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Exact Test for Diagonals-Parameter Symmetry Model in a 3×3 Table with Ordered Categories

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Summary

This short note gives an exact test for Goodman's (1979) diagonals-parameter symmetry model in a 3×3 table with ordered categories, which can be easily conducted applying Fisher's exact test of independence for a 2×2 table. An example is given.

Key words: Conditional distribution; Fisher's exact test.

1. Introduction

Fisher's exact test of independence in a 2×2 table is widely known (e.g., Plackett, 1981, p.47). For the analysis of square contingency tables with ordered categories, Goodman (1979) proposed the diagonals-parameter symmetry (DPS) model. The purpose of this short note is to give an exact test for the DPS model in a 3×3 table, which can be easily conducted applying Fisher's exact test.

2. Exact test for diagonals-parameter symmetry model

Consider a 3×3 table with cell probabilities $\{p_{ij}\}$. The DPS model is defined by

$$p_{ij} = \begin{cases} \tau_{j-i}\omega_{ij} & \text{for } i<j, \\ \omega_{ij} & \text{for } i\geq j, \end{cases}$$

where $\omega_{ij}=\omega_{ji}$. This model is equivalent to

$$\psi = 1,$$

where $\psi=(p_{12}p_{32})/(p_{21}p_{23})$. Suppose the cell counts $\{n_{ij}\}$ have a multinomial distribution with $n=\Sigma\Sigma n_{ij}$ fixed. The probability function of $\{n_{ij}\}$ is

$$P(\{n_{ij}\}|n) = \frac{n!}{\Pi\Pi n_{ij}!}\Pi\Pi p_{ij}^{n_{ij}}$$

$$= \frac{n!}{(\prod_{k=1}^{3} n_{kk}!)n_{13}!n_{31}!n_{1\cdot}^{*}!n_{2\cdot}^{*}!} \gamma^{n}(\prod_{k=1}^{3}\theta_k^{n_{kk}})\delta_1^{n_{13}}\delta_2^{n_{31}}$$

$$\times(\prod_{s=1}^{2}\phi_s^{n_{s\cdot}^{*}}\lambda_s^{n_{\cdot s}^{*}})h(n_{12};\{n_{i\cdot}^{*}\},\{n_{\cdot j}^{*}\},\psi),$$

where $n_{1\cdot}^{*}=n_{12}+n_{23}$, $n_{2\cdot}^{*}=n_{21}+n_{32}$, $n_{\cdot 1}^{*}=n_{12}+n_{21}$, $n_{\cdot 2}^{*}=n_{23}+n_{32}$,

$\gamma=p_{32}$, $\theta_k=p_{kk}/p_{32}$, $\delta_1=p_{13}/p_{32}$, $\delta_2=p_{31}/p_{32}$,

$\phi_1=p_{23}/p_{32}$, $\phi_2=1$, $\lambda_1=p_{21}/p_{32}$, $\lambda_2=1$,

and $$h(n_{12};\cdot,\psi) = \binom{n_{1\cdot}^{*}}{n_{12}}\binom{n_{2\cdot}^{*}}{n_{\cdot 1}^{*}-n_{12}}\psi^{n_{12}}.$$

[Note that γ is a function of $\{\theta_k\}$, $\{\delta_s\}$, $\{\phi_s\}$, $\{\lambda_s\}$ and ψ.] The sufficient statistics for the nuisance parameters are $\{n_{kk}\}$, n_{13}, n_{31}, $\{n_{s\cdot}^{*}\}$ and $\{n_{\cdot t}^{*}\}$. Given them (then n_{12} determines all other cell counts), the conditional probability function of n_{12} is

$$P(n_{12}|\{n_{kk}\},n_{13},n_{31},\{n^*_{s\cdot}\},\{n^*_{\cdot t}\}) = h(n_{12};\cdot,\psi)/\sum_{u=A}^{B} h(u;\cdot,\psi), \tag{2.1}$$

where n_{12}=A,A+1,$\cdots$,B; A=max(0, $n^*_{\cdot 1}-n^*_{2\cdot}$) and B=min($n^*_{1\cdot}$, $n^*_{\cdot 1}$). The exact test for the DPS model could be conducted using this conditional distribution. When ψ=1, (2.1) may be expressed as

$$P(n_{12}|\bullet) = \binom{n^*_{1\cdot}}{n_{12}}\binom{n^*_{2\cdot}}{n^*_{\cdot 1}-n_{12}}\Big/\binom{n^*_{1\cdot}+n^*_{2\cdot}}{n^*_{\cdot 1}}.$$

Denote the 2×2 table with the first row (p_{12}, p_{23}) and the second row (p_{21}, p_{32}) by Δ, and the 2×2 table with the first row (n_{12},n_{23}) and the second row (n_{21},n_{32}) by Δ^*. We note that (i) the DPS model applied to the original table is equivalent to the independence model applied to the Δ table, and (ii) the method of the exact test for the DPS model based on (2.1) is identical to the method of Fisher's exact test of independence applied to the Δ^* table.

3. An example

The data in Table 1 is taken from Bishop et al.(1975,p.288).

Table 1. Cross-Classification of Ewes According to Number of Lambs Born in Consecutive Years

Number of Lambs 1953	Number of Lambs 1952			Totals
	0	1	2	
0	58	52	1	111
1	26	58	3	87
2	8	12	9	29
Totals	92	122	13	227

An exact test of H_0: $\psi=1$ against H_1: $\psi\neq1$ is conducted. We get

$$h(u;\cdot,\psi) = \binom{55}{u}\binom{38}{78-u}\psi^u \qquad \text{for } u=40,41,\cdots,55.$$

Next, we can form the following table.

<u>Table 2.</u> Values of $P(n_{12}|\{n_{kk}\},n_{13},n_{31},\{n^*_{s\cdot}\},\{n^*_{\cdot t}\})$ with $\psi=1$, given $\{n_{kk}\},n_{13},n_{31},\{n^*_{s\cdot}\},\{n^*_{\cdot t}\}$ in Table 1

n_{12}	$P(n_{12}\mid\circ)$
40	0.0001523
41	0.0021178
42	0.0130599
43	0.0473801
44	0.1130661
45	0.1879411
46	0.2247121
47	0.1967085
48	0.1270409
49	0.0604957
50	0.0210525
51	0.0052538
52	0.0009093
53	0.0001029
54	0.0000068
55	0.0000002

From Table 2, for the usual two-sided test at 5% level of significance (whose critical region is determined such that each tail area is at most 0.025), the critical region consists of the values 40,41,42,51,52,53,54,55 of n_{12} (the actual significance level is less than 0.05.) Since n_{12} for the observed table is 52, we now reject H_0 (twice the minimum of the exact upper and lower p-values is 0.0020384). Therefore, the odds that the number of lambs born to a ewe in 1953 is more, instead of less, than the number in 1952 when a pair of numbers in 1952 and 1953 is (0,1), is *not* equal to the odds when a pair of numbers in 1952 and 1953 is (1,2).

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