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ALTERNATIVE APPROXIMATION TO THE EXTREME NORMAL TAIL PROBABILITY AND ITS INVERSE

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ABSTRACT

This paper proposes a simple approximation to the normal tail probability and its inverse.Results show that it works well on a larger interval compared with similar methods.Its accuracy is better than that of other schemes especially in approximating smaller values of the normal tail probability and its inverse.

Keywords: *Approximation;Normal tail probability;Inverse cumulative normal distribution.*

INTRODUCTION

The normal tail probabilities and its inverse has practical and theoritical importance.There are Many sources,such as Hamaker(1978);Lin (1989);Lin(1990) and Revfiem(1990),in which approximating the standard normal tail probability and its inverse has appeared.Let Z be the standard normal random variable with density and distribution functions $\phi(z)$ and $\Phi(z)$ respectively, $p=Q(z)=1-\Phi(z)$ and $z=Q^{-1}(p)$.Hamaker (1978) proposes the following simple approximation to the normal tail probability and its inverse denoted by p_1 and z_1:

$$p_1=1/2[1-\{1-\exp(-y^2)\}^{1/2}], \tag{1}$$

with $y=0.806\ z(1-0.018z)$ and

$$z_1=1.238y(1+0.0262y), \tag{2}$$

where $y=[-\ln\{4p(1-p)\}]^{1/2}$. Another simple approximation to p and z, denoted by p_2 and z_2 is given by Lin(1989) where

$$p_2=1/2\ \exp(bz+az^2), \tag{3}$$

with a=-0.416 and b=-0.717, and

$$z_2=[-b-\sqrt{(b^2-4ac)}]/2a, \tag{4}$$

with c=-ln(2p). Lin(1990) offers a simpler logistic approximation to p and z denoted by p_3 and z_3 where

$$p_3=(1+\exp\ y)^{-1}, \tag{5}$$

with $y=4.2\pi z/(9-z)$, and

$$z_3=9y/(4.2\pi+y), \tag{6}$$

where y=ln(1/p-1). Revfiem(1990) gives an approximation up to z=3 and suggests to use the approximation given by Feller(1968) for z>3. Revfiem's approximation denoted by p_4 and z_4 is given by

$$p_4=\begin{cases} 1-1/\{1+\exp(-2y)\}, & z\leq 3 \\ \phi(z)/z, & z>3 \end{cases} \tag{7}$$

with $y=z\{(2/\pi)^{1/2}+z^2/28\}$, and

$$z_4=\begin{cases} v-7.45/v, & z\leq 3 \\ 2r[\{1-(r^2-1)c/2\}^{1/2}-1]/(r^2-1), & z>3 \end{cases} \tag{8}$$

with $v=[14\{(2.11+s^2)^{1/2}-s\}]^{1/3}$, $s=0.5\ \ln\{(1-\Phi)/\Phi\}$;

$c=\ln\{r(2\pi)^{1/2}(1-\Phi)\}-3/2$ and $r=1-\log_{10}(1-\Phi)$.

PROPOSED MODEL AND CONCLUSIONS

This paper proposes the following approximation to p and z denoted by p_5 and z_5 where

$$p_5=1/2\ \exp(bz^d+az^{2d}) \tag{9}$$

Replacing p_5 by p and solving equation (9) for z we get the following approximation for z:

$$z_5=(x)^{(1/d)} \tag{10}$$

where $x=[-b-\sqrt{(b^2-4ac)}]/2a$, and $c=-\ln(2p)$.

An effort was made to fit a form bz^d+az^{2d} to ln(2p) using the minimization of the sum of absolute deviations criterion.The obtaind values are; a=-0.307; b=-0.82; and d=1.074.Comparisons are made in Table 1 and Table 2 (E-05=10^{-5},etc.) to show where the approximations differ.Table 1 shows that the proposed approximations p_5 have negligible absolute relative errors over the interval $2\leq z\leq 4$ and absolute relative errors of 2% or less over the rest of the interval $0\leq z\leq 7$. Table 2 shows that the present approximations z_5 have negligible absolute relative errors except for z=0.5 and z=1, in which the relative absolute errors of new approximation $z_5$are 4% and 1% respectively The new approximations can easily be computed using ordinary scientific calculators.The superiority of the proposed approximations p_5 and z_5 on the interval $2\leq z\leq 7$ is clear.We can note that p_1,p_2 and p_3 have increasing relative absolute errors over the interval $4\leq z\leq 7$ (greater than 10% for $z\geq 4.5$).Although approximations p_4 have decreasing relative absolute errors on the interval $3\leq z\leq 7$ but its absolute relative errors on this interval are always greater than that of p_5(p_4 has an absolute relative errors of 10% at z=3).Also the new approximations

TABLE 1 Estimates of $p=1-\Phi(z)$

Exact		Approximations				
z	p	p_1	p_2	p_3	p_4	p_5
0.0	.5000000	.5000000	.5000000	.5000000	.5000000	.5000000
0.5	.3085375	.3080176	.3148537	.3151485	.3085700	.3160312
1.0	.1586553	.1588551	.1610328	.1611987	.1587982	.1620019
1.5	.0668072	.0671664	.0668939	.0666741	.0669367	.0676044
2.0	.0227501	.0228689	.0225697	.0225346	.0226881	.0228198
2.5	.0062097	.0062006	.0061849	.0062130	.0060270	.0061927
3.0	.0013499	.0013371	.0013766	.0013621	.0012100	.0013437
3.5	.0002326	.0002310	.0002489	.0002256	.0002493	.0002320
4.0	.0000317	.0000324	.0000365	.0000260	.0000335	.0000317
4.5	.340E-05	.376E-05	.436E-05	.186E-05	.355E-05	.343E-05
5.0	.287E-06	.358E-06	.422E-06	.687E-07	.297E-06	.291E-06
6.0	.987E-09	.207E-08	.212E-08	.346E-11	.101E-08	.100E-08
7.0	.128E-11	.688E-11	.464E-11	.878E-20	.130E-11	.128E-11

NOTE: Entries in the P column are based on Staff of Research and Education Association, Dr. Fogiel, Director (1984).

TABLE 2 Estimates of z

Exact		Approximations				
z	p	z_1	z_2	z_3	z_4	z_5
0.0	.5000000	0.0	0.0	0.0	0.0	0.0
0.5	.3085375	.4981156	0.517770	0.518702	0.499999	0.521
1.0	.1586553	1.001018	1.009578	1.010900	1.000302	1.013505
1.5	.0668072	1.503993	1.500660	1.498988	1.500572	1.506051
2.0	.0227501	2.004328	1.996654	1.995979	1.998355	2.001280
2.5	.0062097	2.501913	2.498570	2.500192	2.489195	2.499030
3.0	.0013499	2.998658	3.006091	3.002738	2.968205	2.998600
3.5	.0002326	3.497160	3.518538	3.492146	3.516780	3.499273
4.0	.0000317	3.999670	4.035214	3.958463	4.010938	4.000468
4.5	.340E-05	4.507750	4.555478	4.394899	4.506833	4.501723
5.0	.287E-06	5.022353	5.078758	4.797798	5.003715	5.002674
6.0	.987E-09	6.073502	6.132820	5.500232	5.999209	6.002906
7.0	.128E-11	7.155605	7.194528	6.073554	6.995810	6.999794

p_5 and z_5 are more simple than p_4 and z_4 (the proposed approximation uses only one simple formula while the other uses two). The present approximation can easily be used for simulating the tail portion of the normal distribution and for generating normal random variates.

A reasonable conclusion is that the new approximations are simple and have reasonable accuracy over a wide range and can be recommended over the interval $2 \leq z \leq 7$.

References:

Feller, W. (1968) An introduction to probability theory and its applications(vol. 1),New York: John Wiley.

Hamaker, H.C. (1978).Approximating the cumulative normal distribution and its inverse. Applied Statistics,27, 76-77.

Lin, J.T. (1989).Approximating the normal tail probability and its inverse for use on a pocket calculator.Applied Statistics,38,69-70.

Lin, J.T. (1990).A simpler logistic Approximation to the normal tail probability and its inverse. Applied Statistics,39,255-257.

Revfeim,K.J.A. (1990).More approximations for the cumulative and inverse normal distribution. The American Statistician,44,63-63.

Staff of Research and Education Association,Dr.M.Fogiel director (1984).Handbook of Mathematical,scientific,and Engineering Formulas,Tables,Functions,Graphs,Transformations.Research and Education Association,61 Ethel Road West Piscataway,New Jersey 08854.