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A Characteristic Property of The Generalized Hyperexponential Distribution

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Abstract

This paper presents a property of the generalized hyperexponential distribution by considering the conditional expectation of the spacing of two consecutive order statistics.

1. Introduction

The problem of mixtures of distributions have received some attention in the literature in recent years. However, due to the complexity of mixtures, many of the related problems have been ignored.

In some practical situations, population characterizations have to be inferred from order statistics, the difference of any two consecutive order statistics being a case of particular interest. Nevertheless, the extension to the case of a linear combination of the differences between each order statistic and the smallest one is another goal.

In this paper we present characterization theorems of generalized hyperexponential (GH) distribution function, which is - as defined by Botta and Harris [1] - a linear combination of a finite number of exponential Cdf's with mixing parameters (positive and negative) that sum to unity, i.e.

$$F(x) = \Sigma_{i=1}^{n} \lambda_i (1 - \exp(-\alpha_i x)), \qquad (1)$$

with $\Sigma_{i=1}^{n}\lambda_i = 1$, λ_i real, $\alpha_i > 0$.

This distribution arises from a desire to preserve the computationally attractive feature of "memorylessness" possessed by the exponential probability distribution while extending the representations to a broader class in order to approximate an arbitrary probability distribution function.

The simplest structure of the GH probability distribution considered in this paper is the case $n = 2$ given by

$$F(x) = \lambda\ (\ 1 - \exp(-\alpha_1 x)) + (1-\lambda)(\ 1 - \exp(-\alpha_2 x)),\ \alpha_1, \alpha_2 > 0, \quad (2)$$

i.e. a mixture of two exponentials where we permit the intermediate mixing "probabilities" -that sum to unity- to have positive or negative values. This kind of freedom makes the GH distributions extremely versatile.

2. Characterization Theorems

Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with common distribution function $F(x)$. Let $X_{r,n}$ denote the rth order statistics from this sample of size n.

We derive necessary and sufficient conditions for a random variable to be distributed as a generalized hyperexponential distribution using conditional expectation of the difference between order statistics.

Theorem (1):

Let X be a non-negative random variable with an absolutely continuous Cdf $F(x)$ and assume $EX < \infty$. Let $X_{1,n} < X_{2,n} < ... < X_{n,n}$ be the order statistics from a sample of size n from this distribution. Let α and β be constants such that $\alpha > 0 > \beta$ and $(\alpha/2)^2 > |\beta|$ with $r(.)$ is the failure rate. If, for $y > 0$,

$$E(X_{n,n} - X_{n-1,n} | X_{n-1,n} = y) = \alpha + \beta\, r(y), \tag{3}$$

then F(x) is a generalized hyperexponential distribution.

Proof

The joint pdf of $X_{n-1,n}$ and $X_{n,n}$ is

$$g(X_{n-1,n}, X_{n,n}) = n(n-1)\, [F(x_{n-1,n})]^{n-2}\, f(x_{n-1,n})\, f(x_{n,n}),$$

from which, we obtain

$$E(X_{n,n} | X_{n-1,n} = y) = \int_y^{\infty} \frac{x\, f(x)}{1-F(y)}\, dx\ . \tag{4}$$

which implies, from (3),

$$\int_0^{\infty} x\, f(x)\, dx = (y + \alpha)(1 - F(y)) + \beta\, f(y).$$

Differentiating twice with respect to y ,

$$\beta\, f''(y) - \alpha\, f'(y) - f(y) = 0, \tag{5}$$

whose characteristic equation is

$$\beta\, \omega^2 - \alpha\, \omega - 1 = 0,$$

with roots $\omega = \frac{\alpha}{2\,\beta} \pm \sqrt{(\frac{\alpha}{2\,\beta})^2 + \frac{1}{\beta}}$.

Clearly $\omega_1, \omega_2 < 0$, so set $\omega_1 = -a$, $\omega_2 = -b$, where $a, b > 0$.

Then we have

$$f(y) = c_1\, e^{-ay} + c_2\, e^{-by}, \qquad y > 0.$$

Noting that $\int_0^{\infty} x\, f(x)\, dx = 1$, and putting $c_1 = \lambda\, a$, we obtain

$$c_2 = (1 - \lambda)\, b\ .$$

Thus the distribution of X is a mixture of two exponentials, i.e. the distribution of X is a generalized hyperexponential distribution given by (2).

Corollary (2):

Let X_1 and X_2 be two independent copies of non-negative random variable X having an absolutely continuous Cdf F(x). If for $y > 0$,

$$E(X_{2,2} - X_{1,2} | X_{1,2} = y) = \alpha + \beta\, r(y) \tag{6}$$

where α and β are constants such that $\alpha > 0 > \beta$ and $(\alpha/2)^2 > |B|$ with

$r(.)$ the failure rate, then $F(x)$ is a generalized hyperexponential distribution defined in (2).

The result given in the corollary reduces to the lack of memory property given by Nassar and Mahmoud [3] under somewhat different assumptions.

To extend this result to functions of order statistics, we define

$$S_1 = \frac{1}{n}\Sigma_{i=1}^{n}(X_{i,n}-X_{1,n}).$$

We now pose the question whether there is a necessary and sufficient condition using S_1 for the population to be a generalized hyperexponential. The answer to this question is given by the following result.

<u>Theorem (3)</u>:

Let X be a non-negative random variable with an absolutely continuous Cdf $F(x)$ and assume $EX < \infty$. Let $X_{1,n} < X_{2,n} < \dots < X_{n,n}$ be the order statistics from a sample of size n from this distribution. Let S_1 be defined as above and let α and β be constants such that $\alpha > 0 > \beta$ and $(\alpha/2)^2 > |\beta|$ with $r(.)$ the failure rate. If, for $y > 0$,

$$E(S_1 \mid X_{1,n}=y) = \alpha + \beta\, r(y) \tag{7}$$

then $F(x)$ is a generalized hyperexponential distribution.

<u>Proof</u>

$$\begin{aligned} E(S_1 \mid X_{1,n}=y) &= E\,[\frac{1}{n}\,\Sigma_{i=1}^{n}(X_{i,n} - X_{1,n})\ |X_{1,n} = y\,] \\ &= E\,[\frac{1}{n}\,(\Sigma_{i=1}^{n}X_{i,n} - n\,X_{1,n})\ |X_{1,n} = y\,] \\ &= E\,[\frac{1}{n}\ \Sigma_{i=1}^{n}X_{i,n} - X_{1,n}|X_{1,n} = y\,] \\ &= E\,[\,\overline{X} - y \mid X_{1,n} = y\,] = \alpha + \beta\, r(y) \end{aligned} \tag{8}$$

Since, for $s > r$, the distribution of the s<u>th</u> order statistic from a sample of size n, given the $r\underline{th}$ order statistic is the same as the distribution of the $(s-r)\underline{th}$ order statistic from a sample of size $(n-r)$, i.e.

$$P(X_{s,n} < x \mid X_{r,n} = y) = P(X^{*}_{s-r,n-r} < x),$$

with $$F^{*}(x) = \frac{F(x) - F(y)}{1 - F(y)}, \qquad x \geq y,$$

Then

$$P(X_j<x \mid X_{1,n}= y) = \frac{1}{n} P [X_{1,n}< x \mid X_{1,n}= y] + \frac{n-1}{n} \frac{F(x)- F(y)}{1 - F(y)}$$

from which

$$E(X_j \mid X_{1,n} = y) = \frac{y}{n} + \frac{n - 1}{n(1-F(y))} \int_y^{\infty} x \, dF(x) \qquad (9)$$

Now R.H.S. of (9) does not depend on j, j = 1, 2, $\cdots$, n, so we let

$$E [\overline{X} \mid X_{1,n}= y]= \frac{1}{n} \Sigma^{n}_{j=1}E(X_j \mid X_{1,n}= y) = y + \alpha_n+\beta_n r(y). \quad (10)$$

But from (9)

$$\frac{1}{n-1} E(nX_j - y \mid X_{1,n} = y) = \frac{1}{1-F(y)} \int_y^{\infty} x \, dF(x)$$

which does not depend on n. So set n = 2 to obtain

$$E(2X_j- X_{1,2}\mid X_{1,2} = y) = 2 E(X_j \mid X_{1,2} = y) - y = y + \alpha_2 +\beta_2 r(y)$$

But

$$E(X_1\mid X_{1,2} = y) = E(X_2\mid X_{1,2} = y)$$

Thus we have

$$\begin{aligned} E(2X_j- X_{1,2}\mid X_{1,2} = y) &= E(X_1+X_2-X_{1,2}\mid X_{1,2} = y) \\ &= E(X_{1,2}+ X_{2,2}- X_{1,2}\mid X_{1,2} = y) \\ &= E(X_{2,2}\mid X_{1,2} = y) \\ &= y + \alpha_2 + \beta_2 \, r(y) \qquad \text{(from (10))} \end{aligned}$$

Using corollary (2), we obtain that F(x) is a generalized hyperexponential distribution, defined by (2).

Wang and Srivastava [4] obtained characterizations of several distributions using the property $E(S_k\mid X_{k,n}= y) = \alpha + \beta y$ with

$$S_k = \frac{1}{n-k} \Sigma^{n}_{i=k+1} (X_{i,n} - X_{k,n}).$$

Note that theorems (1) and (3) could be extended to the general form of GH distribution, given by (1), i.e. the mixture of any finite number of exponential terms, but the characterization would involve derivatives of the failure rate.

3. Applications

GH distributions are extremely attractive distributions for use in applied probability modeling, reliability theory, and queueing theory because of: (i) their simple mathematical structure that facilitates such operations as differentiating, integrating, and taking Laplace transforms, (see Nassar [2] and Nassar and Mahmoud [3]) and (ii) their uniqueness of representation as a linear combination of exponential terms.

For example, the Cdf of the time until absorption in a finite state continuous time Markov chain is

$$F(t) = 1 - \lambda \exp(Qt).e,$$

where Q is the generator matrix corresponding to an (n+1)-state chain with state (n+1) absorbing, the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the vector of initial state probabilities at $t = 0$, and the vector e is an n-dimensional column vector of all ones. The entries q_{ij} in the generator matrix represent the instantaneous rate of transition from state i to state j. Taking Q diagonal with $q_{ii} = -\alpha_i$ gives the GH distribution in (1), i.e.

$$F(t) = \Sigma_{i=1}^{n} \lambda_i \, (1 - \exp(-\alpha_i t)).$$

References

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