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RECURRENCE RELATIONS BETWEEN ANY CONTINUOUS FUNCTION OF ORDER STATISTICS FOR DOUBLY TRUNCATED CONTINUOUS DISTRIBUTIONS AND ITS APPLICATIONS

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ABSTRACT

General results for obtaining recurrence relations for doubly truncated continuous distributions, are established, in the single and joint order statistics. We also present two applications. The first application is: recurrence relations between factorial generating functions, the examples considered are exponential, logistic, extreme-value, and Laplace distributions. The second application is: some recurrence relations between reliability and failure rate functions.

1-INTRODUCTION

Khan et al (1983a,1983b,1987), obtained general results for obtaining recurrence relations for moments of order statistics from doubly truncated continuous distributions and they derived the single and product moments of order statistics from doubly truncated Burr distribution and its characterizations. Mohie El-Din et al (1991a,1991b,1992) obtained the single and product moments of order statistics from doubly truncated parabolic, skewed and U-shape distributions, they also characterized the Weibull, Pareto and the power function distributions. They also obtained some results for obtaining recurrence relations between single moment generating functions and between moment generating functions of the sum of two order statistics from doubly truncated continuous distributions and presented some applications for these results to exponential, logistic, extreme-value and Weibull distributions. In this paper, we derived some general results for obtaining recurrence relations between the expectations of any continuous function in both, single and joint order statistics. We also specialized these results to the factorial generating functions, these results are applied to exponential, logistic, extreme-value, and Laplace distributions. Other applications in life testing are also given.

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics from a continuous distribution function (c.d.f) F(x) and probability density function (p.d.f) f(x). Let,

$g_{r:n}(x) = g(x_{r:n})$, $\quad 1 \le r \le n$

any continuous functions of $X_{r:n}$.

$\zeta_{r,s:n}(x,y) = \zeta(x_{r:n}, x_{s:n})$, $\quad 1 \le r \le s \le n$, $(X_{r:n} = X, X_{s:n} = Y)$

any continuous functions of $X_{r:n}$ and $X_{s:r}$.

$\psi_{r:n}(t) = \psi_{X_{r:n}}(t)$, $\quad 1 \le r \le n$

[see (1.3)].

$\psi_{r,s:n}(t_1,t_2) = \psi_{X_{r:n},X_{s:n}}(t_1,t_2)$, $\quad 1 \le r \le s \le n$

[see (1.6)].

$R_{r:n}(x) = R(x_{r:n})$ $\quad 1 \le r \le n$

$h_{r:n}(x) = h(x_{r:n})$ $\quad 1 \le r \le n$

then,

$$E(g_{r:n}(x)) = C_{r:n} \int_{-\infty}^{\infty} g(x)[F(x)]^{r-1}[1-F(x)]^{n-r} f(x)dx \,, \tag{1.1}$$

where,

$$C_{r:n} = \frac{n!}{(n-r)!(r-1)!} \,, \tag{1.2}$$

putting $g(x) = t^x$ in (1.1) yields,

$$\psi_{r:n}(t) = C_{r:n} \int_{-\infty}^{\infty} t^x [F(x)]^{r-1}[1-F(x)]^{n-r} f(x)dx \,, \tag{1.3}$$

which, is the factorial generating function in the order statistics [see Patel et.al. (1976)]. Let $R_{r:n}(x) = 1 - F(x_{r:n})$ denote the reliability (survival) function. Thus, the failure rate function at $X_{r:n}$ is given by,

$$h_{r:n}(x) = \frac{f(x_{r:n})}{1 - F(x_{r:n})} \,,$$ [see Nagaraja, (1990)].

Given two integers r and s such that ($1 \le r < s \le n$), then

$$E(\zeta_{r,s:n}(x,y)) = C_{r,s:n} \int_{-\infty}^{\infty} \int_{x}^{\infty} \zeta(x,y)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)f(y)dydx, \quad x \le y, \tag{1.4}$$

where,

$$C_{r,s:n} = \frac{n!}{(r-1)!(n-s)!(s-r-1)!} \,. \tag{1.5}$$

Putting $\zeta(x,y) = t_1^x \, t_2^y$ in (1.4), yields,

$$\psi_{r,s:n}(t_1,t_2) = C_{r,s:n} \int_{-\infty}^{\infty} \int_{x}^{\infty} t_1^x \, t_2^y \, [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)f(y)dydx, \quad x \le y, \tag{1.6}$$

which is the joint factorial generating function in the order statistics.

The doubly truncated p.d.f $f_t(x)$ at Q_1 and P_1, where $Q_1 \leq P_1$ is,

$$f_t(x) = \frac{f(x)}{P - Q}, \quad Q_1 \leq x \leq P_1, \tag{1.7}$$

where, $F(Q_1) = Q$ and $F(P_1) = P$ [see Mood et al (1974)].

Remark: Truncation can be utilized to form other families of distributions.

2 - Recurrence relations between the expected values for any continuous function of order statistics.

(I): For doubly truncated continuous distributions, the equation (1.1) takes the form,

$$E(g_{r:n}(x)) = C_{r:n} \int_{Q_1}^{P_1} g(x)[F_t(x)]^{r-1}[1-F_t(x)]^{n-r} f_t(x)dx \tag{2.1}$$

where, $f_t(.)$ is given by (1.7) and $F_t(.)$ is the corresponding c.d.f.

Theorem (2.1)

For any arbitrary continuous distribution and for any continuous function $g_{r:n}(x)$, $2 \leq r \leq n$, $Q_1 \leq X_{r:n} \leq P_1$, we have,

$$E(g_{r:n}(x)) - E(g_{r-1:n-1}(x)) = \frac{C_{r:n}}{n} \int_{Q_1}^{P_1} [F_t(x)]^{r-1}[1-F_t(x)]^{n-r+1} dg(x), \tag{2.2}$$

where, $C_{r:n}$ is given by (1.2).

Proof: From (2.1), one can see that,

$$E(g_{r:n}(x)) - E(g_{r-1:n-1}(x)) = \frac{C_{r:n}}{n} \int_{Q_1}^{P_1} g(x)[F_t(x)]^{r-2}[1-F_t(x)]^{n-r} [nF_t(x) - (r-1)]f_t(x)dx, \tag{2.3}$$

let $\tau(x) = -[F_t(x)]^{r-1}[1-F_t(x)]^{n-r+1}$, (2.4)

hence $d\tau(x) = [F_t(x)]^{r-2}[1-F_t(x)]^{n-r}[nF_t(x) - (r-1)]f_t(x)dx$, (2.5)

form (2.3) and (2.5) we get,

$$E(g_{r:n}(x)) - E(g_{r-1:n-1}(x)) = \frac{C_{r:n}}{n} \int_{Q_1}^{P_1} g(x)d\tau(x). \tag{2.6}$$

Integrating by parts we get,

$$\int_{Q_1}^{P_1} g(x)d\tau(x) = g(x)\tau(x)\Big|_{Q_1}^{P_1} - \int_{Q_1}^{P_1} \tau(x)dg(x) = -\int_{Q_1}^{P_1} \tau(x)dg(x), \tag{2.7}$$

form (2.4), (2.6) and (2.7), the proof is complete.

Remarks: 1- putting $g(x) = \exp(tx)$ in (2.2), yields the results by Mohie El-Din et.al.(1992).

2- putting $g(x) = x^k$ in (2.2), yields the results by Khan el.al. (1983a).

(II) For doubly truncated continuous distributions, the equation (1.4) takes the form,

$$E(\zeta_{r,s:n}(x,y)) = C_{r,s:n}\int_{Q_1}^{P_1}\int_{x}^{P_1}\zeta(x,y)[F_t(x)]^{r-1}[F_t(y)-F_t(x)]^{s-r-1}[1-F_t(y)]^{n-s} f_t(x)f_t(y)dydx,\quad x\le y, \tag{2.8}$$

Theorem (2.2)

For any arbitrary continuous distribution and for any continuous function $\zeta(X,Y)$, $1\le r\le s\le n-1$, $Q_1\le x\le y\le P_1$, we have,

$$E(\zeta_{r,s:n}(x,y)) - E(\zeta_{r,s-1:n}(x,y)) = \frac{C_{r,s:n}}{n-s+1}\int_{Q_1}^{P_1}\int_{x}^{P_1}[F_t(x)]^{r-1} [F_t(y)-F_t(x)]^{s-r-1}[1-F_t(y)]^{n-s+1}f_t(x)d\zeta_x(x,y)dx, \tag{2.9}$$

where, $C_{r,s:n}$ is given by (1.5) and $d\zeta_x(x,y)$ means that the differentiation is with respect to y.

Proof:

From (2.8), one can see that,

$$E(\zeta_{r,s:n}(x,y)) - E(\zeta_{r,s-1:n}(x,y)) = \frac{C_{r,s:n}}{n-s+1}\int_{Q_1}^{P_1}\int_{x}^{P_1}\zeta(x,y)[F_t(x)]^{r-1} [F_t(y)-F_t(x)]^{s-r-2}[1-F_t(y)]^{n-s}\,[(n-r)F_t(y)-(n-s+1)F_t(x)-(s-r-1)] f_t(x)f_t(y)dydx, \tag{2.10}$$

let $Z(x,y) = -[F_t(y) - F_t(x)]^{s-r-1}[1 - F_t(y)]^{n-s+1}$, (2.11)

then,

$$dZ_x(x,y)=[F_t(y)-F_t(x)]^{s-r-2}[1-F_t(y)]^{n-s} [(n-r)F_t(y)-(s-r-1)-(n-s+1)F_t(x)]f_t(y)dy. \tag{2.12}$$

From (2.10) and (2.12) we get

$$E(\zeta_{r,s:n}(x,y))-E(\zeta_{r,s-1:n}(x,y))=\frac{C_{r,s:n}}{n-s+1}\int_{Q_1}^{P_1}[F_t(x)]^{r-1}[\int_{x}^{P_1}\zeta(x,y)dZ_x(x,y)] f_t(x)dx. \tag{2.13}$$

Integrating by parts yields,

$$\int_{x}^{P_1} \zeta(x,y)\,dZ_x(x,y) = -\int_{x}^{P_1} Z(x,y)\,d\zeta_x(x,y), \tag{2.14}$$

From (2.11), (2.13) and (2.14), the proof is complete.

Remarks: 1- Putting $\zeta(x,y) = \exp(t(x+y))$ in (2.9), yields the results of Mohie EL-Din et. al. (1992).

2- Putting $\zeta(x,y)=X^jY^k$ in (2.9), yields the results of Khan el. al. (1983b).

3 - Application (I): Recurrence relations between factorial generating functions.

The results in (2.2) and (2.9) can be specialized to factorial generating functions as follows,

Proposition (3.1):

For any continuous distribution function, $2 \le r \le n$, $Q_1 \le x_r \le P_1$, the factorial generating function $\psi_{r:n}(t)$ satisfy,

$$\psi_{r:n}(t) - \psi_{r-1:n-1}(t) = \frac{C_{r:n}}{n}(\ln(t))\int_{Q_1}^{P_1} t^x [F_t(x)]^{r-1}[1-F_t(x)]^{n-r+1}dx, \tag{3.1}$$

Proof: Putting $g(x) = t^x$ in theorem (2.1), we get the result.

Proposition (3.2):

For any continuous distribution function, $1 \le r \le s \le n$, $Q_1 \le x_r \le P_1$, the factorial generating function $\psi_{r,s:n}(t_1,t_2)$ satisfy,

$$\psi_{r,s:n}(t_1,t_2) - \psi_{r,s-1:n}(t_1,t_2) = \frac{C_{r,s:n}}{n-s+1}(\ln(t_2))\int_{Q_1}^{P_1}\int_{x}^{P_1} t_1^x t_2^y\,[F_t(x)]^{r-1}$$
$$[F_t(y) - F_t(x)]^{s-r-1}[1-F_t(y)]^{n-s+1} f_t(x)\,dy\,dx, \tag{3.2}$$

Proof: Putting $\zeta(x,y) = t_1^x t_2^y$ in theorem (2.2), we get the result.

Letting $t_1 = t_2 = t$ in (3.2), yields the recurrence relations between the factorial generating function of the sum of two order statistics.

Proposition (3.3):

For any continuous distribution function, the factorial generating function $\psi_{r:n}(t)$, satisfies

$$r\psi_{r+1:n}(t) = n\psi_{r:n-1}(t) - (n-r)\psi_{r:n}(t), \tag{3.3}$$

Proof: Since for $1 \le r \le n$

$$rE[B(x_{r+1:n})] = nE[B(x_{r:n-1})] - (n-r)E[B(x_{r:n})], \quad (3.4)$$

[see Patel et.al.(1976)],

putting $B(x)=t^x$ in (3.4), then the proof is complete.

In the following some examples are presented.

Example(3.1): Doubly truncated exponential distribution.

The p.d.f of doubly truncated exponential distribution is,

$$f_t(x) = \frac{e^{-x/\theta}}{\theta(P-Q)}, \quad Q_1 \le x \le P_1, \ \theta > 0, \quad (3.5)$$

where, $1-P = e^{-P_1/\theta}$ and $1-Q = e^{-Q_1/\theta}$.

Let $P_2 = \frac{1-P}{(P-Q)}$ and $Q_2 = \frac{1-Q}{(P-Q)}$, $1+P_2 = Q_2$, then,

$$1 - F_t(x) = -P_2 + \theta f_t(x). \quad (3.6)$$

First, we derive a recurrence relation for $\psi_{r:n}(t)$.

From (3.1) and using (3.6), one can see that,

$$(1 - \frac{\theta \ln(t)}{n})\psi_{r:n}(t) = \frac{-P_2(n-1)}{(n-r)}[\psi_{r:n-1}(t) - \psi_{r-1:n-2}(t)] + \psi_{r-1:n-1}(t),$$
$$n \ne \theta \ln(t). \quad (3.7)$$

Using (3.3) in (3.7), we get,

$$(1 - \frac{\theta \ln(t)}{n})\psi_{r:n}(t) = -P_2\psi_{r:n-1}(t) + Q_2\psi_{r-1:n-1}(t), \quad n \ne \theta \ln(t). \quad (3.8)$$

Next, we obtain a recurrence relation for $\psi_{r,s:n}(t_1,t_2)$.

From (3.6), we have,

$1 - F_t(y) = -P_2 + \theta f_t(y)$ and substituting with it in (3.2), one can see that,

$$[1 - \frac{\theta \ln(t_2)}{n-s+1}]\psi_{r,s:n}(t_1,t_2) = \psi_{r,s-1:n}(t_1,t_2) - \frac{nP_2}{n-s+1}[\psi_{r,s:n-1}(t_1,t_2)$$
$$- \psi_{r,s-1:n-1}(t_1,t_2)], \ \theta \ln(t_2) \ne n-s+1,$$
$$1 \le r \le s \le n-1. \quad (3.9)$$

Example (3.2): Doubly truncated logistic distribution.

The p.d.f of doubly truncated logistic distribution is

$$f_t(x) = \frac{e^{-(x-\alpha)/\beta}}{\beta(P-Q)[1+e^{-(x-\alpha)/\beta}]^2}, \quad Q_1 \le x \le P_1, \ -\infty < \alpha < \infty, \ \beta > 0, \quad (3.10)$$

where, $\frac{1}{P} - 1 = e^{-(P_1-\alpha)/\beta}$ and $\frac{1}{Q} - 1 = e^{-(Q_1-\alpha)/\beta}$.

Let $P_2 = \frac{P}{P-Q}$ and $Q_2 = \frac{Q}{P-Q}$, from (3.10), we have

$$1 - F_t(x) = P_2 - \beta\, e^{(x-\alpha)/\beta} f_t(x) - \beta\, f_t(x). \tag{3.11}$$

First, we obtain a recurrence relation for $\psi_{r:n}(t)$.
Using (3.11) and (3.3) in (3.1), one can see that,

$$(1+ \frac{\beta\, \ln(t)}{n})\, \psi_{r:n}(t) = P_2\, \psi_{r:n-1}(t) - Q_2\, \psi_{r-1:n-1}(t)$$
$$- \frac{\beta\, e^{-\alpha/\beta} \ln(t)}{n}\, \psi_{r:n}(te^{1/\beta}),\quad 1 \le r \le n-1. \tag{3.12}$$

Next, we obtain the recurrence relation for $\psi_{r:s:n}(t_1, t_2)$.
From (3.11), we get

$$1 - F_t(y) = P_2 - \beta\, e^{(y-\alpha)/\beta} f_t(y) - \beta\, f_t(y). \tag{3.13}$$

From (3.2) and (3.13), one can see that.

$$[1+ \frac{\beta\, \ln(t_2)}{n-s+1}]\, \psi_{r,s:n}(t_1, t_2) = \frac{nP_2}{n-s+1}\, [\, \psi_{r,s:n-1}(t_1, t_2) - \psi_{r,s-1:n-1}(t_1, t_2)]$$
$$- \frac{\beta e^{-\alpha/\beta} \ln(t_2)}{n-s+1}\, \psi_{r,s:n}(t_1, t_2 e^{1/\beta}),\ 1 \le r \le s \le n-1. \tag{3.14}$$

Example (3.3): Doubly truncated extreme-value distribution.

The p.d.f of doubly truncated extreme-value distribution is,

$$f_t(x) = \frac{EXP(-x-e^{-x})}{P-Q}, \qquad Q_1 \le x \le P_1, \tag{3.15}$$

where, $P= EXP(-e^{-P_1})$ and $Q= EXP(-e^{-Q_1})$.

Let $P_2= \frac{P}{P-Q}$ and $Q_2= \frac{Q}{P-Q}$, then by (3.15), we have

$$1 - F_t(x) = P_2 - e^{x} f_t(x). \tag{3.16}$$

First, we obtain a recurrence relation for $\psi_{r:n}(t)$.
Using (3.16) and (3.3) in (3.1), yields,

$$\psi_{r:n}(t) + \frac{\ln(t)}{n}\, \psi_{r:n}(te) = P_2 \psi_{r:n-1}(t) - Q_2 \psi_{r-1:n-1}(t). \tag{3.17}$$

Next, we obtain a recurrence relation for $\psi_{r,s:n}(t_1, t_2)$.

From (3.16) we have, $1 - F_t(y) = P_2 - e^{y} f_t(y)$. (3.18)

It follows from (3.18) and (3.2), that

$$\psi_{r,s:n}(t_1,t_2) = \psi_{r,s-1:n}(t_1,t_2) + \frac{nP_2}{n-s+1}\,[\psi_{r,s:n-1}(t_1,t_2) - \psi_{r,s-1:n-1}(t_1,t_2)$$
$$- \frac{\ln(t_2)}{n-s+1}\,\psi_{r,s:n}(t_1,t_2 e). \qquad (3.19)$$

Example (3.4): Doubly truncated Laplace distribution.

The p.d.f of doubly truncated Laplace distribution is,

$$f_t(x) = \frac{e^{-\frac{|x-\alpha|}{\beta}}}{2\beta(P-Q)}, \qquad Q_1 \le x < P_1 \qquad (3.20)$$

where,

$$P = \begin{cases} 1 - \frac{1}{2} e^{-(P_1-\alpha)/\beta} & , P_1 > \alpha \\ \frac{1}{2} e^{(P_1-\alpha)/\beta} & , P_1 \le \alpha \end{cases}$$

and

$$Q = \begin{cases} 1 - \frac{1}{2} e^{-(Q_1-\alpha)/\beta} & , Q_1 > \alpha \\ \frac{1}{2} e^{(Q_1-\alpha)/\beta} & , Q_1 \le \alpha . \end{cases}$$

Let $P_2 = \frac{P}{P-Q}$, $Q_2 = \frac{Q}{P-Q}$, $P_3 = \frac{P-1}{P-Q}$ and $Q_3 = \frac{Q-1}{P-Q}$.

From (3.20), we have

$$1 - F_t(x) = \begin{cases} \int_x^{P_1} \frac{e^{(x-\alpha)/\beta}}{2\beta(P-Q)}\,dx & x \le \alpha \\ \int_x^{P_1} \frac{e^{-(x-\alpha)/\beta}}{2\beta(P-Q)}\,dx, & x > \alpha \end{cases}$$

$$= \begin{cases} P_2 - \beta f_t(x) & , \quad x \le \alpha \\ P_3 + \beta f_t(x) & , \quad x > \alpha. \end{cases} \qquad (3.21)$$

First, we derive a recurrence relation for $\psi_{r:n}(t)$,

Using (3.21) and (3.3) in (3.1), one can see that,

$$\left.\begin{aligned} &(1 + \frac{\beta\ln(t)}{n})\psi_{r:n}(t) = P_2\psi_{r:n-1}(t) - Q_2\psi_{r-1,n-1}(t), \qquad x \le \alpha \\ &\text{and} \\ &(1 - \frac{\beta\ln(t)}{n})\psi_{r:n}(t) = P_3\psi_{r:n-1}(t) - Q_3\psi_{r-1,n-1}(t),\ n \ne \beta\ln(t),\ x > \alpha. \end{aligned}\right\} (3.22)$$

Next, we obtain a recurrence relation for $\psi_{r,s:n}(t_1,t_2)$.

From (3.21), we can write,

$$1 - F_t(y) = \begin{cases} P_2 - \beta f_t(y) , & x \le \alpha \\ P_3 + \beta f_t(y) , & x > \alpha. \end{cases} \tag{3.23}$$

From (3.23) and (3.2), one can see that,

$$\left.\begin{array}{l} (1+ \frac{\beta \ln(t_2)}{n-s+1})\psi_{r,s:n}(t_1,t_2) = \psi_{r,s-1,n}(t_1,t_2) + \\ \frac{np_2}{n-s+1} [\psi_{r,s:n-1}(t_1,t_2) - \psi_{r,s-1:n-1}(t_1,t_2)], \quad x \le \alpha \\ \text{and} \\ (1- \frac{\beta \ln(t_2)}{n-s+1})\psi_{r,s:n}(t_1,t_2) = \psi_{r,s-1,n}(t_1,t_2) + \\ \frac{np_3}{n-s+1} [\psi_{r,s:n-1}(t_1,t_2) - \psi_{r,s-1:n-1}(t_1,t_2)], \\ \qquad \beta \ln(t_2) \ne n-s+1, \quad x > \alpha. \end{array}\right] \tag{3.24}$$

4- Application (II): Recurrence relations in life testing.

Theorem (4.1)

For any arbitrary continuous distribution, the reliability function $R_{r:n}(x)$, satisfies,

$$E(R_{r:n}(x)) - E(R_{r-1:n-1}(x)) = \frac{-(n-r+1)}{n(n+1)}, \; 0 \le x_r \le 1, \; 2 \le r \le n.$$

Proof:

Putting $g(x)=R(x)= 1- F_t(x)$,$Q_1=0$ and $P_1=1$ in theorem (2.1), we have,

$$E(R_{r:n}(x)) - E(R_{r-1:n-1}(x)) = \frac{-C_{r:n}}{n}\int_0^1 [F_t(x)]^{r-1}[1-F_t(x)]^{n-r+1}d(1-F_t(x))$$
$$= \frac{-C_{r:n}}{n} \beta(r,n-r+2), \tag{4.1}$$

from (1.2) and (4.1), the proof is complete.

Theorem (4.2)

For any arbitrary continuous distribution, the failure rate function $h_{r:n}(x)$, satisfies,

$$(n-1)E(h_{r:n}(x)) - nE(h_{r-1:n-1}(x)) = \frac{n(n-1)}{(n-r)}[E[f_t(x_{r:n-1})] - E[f_t(x_{r-1:n-2})],$$
$$0 \le x_r \le 1 \text{ and } 2 \le r \le n. \tag{4.2}$$

Proof:

Putting $g(x) = h(x) = \frac{f_t(x)}{1-F_t(x)}$, $Q_1 = 0$ and $P_1 = 1$ in theorem (2.1), we have,

$$E(h_{r:n}(x)) - E(h_{r-1:n-1}(x)) = \frac{C_{r:n}}{n}\int_0^1 [F_t(x)]^{r-1}[1-F_t(x)]^{n-r+1}dh(x),$$

but, $dh(x) = (1 - F_t(x))^{-1} df_t(x) + h(x)f_t(x)(1 - F_t(x))^{-1} dx$, then,

$$E(h_{r:n}(x)) - E(h_{r-1:n-1}(x)) = \frac{C_{r:n}}{n}\int_0^1 [F_t(x)]^{r-1}[1-F_t(x)]^{n-r}df_t(x),$$

$$+ \frac{C_{r:n}}{n}\int_0^1 h(x)[F_t(x)]^{r-1}[1-F_t(x)]^{n-r}f_t(x)dx.$$

Hence,

$$(n-1)E(h_{r:n}(x)) - nE(h_{r-1:n-1}(x)) = \frac{n(n-1)}{(n-r)}[E[f_t(x_{r:n-1})] - E[f_t(x_{r-1:n-2})]].$$

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