 0. This implies that L at $\hat{\sigma}_a(1)$ is close to its maximum value. Therefore the PMLE of σ and MLP of $X_{r+1:n}$ are, respectively,

$$\hat{\sigma}_a(1) = \sqrt{\frac{\sum_{i=1}^{r} X_{i:n}^2 + (n-r)\,X_{r:n}^2}{2(r+1)}} \qquad \text{and} \qquad \delta_a(1) = X_{r:n}.$$

Type I Modification

Clearly (3.3) and (3.4) do not yield explicit solutions for σ and $X_{s:n}$ $(s > r+1)$. We will now derive the MMLP by replacing $h_1(Z_{r:n}, Z_{s:n})$ and $Z_{r:n}h_1(Z_{r:n}, Z_{s:n}) - Z_{s:n}h_2(Z_{r:n}, Z_{s:n})$ by their respective expected values. From Lemma 2.3 of Raqab (1993) and Lemma 2.1 of Mehrotra and Nanda (1974), respectively, we have

$$Eh_2(Z_{r:n}, Z_{s:n}) = \frac{1}{s-r-1} \sum_{i=s}^{n} E(Z_{i:n} - \frac{1}{Z_{i:n}}), \; (s - r \ge 2), \tag{3.9}$$

$$E\{(Z_{r:n}\, h_1(Z_{r:n}, Z_{s:n}) - Z_{s:n}\, h_2(Z_{r:n}, Z_{s:n})\} = \frac{1}{s-r-1}\sum_{i=r+1}^{s-1} E(Z_{i:n}^2 - 2), \quad (s - r \geq 2). \tag{3.10}$$

On using (3.9) and (3.10), we may modify the PLE's in (3.3) and (3.4) as

$$-(n-s+1)\, Z_{s:n} + \frac{1}{Z_{s:n}} + \sum_{i=s}^{n} E(Z_{i:n} - \frac{1}{Z_{i:n}}) = 0, \tag{3.11}$$

$$-\frac{1}{\sigma}\{2s - (n-s+1)\, Z_{s:n}^2 - \sum_{i=1}^{r} Z_{i:n}^2 - \sum_{i=r+1}^{s-1} E(Z_{i:n}^2)\} = 0. \tag{3.12}$$

Letting $b_1 = n-s+1$, $b_2 = \sum_{i=s}^{n} E(Z_{i:n} - \frac{1}{Z_{i:n}})$ and $b_3 = 2s - \sum_{i=r+1}^{s-1} E(Z_{i:n}^2)$, the equations in (3.11) and (3.12) can be rewritten in the following form:

$$b_1\, X_{s:n}^2 - b_2\, \sigma\, X_{s:n} - \sigma^2 = 0, \tag{3.13}$$

$$b_1\, X_{s:n}^2 - b_3\, \sigma^2 + \sum_{i=1}^{r} X_{i:n}^2 = 0. \tag{3.14}$$

Upon solving equation (3.13), we obtain two roots; however, the negative root drops out, since $X_{s:n} > 0$ almost surely. Using (3.14), the admissible solution of (3.13) yields the MMLP of $X_{s:n}$ as

$$\delta_a(1) = \frac{b_2 + \sqrt{b_2^2 + 4b_1}}{2b_1}\, \hat{\sigma}_a(1), \tag{3.15}$$

where

$$\hat{\sigma}_a(1) = \left\{ \frac{4b_1 \sum_{i=1}^{r} X_{i:n}^2}{[4b_1 b_3 - (b_2 + \sqrt{b_2^2 + 4b_1})^2\,]} \right\}^{1/2}, \quad (r+1 < s \leq n).$$

Thus, we take the Type I MMLP of $X_{s:n}$ as

$$\delta_a^*(1) = \begin{cases} \delta_a(1) & \text{if } \delta_a(1) > X_{r:n} \text{ and } r < s \le n \\ X_{r:n} & \text{if } \delta_a(1) \le X_{r:n} \text{ or } s = r+1. \end{cases}$$

Type II Modification

Here, in the first stage, the likelihood equation in (3.5) can be solved without any modification to give the PMLE of σ. This estimator has been obtained by Harter and Moore (1965), and is given by

$$\hat{\sigma}_a(2) = \frac{1}{\sqrt{2r}} \sqrt{\sum_{i=1}^{r} X_{i:n}^2 + (n-r)\, X_{r:n}^2}. \tag{3.16}$$

Harter and Moore showed that $2r \frac{\hat{\sigma}^2_a(2)}{\sigma^2} \overset{d}{=} W$, where W is a chi-square random variable with 2r degrees of freedom. Hence $\hat{\sigma}^2_a(2)$ is an unbiased estimator of σ^2 but $\hat{\sigma}_a(2)$ is not unbiased for σ. As they point out, an unbiased estimator of σ is

$$\tilde{\sigma}_a(2) = \frac{\sqrt{r}\,\Gamma(r)}{\Gamma(r+1/2)}\, \hat{\sigma}_a(2).$$

Using (3.16) and replacing the extended hazard function $h_2(Z_{r:n}, Z_{s:n})$ appearing in (3.6) with its expected value in (3.9), we obtain the following modified PLE:

$$-(n-s+1)\, Z_{s:n} + \frac{1}{Z_{s:n}} + \sum_{i=s}^{n} E(Z_{i:n} - \frac{1}{Z_{i:n}}) = 0. \tag{3.17}$$

Upon solving equation (3.17) we obtain a quadratic equation in $X_{s:n}$ which has two roots. One of them is inadmissible, since $X_{s:n} > 0$ almost surely. The admissible solution is given by

$$\delta_a(2) = \frac{b_2 + \sqrt{b_2^2 + 4b_1}}{2b_1}\ \hat{\sigma}_a(2), \quad r+1 < s \le n,$$

where $\hat{\sigma}_a(2)$ is defined in (3.16). At $\delta_a(2)$, we have $\Delta_{11}(L2) \equiv -\{b_1 + (\hat{\sigma}_a(2)/\delta_a^2(2))^2\} < 0$. Then L_2 at $\delta_a(2)$ is close to the maximum value of L_2. So the MMLP of $X_{s:n}$ is given by

$$\delta_a^*(2) = \begin{cases} \delta_a(2) & \text{if } \delta_a(2) > X_{r:n} \text{ and } r+1 < s \le n \\ X_{r:n} & \text{if } \delta_a(2) \le X_{r:n} \text{ or } s = r+1. \end{cases}$$

Type IV Modification

In this approach, we use the estimator of σ given in (3.16) to be substituted in (3.6). Also we expand $h_2(Z_{r:n}, Z_{s:n})$ appearing in (3.6) in the Taylor series around the point $(\eta(pr), \eta(ps))$; that is,

$$h_2(Z_{r:n}, Z_{s:n}) \cong \gamma + \rho\, Z_{r:n} + \nu\, Z_{s:n},$$

where
$$\eta_i = F^{-1}(p_i) = [-2 \log q_i]^{1/2};\ q_i = 1 - p_i,$$
$$\gamma = q_r\, q_s\, \eta_s\, (\eta_s^2 - \eta_r^2)/p_{sr}^2,$$
$$\rho = (q_r\, q_s\, \eta_r \eta_s)/p_{sr}^2,$$
$$\nu = (q_s/p_{sr})\{1 - (q_r\, \eta_s^2)/p_{sr}\}.$$

So the modified PLE corresponding to L_2 is

$$-(n-s+1)\, Z_{s:n} + \frac{1}{Z_{s:n}} + d\{\gamma + \rho\, Z_{r:n} + \nu\, Z_{s:n}\} = 0. \tag{3.18}$$

where d=s-r-1. Equation (3.18) can be reduced to a quadratic equation with only one admissible root and it is given by

$$\delta_a(4)= \frac{1}{2u_1}\{ d(\gamma\hat{\sigma}_a(2)+ \rho X_{r:n})+\sqrt{d^2(\gamma\hat{\sigma}_a(2)+\rho X_{r:n})^2+ 4u_1 \hat{\sigma}^2_a(2)} \}.$$

where $u_1 = [(n-s+1) - (s-r-1)v]$. We have also, at $\delta_a(4)$,

$$\Delta_{22}(L_2) \equiv - \{b_1+ (\hat{\sigma}_a(2)/\delta_a(4))^2- dv\}. \tag{3.19}$$

To show that this derivative is negative, we introduce the following lemma.

Lemma: Assume that F is the cdf of the Rayleigh distribution and $v = (q_s/p_{sr})\{1 - (q_r \eta_s^2)/p_{sr}\}$, where $\eta_s = F^{-1}(p_s)$, $0 < p_r < p_s < 1$. Then $v < 0$.

Proof: Let $h(x) = \dfrac{-\log(1-x)}{x-p_r}, \quad p_r < x < 1.$

On differentiating h(x), we obtain

$$h'(x) = \{g_1(x)/g_2(x)\},$$

where $g_1(x) = \dfrac{x-p_r}{1-x} + \log(1-x)$, and $g_2(x) = (x-p_r)^2$.

It is easily seen that $g_1(x)$ is a monotonically increasing function and $g_1(p_r) = \log(1-p_r) < 0$, $g_1(1) = \infty$. Therefore $g_1(x) = 0$ has a unique solution in $(p_r, 1)$, say, x_0. So

$$\frac{x_0-p_r}{1-x_0} + \log(1-x_0) = 0. \tag{3.20}$$

Note that h(x) attains its minimum at x_0. By making use of (3.20), we have

$$h(x_0) = \frac{-\log(1-x_0)}{x_0-p_r} = \frac{1}{1-x_0}.$$

Since $x_0 \varepsilon(p_r, 1)$, it follows that $h(x_0)$ is greater than $1/[2(1-p_r)]$. Hence h(x) is greater than $1/[2(1-p_r)]$ for all x in $(p_r, 1)$. This completes the proof.

From (3.19) and the above Lemma, it follows that L_2 at $\delta_a(4)$ is close to the maximum value of L_2. Hence, the MMLP can be written as

$$\delta_a^*(4) = \begin{cases} \delta_a(4) & \text{if } \delta_a(4) > X_{r:n} \text{ and } r+1 < s \le n \\ X_{r:n} & \text{if } \delta_a(4) \le X_{r:n} \text{ or } s = r+1. \end{cases}$$

4. Numerical Comparisons

In order to use the MMLP's $\delta_a^*(1)$ and $\delta_a^*(2)$, the second moment of standardized order statistics and the values of $E(Z_{i:n} - \frac{1}{Z_{i:n}})$ are needed. The second moment of $Z_{i:n}$ can be computed from Govindarajulu and Joshi (1968). In Table 1 below, present the values of $E(Z_{i:n} - \frac{1}{Z_{i:n}})$, $i = 1, 2, \ldots, n$, for $n = 2(1)10$.

A simulation study based on 10,000 Monte Carlo runs was performed in double precision on an IBM 3081 computer in FORTRAN. We used IMSL subroutine DRNUNO to simulate the first r uniform order statistics. To generate the s-th uniform statistic, we used the fact that its conditional distribution given the r-th order statistic, is linearly related to a beta random variable with s-r and n-s+1 (see Devroye (1986) for some further details and relevant references). Then we employed inverse cdf transformation to obtain necessary Rayleigh order statistics. This procedure was done for the sample sizes 5, 10 when $\sigma = 1$ and with different choices of r (the number of observed values) and s (rank of the value to be predicted).

We have to point out that the Type III technique fails to give an admissible predictor of $X_{s:n}$. ZPORC subroutine in IMSL-FORTRAN is used to find out the roots of the likelihood equations. Unfortunately, the roots of these equations are inadmissible because they are either complex or negative. Raqab (1993) applied all techniques of modification discussed in this paper to obtain the MMLP's for the normal distribution. He obtained four modified predictors along with estimating the location parameter μ. Whereas her we predict a future order statistic along with estimating the scale parameter σ. This indicates that the parameter to be estimated has important role in obtaining a reasonable modified predictor.

For the purpose of comparison, we compute the following values:

(1) Bias of $\delta_a^*(i)$, $E(\delta_a^*(i) - Z_{s:n})$, (2) MSPE of $\delta_a^*(i)$, $E(\delta_a^*(i) - Z_{s:n})^2$,

(3) Likelihood function at $(\delta_a^*(i), \hat{\sigma}_a(i))$,

where $i = 1, 2, 4$. Table 2 presents the values of the bias and MSPE of the so obtained predictors for the Rayleig distribution. As we see in Table 2, all these modified predictors are negatively biased. On comparing with the other modified predictors, the values of the bias and MSPE of $\delta_a^*(2)$ are found to be smaller for all choices of r, s considered. For this reason, we define the following ratios:

$$\text{Ratio (I)} = \frac{L(\delta_a^*(1),\ \hat{\sigma}_a(1))}{L(\delta_a^*(2),\ \hat{\sigma}_a(2))},\ \text{and}\ \ \text{Ratio (IV)} = \frac{L(\delta_a^*(4),\ \hat{\sigma}_a(4))}{L(\delta_a^*(2),\ \hat{\sigma}_a(2))}.$$

Ratio(I) and Ratio(IV) are also presented in Table 2. Based on all criteria (Bias, MSPE, Likelihood Function), we consider $\delta_a^*(2)$ to be the best one among the other modified predictors.

In Tables 3, we compute the values of the MSPE's of the BLUP ($\delta_L(1)$), ALUP ($\delta_L(2)$). These values of MSPE's are computed based on the moments of Rayleigh order statistics given in Govindarajulu and Joshi (1968). Table 3 also contains the efficiencies of $\delta_a^*(2)$ which are defined by

$$\text{EFF1} = \frac{\text{MSPE}(\delta_L(1))}{\text{MSPE}(\delta_a^*(2))},\ \text{and EFF2} = \frac{\text{MSPE}(\delta_L(2))}{\text{MSPE}(\delta_a^*(2))}.$$

From these tables, there is a clear evidence that the efficiencies of Type II MMLP are remarkably high when compared to the BLUP, and exceed that of the BLUP in most of the considered cases of r, s. However, the Type II MMLP is always more efficient than the ALUP in all these considered cases.

Table 1 Values of $E(Z_{i:n} - \frac{1}{Z_{i:n}})$ for Rayleigh Distribution

n	i	Value	n	i	Value	n	i	Value	n	i	Value
2	1	- 0.8862	6	1	- 2.5583	8	2	- 1.0254	9	8	1.3201
	2	0.8862		2	- 0.6604		3	- 0.3426		9	1.8756
3	1	- 1.4472		3	0.0253		4	0.1054	10	1	- 3.5670
	2	0.2357		4	0.5259		5	0.4776		2	- 1.3188
	3	1.2115		5	1.0123		6	0.8363		3	- 0.6189
4	1	- 1.8800		6	1.6552		7	1.2366		4	- 0.1851
	2	- 0.1489	7	1	- 2.8422		8	1.8138		5	0.1519
	3	0.6203		2	- 0.8547	9	1	- 3.3422		6	0.4483
	4	1.4085		3	- 0.1746		2	- 1.1788		7	0.7339
5	1	- 0.2420		4	0.2915		3	- 0.4887		8	1.0348
	2	- 0.4319		5	0.7018		4	- 0.0503		9	1.3914
	3	0.2756		6	1.1365		5	0.3007		10	1.9294
	4	0.8502		7	1.7417		6	0.6197			
	5	1.5481	8	1	- 3.1018		7	0.9445			

Table 2 Bias and MSPE of MMLP's and Associated Likelihood Ratios for Standard Rayleigh Distribution.

a. Sample Size 5

r	s	Bias(I)	Bias(II)	Bias(IV)	MSPE(I)	MSPE(II)	MSPE(IV)	Ratio(I)	Ratio(IV)
2	3	- 0.3077	- 0.0955	- 0.3077	0.1658	0.0982	0.1658	1.0000	1.0000
	4	- 0.3237	- 0.1158	- 0.2659	0.3118	0.2521	0.2764	1.1384	0.8252
	5	- 0.4024	- 0.1539	- 0.2977	0.6495	0.5837	0.5943	1.0828	0.8492
3	4	- 0.3499	- 0.0848	- 0.3499	0.2131	0.1169	0.2131	1.0000	1.0000
	5	- 0.2437	- 0.1137	- 0.3348	0.4291	0.3887	0.4443	1.1136	0.8153

b. Sample Size 10

r	s	Bias(I)	Bias(II)	Bias(IV)	MSPE(I)	MSPE(II)	MSPE(IV)	Ratio(I)	Ratio(IV)
3	4	- 0.1646	- 0.0505	- 0.1646	0.0486	0.0278	0.0486	1.0000	1.0000
	6	- 0.2301	- 0.0588	- 0.1328	0.1403	0.0991	0.1043	0.9905	0.8208
	8	- 0.2695	- 0.0767	- 0.1376	0.2690	0.2200	0.2207	0.9629	0.8653
	10	- 0.3524	- 0.1107	- 0.2339	0.6752	0.6051	0.6054	0.9970	0.8221
4	5	- 0.1608	- 0.0431	- 0.1608	0.0472	0.0267	0.0472	1.0000	1.0000
	7	- 0.1643	- 0.0492	- 0.1360	0.1241	0.0948	0.1027	1.0261	0.8009
	9	- 0.1960	- 0.0667	- 0.1491	0.2750	0.2380	0.2417	1.0014	0.8338
	10	- 0.2290	- 0.0803	- 0.2115	0.5247	0.4763	0.4829	1.0166	0.8003
5	6	- 0.1651	- 0.0390	- 0.1651	0.0503	0.0284	0.0503	1.0000	1.0000
	8	- 0.1327	- 0.0492	- 0.1509	0.1380	0.1126	0.1237	0.9954	0.7853
	10	- 0.1737	- 0.0720	- 0.2128	0.4284	0.3900	0.4056	0.9970	0.7848
6	8	- 0.0947	- 0.0408	- 0.2050	0.0994	0.0786	0.1089	0.9603	0.7291
	10	- 0.1403	- 0.0720	- 0.2286	0.3703	0.3392	0.3649	0.9653	0.7723
7	9	- 0.0893	- 0.0483	- 0.2434	0.1364	0.1137	0.1572	0.9261	0.7317
	10	- 0.1012	- 0.0506	- 0.2363	0.3151	0.2872	0.2872	0.9443	0.7628

Table 3 Efficiencies of the Type II MMLP When Compared to Linear Predictors for the Sample Sizes 5, 10 from the Rayleigh Distribution, Assuming $\sigma=1$.

			MSPE's			EFFICIENCIES	
n	r	s	$\delta_L(1)$	$\delta_L(2)$	$\delta^*_{a}(2)$	EFF1	EFF2
5	2	3	0.0974	0.1125	0.0982	0.9919	1.1456
		4	0.2615	0.2832	0.2521	1.0373	1.1234
		5	0.6250	0.6596	0.5837	1.0708	1.1300
	3	4	0.1150	0.1415	0.1169	0.9837	1.2104
		5	0.3911	0.4269	0.3887	1.0062	1.0983
10	3	4	0.0273	0.0387	0.0278	0.9820	1.3921
		6	0.1012	0.1192	0.0991	1.0212	1.2028
		8	0.2263	0.2554	0.2200	1.0286	1.1609
		10	0.6142	0.6688	0.6051	1.0150	1.1053
	4	5	0.0258	0.0401	0.0267	0.9663	1.5019
		7	0.0999	0.1206	0.0948	1.0538	1.2722
		9	0.2569	0.2880	0.2380	1.0794	1.2100
		10	0.4745	0.5248	0.4763	0.9962	1.1018
	5	6	0.0270	0.0443	0.0284	0.9507	1.5599
		8	0.1137	0.1365	0.1126	1.0098	1.2123
		10	0.4060	0.4400	0.3900	1.0410	1.1282
	6	8	0.0742	0.1031	0.0786	0.9440	1.3117
		10	0.3233	0.3605	0.3392	0.9531	1.0628
	7	9	0.1049	0.1433	0.1137	0.9226	1.2603
		10	0.2859	0.3130	0.2872	0.9955	1.0898

References:

Balakrishnan, N. and Clifford, A.C. (1991). *Order Statistics and Inference Estimation methods*, Boston: Academic Press.

David, H. A. (1981). *Order Statistics*, 2nd Edition, New York: John Wiley.

Goldberger, A. S. (1962). Best Linear Unbiased Prediction in the Generalized Linear Regression Model, *J. Amer. Statist. Ass.* 57, 369- 75.

Devroye, L. (1986). *Non-Uniform Random Variate Generation*, New York: Springer-Verlag.

Govindarajulu, Z. and Joshi, M. (1968). Best Linear Unbiased Estimation of Location and Scale Parameters of Weilbull Distribution Using Ordered Observations, *Rep. Stat. App. Res.*, Juse 15, 1-14.

Gupta, A. K. (1952). Estimation of the Mean and Standard Deviation of Normal Population from a Censored Sample, *Biometrika* 39, 260- 73.

Harter, H. L. and Moore, A. H. (1965). Point and Interval Estimators Based on m Order Statistics for the Scale Parameter for a Weibul Population with Known Shape Parameter, *Technometrics* 7, 405-22.

Johnson, R. A. and Mehrotra, K. G. (1972). Locally Most Powerful Rank Tests for the Two Sample Problem with Censored Data, *Ann. Math. Statist.* 3, 823-31.

Kaminsky, K. S. and Nelson, P.I. (1975). Best Linear Unbiased Prediction of Order Statistics in Location and Scale Families, *J. Amer. Statist. Ass.* 70,145- 50.

Kaminsky, K. S. and Rhodin, L. S. (1985). Maximum Likelihood Prediction, *Ann. Inst. Statist. Math.* 37, Part A, 707- 17.

Mehrotra, K. G. and Nanda, P. (1974). Unbiased Estimation of Parameters by Order Statistics in the Case of Censored Samples, *Biometrika* 61, 601-606.

Raqab, M. Z. (1993). Modified Maximum Likelihood predictors of Future Order Statistics from Normal Samples, Submitted for Publication.

Raqab, M. Z. (1993). Linear Predictors of Future Order Statistics, Unpublished Manuscipt.

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