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ON THE LOSS OF INFORMATION THROUGH THE USE OF HYBRID CENSORING

Moshira A. Ismail[1]

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ABSTRACT

In this paper two expressions are derived for the expected gain in information provided by a hybrid censored experiment using Shannon-Lindley's measure. The first expression is used to provide information about the scale parameter of the exponential model or any one-to-one transformation of it whereas the second one is appropriate when the objective is to gain knowledge about a future observation. Attention is restricted to the case where sampling is with replacement. Properties of the derived expressions are studied and some applications are considered particularly in assessing the loss of information through the use of hybrid censoring in comparison to type I and type II censoring. Numerical results are provided for illustration.

1- Department of Statistics, Faculty of Economics and Political Sciences, Cairo University

1. INTRODUCTION

Suppose that n items whose lifetimes are independent and identically distributed (iid) exponential random variables with probability density function

$$f(x|\theta) = \frac{1}{\theta} e^{\left(-\frac{x}{\theta}\right)} \qquad x > 0,\ \theta > 0 \qquad (1.1)$$

are placed on a life test. Due to several practical considerations, it is often desirable to terminate the test before all items have failed. Some possibilities are :

1- Testing is terminated after a fixed number, r, of failures have been observed (type II censoring) .

2- Testing is terminated at a fixed time t_o, (type I censoring) .

3- Testing is terminated at min (t_o, x_r) where x_r denotes the lifetime of the r^{th} failure. Sampling plan (3) is called a hybrid test scheme. It combines the advantages of both type I and type II censoring and is used as a reliability acceptance test in MIL-STD-781(C)(1977).

Because of the lack of memory property of the exponential distribution, one can use it when conducting a life test with replacement. This means that at the time of failure of any item it is immediately replaced by another one or repaired and tested again .

Classical methods for estimation and hypothesis testing based on the exponential model under hybrid censoring and sampling with replacement have been considered by Epstein (1954), Harter (1978), Fairbanks et al. (1982) and Ebrahimi (1986) who dealt with the more-general model

$$f(x|\lambda,\mu)=\lambda e^{(-\lambda(x-\mu))} \qquad x>\mu,\ \lambda>0 \qquad (1.2)$$

which reduces to (1.1) when $\mu=0$ and $\lambda^{-1}=\theta$ Ebrahimi (1992) derived classical methods for predicting future failures under the lifetime model (1.2) when $\mu=0$. Draper and Guttman (1987) considered Bayesian estimation of the mean life θ (defined in (1.1)) and Bayesian prediction of the lifetime of an untested item .

The main advantage of using a hybrid censored plan in comparison to either type I or type II censoring is that it saves time and money through the reduction achieved in the expected testing time and the expected number of failures observed in the experiment. There is, however, a greater loss in the information provided by the experiment.

The objective of this paper is to assess the loss of information due to the use of hybrid censoring. A measure of information is used which incorporates prior knowledge through the adoption of a Bayesian approach. This measure was introduced by Shannon (1948) within communication theory. Lindley (1956) applied it in statistics. Brooks (1982) and (1983) used this measure of information to study the efficiency of type I and type II censored samples as compared to the case of complete sampling. Other applications of this measure in life testing under type I and type II censoring include the work of Barlow and Hsiung (1983), Ismail (1994) and Zaher et al (1995) .

In this paper it will be assumed that the lifetime model is the one-parameter exponential distribution with mean λ^{-1} given by

$$f(x|\lambda)=\lambda e^{-\lambda x} \qquad x>0,\ \lambda>0 \qquad (1.3)$$

In Section 2 an expression is derived for the expected gain in information provided by the experiment about the scale parameter λ or any one-to-one transformation of it under hybrid censoring when sampling is with replacement. The main properties of this measure are also studied. In Section 3 results are derived parallel to the ones given in Section 2 for the case where interest centers on information about a future observation. Section 4 deals with various applications of the measures of information obtained in Sections 2 and 3. In particular they are used to compare the efficiencyof a hybrid censored plan with type I and type II censored samples. Tabulated values for the measures of efficiency are provided in Section 5. Proofs of some of the theoretical results are given in the Appendix .

2. INFORMATION ABOUT THE SCALE PARAMETER OR ANY ONE TO ONE TRANSFORMATION OF IT

Suppose that the amount of information about a parameter λ contained in a density $\pi_1(\lambda)$ is given by :

$$I[\pi_1(\lambda)]=\int \pi_1(\lambda)\ln \pi_1(\lambda)d\lambda \qquad (2.1)$$

Then according to Lindley (1956), the expected gain in information about λ provided by the experimental *E* which results in observing $\underline{x}$ (which is possibly a vector) is given by

$$I[E.\pi(\lambda)] = E_{\underline{x}}\left(I[\pi(\lambda|\underline{x})]\right) - I[\pi(\lambda)] \qquad (2.2)$$

where $\pi(\lambda)$ and $\pi(\lambda|\underline{x})$ denote respectively the prior and posterior distributions of λ .

A very important property of (2.2) is that it is invariant under one-to-one transformations of the parameter space .

Under hybrid censoring with replacement, the likelihood function based on (1.3) is given by

$$L(\lambda|data) \propto \lambda^{k} e^{-\lambda A_k} \qquad (2.3)$$

where k, the number of failures observed in the hybrid censored life test is given by

$$k = \begin{cases} r & , \quad x_r < t_o \\ d & , \quad x_r > t_o \end{cases} \qquad (2.4)$$

d denotes the number of failures observed at time t_o .

A_k , the total observed lifetime, is given by

$$A_k = \begin{cases} s & , \quad k=r \\ nt_o & , \quad 0 \le k \le r-1 \end{cases} \qquad (2.5)$$

where $s = nx_r$

Using the gamma natural conjugate prior for λ given by

$$\pi(\lambda) = \frac{h^g}{\Gamma(g)} e^{-\lambda h} . \lambda^{g-1} \qquad g>0, h>0 \qquad (2.6)$$

the posterior distribution of λ is given by

$$\pi(\lambda|data) = \frac{H^G}{\Gamma(G)} e^{-\lambda H} . \lambda^{G-1} \qquad (2.7)$$

where $H=h+A_k$ and $G=g+k$.

Using (2.1), (2.6) and (2.7) it can be shown that the gain in information about λ provided by the hybrid censored experiment is:

$$I[\pi(\lambda|data)] - I[\pi(\lambda)] =$$
$$\ln H - \ln\Gamma(G) + \Psi(G)[G-1] - G - \{[\ln h - \ln\Gamma(g)] + \Psi(g)[g-1] - g\} \tag{2.8}$$

where $\Psi(v) = \frac{d\ln\Gamma(v)}{dv}$ is the digamma function (see for example Abramowitz and Stegun (1965) .

Note that a sufficient statistic for λ is given by

$$t = \begin{cases} s & , \quad s < nt_o \\ d & , \quad s > nt_o \end{cases} \tag{2.9}$$

The conditional distribution of t is given by :

$$f(t|\lambda) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} e^{-\lambda s} \cdot s^{r-1} & , \quad s < nt_o \\ e^{-nt_o\lambda} \cdot (nt_o\lambda)^d / d! & , \quad d = 0,1,2,\ldots..r-1 \end{cases} \tag{2.10}$$

It follows from (2.6) and (2.10) that the marginal distribution of t defined by

$$f(t) = \int_{\lambda=0}^{\infty} f(t|\lambda)\pi(\lambda)d\lambda$$

is given by

$$f(t) = \begin{cases} \frac{h^g \cdot s^{r-1}}{\beta(r,g)(h+s)^{g+r}} & 0 < s < nt_o \\ \frac{(nt_o)^d \cdot h^g \cdot \Gamma(g+d)}{d!\,\Gamma(g)(h+nt_o)^{g+d}} & d \le r-1 \end{cases} \tag{2.11}$$

Hence from (2.2), the expected gain in information about λ is defined as :

$$E_t\left(I[\pi(\lambda|t)]\right) - I[\pi(\lambda)] \tag{2.12}$$

It can be shown that (2.12) can be expressed as :

$$I(r,a,g) = I_1 + I_2 + I_3 + I_4 - I_5 \tag{2.13}$$

where :

$$I_1 = \frac{1}{\beta(r,g)} \int_{z=0}^{\frac{a}{a+1}} [-\ln(1-z)] \cdot z^{r-1}(1-z)^{g-1} dz$$

$$I_2 = \left[-\ln\Gamma(g+r)+(g+r-1)\Psi(g+r)-(g+r)\right]\frac{1}{\beta(r,g)}\int_{u=0}^{a}\frac{u^{r-1}}{(1+u)^{g+r}}du$$

$$I_3 = \ln(1+a)\sum_{d=0}^{r-1}\frac{\Gamma(g+d)}{\Gamma(g)(1+a)^{g+d}}\frac{a^d}{d!}$$

$$I_4 = \sum_{d=0}^{r-1}\frac{\Gamma(g+d)}{\Gamma(g)(1+d)^{g+d}}\frac{a^d}{d!}\left[-\ln\Gamma(g+d)+(g+d-1)\Psi(g+d)-(g+d)\right]$$

$$I_5 = (g-1)\Psi(g)-\ln\Gamma(g)-g$$

where $a = \dfrac{nt_0}{h}$ ..

Note that :

$$\frac{1}{\beta(r,g)}\int_{u=0}^{a}\frac{u^{r-1}}{(1+u)^{g+r}}du = 1-\frac{1}{\Gamma(g)}\sum_{d=0}^{r-1}\frac{\Gamma(g+d)}{(1+a)^{g+d}}\frac{a^d}{d!} \qquad (2.14)$$

Note also that : $\int_{z=0}^{\frac{a}{1+a}}\{-\ln(1-z)\}z^{r-1}(1-z)^{g-1}dz$

can be expressed as:

$$\sum_{i=0}^{r-1}\frac{(-1)^{i+1}\binom{r-1}{i}}{(g+i)}\left[\frac{\ln(1+a)}{(1+a)^{g+i}}+\frac{1}{g+i}\left[\frac{1}{(1+a)^{g+i}}-1\right]\right]\bullet \qquad (2.15)$$

Refer to the Appendix for derivations of equations (2.13), (2.14) and (2.15).

Due to the invariance property of the measure of information, expression (2.13) is the expected gain in information about any one-to-one transformation of λ such as the mean life $\theta=\lambda^{-1}$ and the reliability function $R(t_m)=e^{(-\lambda t_m)}$ where t_m is a specified mission time .

Properties of the measure of information *I(r,a,g)* :

Limiting cases

i) The case where $nt_0 \rightarrow \infty$.

When $nt_0 \rightarrow \infty$ i.e. $a \rightarrow \infty$, it can be shown that :

$$\lim_{a\to\infty} I_1 = \frac{1}{\beta(r,g)}\frac{d\beta(r,g)}{dg}$$

$$= -\frac{d\,ln\beta(r,g)}{dg} = -\Psi(g) + \Psi(g+r)$$

$$\lim_{a\to\infty} I_2 = -ln\Gamma(g+r) + (g+r-1)\Psi(g+r) - (g+r)$$

$$\lim_{a\to\infty} I_3 = 0$$

$$\lim_{a\to\infty} I_4 = 0$$

Hence the expected gain in information about λ reduces to:

$$I(r,g) = -ln\Gamma(g+r) + (g+r)[\Psi(g+r) - 1] + ln\Gamma(g) - g[\Psi(g) - 1] \qquad (2.16)$$

which is the expression of the expected gain in information under type II censoring with replacement. This result is consistent with the definition of the hybrid censoring plan .

ii) The case were $r \to \infty$

It can be shown that

$$\lim_{r\to\infty} I_1 = 0$$

$$\lim_{r\to\infty} I_2 = 0$$

$$\lim_{r\to\infty} I_3 = (1+a)$$

$$\lim_{r\to\infty} I_4 = \sum_{d=0}^{\infty} \frac{\Gamma(g+d)}{\Gamma(g)(1+a)^{g+d}}\frac{a^d}{d!}\left[-ln\Gamma(g+d) + (g+d-1)\Psi(g+d) - (g+d)\right]$$

Hence the expected gain in information about λ reduces to :

$$I(a,g) = \left\{ln(1+a) + \sum_{d=0}^{\infty}\frac{\Gamma(g+d)}{\Gamma(g)(1+a)^{g+d}}\frac{a^d}{d!}\left[-ln\Gamma(g+d) + (g+d-1)\Psi(g+d) - (g+d\right.\right.$$

$$\left.-\left[(g-1)\Psi(g) - ln\Gamma(g) - g\right]\right\} \qquad (2.17)$$

which is as expected the expression of the expected gain in information about λ under type I censoring with replacement. Expressions (2.16) and (2.17) were derived by Ismail (1988) who also showed that a lower bound $I_L(a,g)$ and an upper bound $I_u(a,g)$ for expression (2.17) are given respectively by :

$$I_L(a,g) = \beta + \frac{1}{2} ln(1+a) + \frac{1}{2} \Psi(g) - g[\Psi(g) - 1] + ln\,\Gamma(g) \quad (2.18)$$

$$I_u(a,g) = \beta + \frac{1}{2} ln(1+a) + \frac{1}{2} \Psi(g) - g[\Psi(g) - 1] + ln\,\Gamma(g) + \frac{1}{4(g-1)(a+1)} \quad (2.19)$$

where $\beta = -\frac{1}{2} ln(2\pi e)$

$e \approx 2.71828$

Note that expressions (2.18) and (2.19) are close for large values of a hence the arithmetic mean $\frac{I_L(a,g) + I_u(a,g)}{2}$ could be a useful approximation to the exact expression (2.17) in this case especially that the convergence of the infinite series given in the exact expression gets slower as the value of a increases which makes its evaluation accurately a rather difficult task .

Theorem (2.1) :

For fixed values of r and g, the expected gain in information about λ defined by equation (2.13) is a concave increasing function in $a = \frac{mt_0}{h}$

<u>*Proof*</u>:

The increasing property of $I\,(r,a,g)$ can be established by verifying that $\frac{\partial I(r,a,g)}{\partial a} > 0$.

Note that $\frac{\partial I(r,a,g)}{\partial a} = \frac{\partial I_1}{\partial a} + \frac{\partial I_2}{\partial a} + \frac{\partial I_3}{\partial a} + \frac{\partial I_4}{\partial a}$

Using the standard result

$$\frac{\partial}{\partial u}\int_b^c f(x,u)dx = \frac{\partial c}{\partial u} f(c,u) - \frac{\partial b}{\partial u} f(b,u) + \int_b^c \frac{\partial f(x,u)}{\partial u} dx$$

we note that

$$\frac{\partial}{\partial a} I_1 = \frac{a^{r-1} \ln(1+a)}{\beta(r,g)(1+a)^{g+r}}$$

$$\frac{\partial}{\partial a} I_2 = \frac{a^{r-1}}{\beta(r,g)(1+a)^{g+r}}\left[-\ln\Gamma(g+r)+(g+r-1)\Psi(g+r-1)-(g+r)\right]$$

Using the recurrence formulae

$$\Gamma(v) = (v-1)\Gamma(v-1) \qquad v>1 \qquad (2.20)$$

$$\Psi(v) = \Psi(v-1) + \frac{1}{v-1} \qquad v>1 \qquad (2.21)$$

we get

$$\frac{\partial I_2}{\partial a} = \frac{1}{\beta(r,g)}\frac{a^{r-1}}{(1+a)^{g+r}}\left[-\ln(g+r-1)-\ln\Gamma(g+r-1)+(g+r-1)\Psi(g+r)-(g+r-1)\right]$$

$$\frac{\partial I_3}{\partial a} = -\frac{\ln(1+a)a^{r-1}}{\beta(r,g)(1+a)^{g+r}} + \frac{1}{a+1}\cdot\frac{1}{\Gamma(g)}\sum_{d=0}^{r-1}\frac{\Gamma(g+d)a^d}{(a+1)^{g+d}d!}$$

$$\frac{\partial I_4}{\partial a} = \frac{1}{\Gamma(g)}\left\{\sum_{d=0}^{r-2}\frac{\Gamma(g+d+1)}{d!}\left[-\ln\Gamma(g+d+1)+(g+d)\Psi(g+d+1)-(g+d+1)\right]\frac{a^d}{(1+a)^{g+d+1}}\right.$$
$$\left.-\sum_{d=0}^{r-1}\frac{\Gamma(g+d+1)}{d!}\left[-\ln\Gamma(g+d)+(g+d-1)\Psi(g+d)-(g+d)\right]\frac{a^d}{(1+a)^{g+d+1}}\right\}$$

Using the formulae (2.20) and (2.21) :

$$\frac{\partial I_4}{\partial a} = I_6 + I_7 + I_8$$

where

$$I_6 = \frac{1}{\Gamma(g)}\sum_{d=0}^{r-2}\frac{\Gamma(g+d+1)}{d!}\left[-\ln(g+d)\right]\frac{a^d}{(1+a)^{g+d+1}}$$

$$I_7 = \frac{1}{\Gamma(g)}\sum_{d=0}^{r-1}\frac{\Gamma(g+d+1)}{d!}\left[\Psi(g+d)\right]\frac{a^d}{(1+a)^{g+d+1}}$$

$$I_8 = -\frac{a^{r-1}}{(a+1)^{g+r}\beta(r,g)}\left[-\ln\Gamma(g+r-1)+(g+r-1)\Psi(g+r-1)-(g+r-1)\right]$$

Using the formula

$$\Psi(v) = \ln v - \frac{1}{2v} + R \qquad (2.22)$$

where R, the remainder term, is negative and $|R| < \frac{1}{12v^2}$

$$I_6 + I_7 = \frac{1}{\beta(r,g)} ln(g+r-1)\frac{a^{r-1}}{(1+a)^{g+r}} + \frac{1}{\Gamma(g)}\sum_{d=0}^{r-1}\frac{\Gamma(g+d)}{d!}\frac{a^d}{(1+a)^{g+d+1}}\left[-\frac{1}{2}+R(g+d)\right]$$

After few simplifications we find that $\frac{\partial I(r,a,g)}{\partial a} > 0$

The concavity property can easily be established by noting that $\frac{\partial^2 I(r,a,g)}{\partial a^2}$ is negative .

It has also been observed numerically that for fixed values of a and g, $I(r,a,g)$ is a monotonic increasing function in r .

3- INFORMATION ABOUT THE MINIMUM LIFETIME OF A FUTURE EXPERIMENT

Based on the hybrid censored experiment consider predicting y_1, the smallest observation in a future independent sample of size m. This statistic has important practical applications in life testing; for example, it represents the lifetime of a system consisting of components connected together in series .

The predictive distribution of y_1, is given by :

$$p(y_1|t) = mGH^G\left[H + my_1\right]^{-(G+1)} , \qquad (3.1)$$

where the sufficient statistic t is defined in equation (2.9) and G and H are defined in (2.7) .

Note that when m=1 expression (3.1) reduces to expression (4.2) given in the paper by Draper and Guttman (1987) .

Amaral and Dunsmore (1985) considered the adaptation of Shannon Lindley's measure of information to situations in which the final objective is to gain knowledge about a future event or observation. In this case, the gain in information about y_l provided by the hybrid censored experiment is given by :

$$I[p(y_1|t)] - p(y_1), \tag{3.2}$$

here $p(y_l)$, the prior predictive distribution of y_l, is defined by (3.1) with g and h replacing G and H respectively .

It can be shown that expression (3.2) is given by :

$$ln G - ln H - \frac{1}{G} - ln g + ln h + \frac{1}{g} \tag{3.3}$$

Taking the expectation of (3.3)with respect to the marginal distribution of t defined by equation (2.11), it can be shown that the expected gain in information about y_l is given by :

$$I_l(r,a,g) = \left\{ \left[ln(g+r) - \frac{1}{(g+r)} \right] \frac{1}{\beta(r,g)} \int_{u=0}^{a} \frac{u^{r-1}}{(1+u)^{g+r}} du + \sum_{d=0}^{r-1} \left[ln(g+d) - \frac{1}{g+d} \right] \frac{\Gamma(g+d)}{\Gamma(g)(1+a)^{g+d}} \frac{a^d}{d!} - I_1 - I_3 - ln g + \frac{1}{g} \right\} \tag{3.4}$$

where a, I_1 and I_3 are defined in equation (2.13)

Properties of the measure of information $I_l(r,a,g)$:

Limiting cases

i) The case where $nt_0 \rightarrow \infty$.

When $nt_0 \rightarrow \infty$ i.e. $a \rightarrow \infty$, it can be shown that expression (3.4) reduces to :

$$I_l(r,g) = \Psi(g+1) - ln g - \Psi(G+1) + \frac{1}{G} \tag{3.5}$$

which is the expected gain in information about y_l under type II censoring with replacement.

ii) The case where $r \to \infty$

when $r \to \infty$, expression (3.4) reduces to

$$I_f(a,g) = \left\{ -ln(1+a) + \sum_{d=0}^{\infty} \left[ln(g+d) - \frac{1}{g+d} \right] \frac{\Gamma(g+d)}{\Gamma(g)(1+a)^{g+d}} \frac{a^d}{d!} - ln\,g + \frac{1}{g} \right\} \quad (3.6)$$

which is the expected gain in information about y_l in the case of type I censoring with replacement.

Expressions (3.5) and (3.6) were derived by Ismail (1988) who gave the following lower and upper bounds for expression (3.6).

$$I_{fL}(a,g) = \Psi(g) - ln\,g + \frac{1}{g} - \frac{1}{2(g-1)}\left[\frac{1}{1+a}\right] \quad (3.7)$$

$$I_{fu}(a,g) = \Psi(g) - ln\,g + \frac{1}{g} \quad (3.8)$$

The arithmetic mean $\frac{\left[I_{fL}(a,g) + I_{fu}(a,g)\right]}{2}$ could be a useful approximation to the exact expression $I_f(a,g)$ in the case of large values of a since the convergence of the infinite series in (3.6) gets slower as a increases.

Theorem 3.1 :

For fixed values of r and g, the expected gain in information about y_l defined by equation (3.4) is a concave increasing function in $a = \frac{mt_0}{h}$

<u>*Proof*</u>:

The increasing property of expression $I_f(r,a,g)$ can be established by showing that $\frac{\partial I_f(r,a,g)}{\partial a} > 0$

Note that

$$\frac{\partial}{\partial a}\left[ln(g+r)-\frac{1}{g+r}\right]\cdot\frac{1}{\beta(r,g)}\int_{u=0}^{a}\frac{u^{r-1}}{(1+u)^{g+r}}du=\left[ln(g+r)-\frac{1}{g+r}\right]\frac{a^{r-1}}{\beta(r,g)(1+a)^{g+r}}$$

$$\frac{\partial}{\partial a}(-I_1)=-\frac{a^{r-1}ln(1+a)}{\beta(r,g)(1+a)^{g+r}}$$

$$\frac{\partial}{\partial a}(-I_3)=\frac{ln(1+a)a^{r-1}}{\beta(r,g)(1+a)^{g+r}}-\frac{1}{(a+1)\Gamma(g)}\sum_{d=0}^{r-1}\frac{\Gamma(g+d)a^d}{(1+a)^{g+d}d!}$$

$$\frac{\partial}{\partial a}\sum_{d=0}^{r-1}\left[ln(g+d)-\frac{1}{g+d}\right]\frac{\Gamma(g+d)}{\Gamma(g)(1+a)^{g+d}}\frac{a^d}{d!}$$

$$=\left\{\sum_{d=0}^{r-2}\left[ln(g+d+1)-\frac{1}{g+d+1}\right]\frac{\Gamma(g+d+1)a^d}{(1+a)^{g+d+1}d!\Gamma(g)}\right.$$

$$\left.-\sum_{d=0}^{r-1}\left[ln(g+d)-\frac{1}{g+d}\right]\frac{\Gamma(g+d+1)a^d}{\Gamma(g)d!(1+a)^{g+d+1}}\right\}$$

Hence $\frac{\partial I_f(r,a,g)}{\partial a}$ can be expressed as :

$$\sum_{d=0}^{r-2}\left[ln\left(1+\frac{1}{g+d}\right)-\frac{1}{g+d+1}\right]\frac{\Gamma(g+d+1)a^d}{(1+a)^{g+d+1}d!\,\Gamma(g)}+\left[ln\left(1+\frac{1}{g+r-1}\right)-\frac{1}{g+r}\right]\frac{a^{r-1}}{\beta(r,g)(1+a)^{g+r}}$$

Noting that $ln\left(1+\frac{1}{v}\right)-\frac{1}{v+1}>0$ for $v>0$

it follows that $\frac{\partial I_f(r,a,g)}{\partial a}>0$

The concavity of expression $I_f(r,a,g)$ follows by noting that:

$$\frac{\partial^2 I_f(r,a,g)}{\partial a^2}<0$$

It has also been observed numerically that for fixed values of a and g, $I_f(r,a,g)$ is a monotonic increasing function in r.

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4- APPLICATIONS OF THE MEASURES OF INFORMATION

4.1 Efficiency of hybrid censored plans

One important application of the measures of expected gain in information for λ and y_l is to assess the loss of information through the use of hybrid censoring as compared to type I and type II censoring. Let us define the following measures .

$$L_1 = \frac{[I(a,g) - I(r,a,g)]}{I(a,g)} \tag{4.1}$$

$$E_1 = (1 - L_1) \times 100 \tag{4.2}$$

$$L_2 = \frac{[I(r,g) - I(r,a,g)]}{I(r,g)} \tag{4.3}$$

$$E_2 = (1 - L_2) \times 100 \tag{4.4}$$

$$FL_1 = \frac{[I_f(a,g) - I_f(r,a,g)]}{I_f(a,g)} \tag{4.5}$$

$$FE_1 = (1 - FL_1) \times 100 \tag{4.6}$$

$$FL_2 = \frac{[I_f(r,g) - I_f(r,a,g)]}{I_f(r,g)} \tag{4.7}$$

$$FE_2 = (1 - FL_2) \times 100 \tag{4.8}$$

L_i , $i=1, 2$ are used to measure the loss in information through the use of a hybrid censored plan as compared to type I and type II censoring plans respectively in the case where information about λ is required. E_i, $i=1, 2$ are used to compare the efficiency of hybrid censoring with type I and type II censoring .

Similar statements hold for Fl_i, $i=1, 2$ and FE_i , $i=1, 2$ in the case where information about y_l is required .

4.2 Designing a hybrid censored plan

Another application of the measures of expected gain in information which could be considered in future research is to design a hybrid censored plan i.e. to select the values of r,n and t_0 which provide a balance between the two conflicting factors .

a. The expected gain in information .

b. The cost of performing the experiment .

Various criteria can be used to achieve the above goal such as :

i) Maximizing expected gain in information under a certain cost constraint

ii) Minimizing expected cost under fixed expected gain in information .

iii) Maximizing the following function

γ [expected gain in information] - cost

where γ denotes the expected utility of one unit of information .

5- NUMERICAL EXAMPLES

In this section some numerical results are provided to illustrate the behaviour of the measures of efficiency E_1, E_2, FE_1 and FE_2 discussed in Section 4. Tables (5-a) and (5-b) give the values of E_1 and E_2 for various combinations of r and a. For the same selected values of the pairs (r, a), tables (5-c) and (5-d) display the values of FE_1 and FE_2. A value of g=3 is assumed .

Table (5-a) : Values of E_1 and E_2 for several combinations of (r, a)

r / a	2		4		6		8		10	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
0.25	93.02	35.53	99.61	23.19	99.98	18.07	99.99	15.36	99.99	13.66
0.50	82.67	57.57	97.38	41.33	99.63	32.84	99.95	27.99	99.99	24.90
1	66.37	79.40	89.33	65.14	96.76	54.79	99.05	47.65	99.73	42.67
2	48.81	93.10	73.67	85.64	86.61	78.19	93.27	71.55	96.66	65.94
3	40.12	96.93	63.10	92.91	77.18	88.24	85.92	83.46	91.36	78.92
4	34.99	98.38	56.03	96.02	69.91	93.02	79.34	89.71	85.83	86.30
5	31.59	99.05	51.04	97.55	64.38	95.55	73.92	93.21	80.86	90.67
7.5	26.46	99.66	43.15	99.06	55.08	98.19	64.10	97.10	71.13	95.82
10	23.66	99.85	38.70	99.54	49.62	99.10	58.05	98.51	64.80	97.79
15	20.42	99.95	33.47	99.85	43.03	99.68	50.53	99.46	56.66	99.17
20	18.58	99.99	30.47	99.93	39.20	99.85	46.09	99.74	51.76	99.61

Table (5-b) : Values of E_1 and E_2 for various pairs of (r, a)

r / a	12		15		20		25		30	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
0.25	99.99	12.48	99.99	11.25	99.99	9.93	99.99	9.08	99.99	8.48
0.50	99.99	22.75	99.99	20.50	99.99	19.31	99.99	16.56	99.99	15.46
1	99.92	39.07	99.99	35.23	99.99	31.11	99.99	28.46	99.99	26.56
2	98.36	61.30	99.44	55.86	99.91	49.56	99.99	45.36	99.99	42.35
3	94.72	74.77	97.51	69.36	99.30	62.39	99.81	57.35	99.95	53.61
4	90.30	82.96	94.54	78.27	97.93	71.61	99.23	66.36	99.72	62.25
5	85.95	88.07	91.19	84.20	95.98	78.27	98.19	73.23	99.19	69.06
7.5	76.70	94.40	83.04	92.11	89.96	88.13	94.03	84.24	96.42	80.64
10	70.32	96.97	76.90	95.57	84.68	92.94	89.80	90.14	93.22	87.34
15	61.79	98.83	68.14	98.21	76.17	96.96	82.03	95.49	86.39	93.88
20	56.55	99.43	62.54	99.11	70.33	98.44	76.25	97.60	80.87	96.63

Table (5-c) : Values of FE_1 and FE_2 for selected pairs of (r, a)

r / a	2		4		6		8		10	
	FE_1	FE_2	FE_1	FE_2	FE_1	FE_2	FE_1	FE_2	FE_1	FE_2
0.25	95.13	37.10	99.77	26.97	99.98	23.03	99.99	21.04	99.99	19.84
0.50	87.76	59.08	98.56	45.98	99.82	39.69	99.97	36.31	99.98	34.25
1	75.81	80.34	93.99	69.03	98.49	61.65	99.61	56.95	99.89	53.86
2	62.61	93.44	84.73	87.62	93.53	82.44	97.21	78.27	98.78	75.01
3	56.07	97.06	78.33	93.96	88.76	90.75	94.01	87.80	96.76	85.23
4	52.27	98.43	74.04	96.63	85.03	94.59	91.09	92.55	94.59	90.65
5	49.80	99.05	71.05	97.93	82.20	96.57	88.66	95.14	92.62	93.74
7.5	46.30	99.63	66.54	99.21	77.60	98.62	84.39	97.96	88.84	97.26
10	42.54	99.80	62.83	99.61	72.98	99.32	80.34	98.96	80.63	98.57
15	40.59	99.90	58.55	99.86	68.63	99.76	75.03	99.63	79.43	99.47
20	40.05	99.93	57.88	99.94	67.87	99.89	74.26	99.82	78.68	99.75

Table (5-d) : Values of FE_1 and FE_2 for various pairs of (r, a)

r / a	12		15		20		25		30	
	FE_1	FE_2	FE_1	FE_2	FE_1	FE_2	FE_1	FE_2	FE_1	FE_2
0.25	99.99	19.04	99.99	18.24	99.99	17.45	99.99	16.97	99.99	16.65
0.50	99.98	32.87	99.99	31.50	99.99	30.12	99.99	29.30	99.99	28.75
1	99.96	51.74	99.98	49.58	99.99	47.41	99.99	46.11	99.99	45.25
2	99.46	72.49	99.83	69.72	99.97	66.76	99.98	64.94	99.99	63.72
3	98.23	83.04	99.27	80.41	99.82	77.32	99.95	75.30	99.98	73.91
4	96.68	88.92	98.37	86.69	99.49	83.84	99.83	81.82	99.94	80.37
5	95.12	92.40	97.33	90.59	98.99	88.11	99.60	86.23	99.84	84.80
7.5	91.87	96.54	94.84	95.41	97.49	93.86	98.74	92.46	99.36	91.29
10	88.99	98.16	90.74	97.52	92.71	96.46	95.30	95.47	97.18	94.58
15	82.62	99.30	85.98	99.02	89.46	98.52	91.51	98.01	92.78	97.51
20	81.90	99.66	8 5.35	99.52	89.02	99.25	91.28	98.96	92.70	98.66

Remarks :

From the above tables the following features are noted :

1- For a fixed value of r, increasing the value of a increases both E_2 and FE_2 which approach the value 100 (where $I(r,a,g) = I(r,g)$ and $I_f(r,a,g) = I_f(r,g)$.) The convergence of $I(r,a,g)$ to $I(r,g)$ and $I_f(r,a,g)$ to $I_f(r,g)$ is faster for smaller values of r. For example when r=2 both E_2 and FE_2 reach 99% when a is only equal to 5, whereas when r = 15, the value 99% is reached for E_2 when $a = 20$ and for FE_2 when $a = 15$.

2- For a fixed value of a, increasing the value of r increases E_1 and FE_1, both approach the value 100 (where $I(r,a,g) = I(a,g)$ and $I_f(r,a,g) = I_f(a,g)$. This convergence is faster for smaller values of a. For example when $a = 0.5$, both E_1 and FE_1 exceed 99% when r is only equal to 6 whereas for a value of a equal to 10, the value 99% is not reached for either E_1 and FE_1 even when r is as large as 30.

Remarks 1 and 2 confirm the theoretical results discussed in Sections 2 and 3 about the limiting behaviour of $I(r,a,g)$ and $I_f(r,a,g)$.

APPENDIX

A1: Derivation of Equation (2.13)

Using equations (2.8) and (2.12), the expected gain in information provided by the hybrid censored plan about the parameter λ is given by :

$$I(r,a,g) = J_1 + J_2 + J_3 + J_4 - J_5 \qquad \text{(A1.1)}$$

where

$$J_1 = \int_{s=0}^{nt_0} ln(h+s) \frac{h^g}{\beta(r,g)} \frac{s^{r-1}}{(h+s)^{g+r}} ds$$

$$J_2 = \sum_{d=0}^{r-1} \frac{ln(h+nt_0)(nt_0)^d h^g \Gamma(g+d)}{(h+nt_0)^{g+d} \Gamma(g) \ d!}$$

$$J_3 = [-ln\Gamma(g+r) + \Psi(g+r)[g+r-1] - (g+r)] \int_{s=0}^{nt_0} \frac{h^g s^{r-1} ds}{\beta(r,g)(h+s)^{g+r}}$$

$$J_4 = \sum_{d=0}^{r-1} \frac{\Gamma(g+d) h^g (nt_0)^d}{\Gamma(g)(h+nt_0)^{g+d} d!} [-ln\Gamma(g+d) + (g+d-1)\Psi(g+d) - (g+d)]$$

$$J_5 = lnh - \Gamma(g) + (g-1)\Psi(g) - g$$

Using the transformation $z = \dfrac{s}{h+s}$ it can be shown that

$$J_1 = lnh \int_{u=0}^{a} \frac{u^{r-1}}{\beta(r,g)(1+u)^{g+r}} du + \frac{1}{\beta(r,g)} \int_{z=0}^{\frac{a}{1+a}} [-ln(1-z)] z^{r-1} (1-z)^{g-1} dz \qquad \text{(A1.2)}$$

J_2 can be expressed as

$$J_2 = lnh \sum_{d=0}^{r-1} \frac{\Gamma(g+d) a^d}{(1+a)^d \Gamma(g) d! (1+a)^g} + \frac{ln(1+a)}{\Gamma(g)} \sum_{d=0}^{r-1} \frac{\Gamma(g+d) a^d}{(1+a)^{g+d} d!} \qquad \text{(A1.3)}$$

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J_3 can be expressed as

$$J_3 = \left[-\ln\Gamma(g+r) + \Psi(g+r)[g+r-1] - (g+r)\right]\int_{u=0}^{a} \frac{1}{\beta(r,g)}\frac{u^{r-1}}{(1+u)^{g+r}}\,du \qquad \text{(A1.4)}$$

J_4 can be expressed as

$$J_4 = \left\{\sum_{d=0}^{r-1} \frac{\Gamma(g+d)}{d!\,\Gamma(g)}\left(\frac{a}{1+a}\right)^{d} \frac{1}{(1+a)^{g}}\left[-\ln\Gamma(g+d) + (g+d-1)\Psi(g+d) - (g+d)\right]\right\} \text{(A1.5)}$$

Noting that $\quad \dfrac{1}{\beta(r,g)}\displaystyle\int_{u=0}^{a} \frac{u^{r-1}}{(1+u)^{g+r}}\,du = 1 - \frac{1}{\Gamma(g)}\sum_{d=0}^{r-1}\frac{\Gamma(g+d)a^{d}}{(1+a)^{g+d}\,d!}$

(Derivation of the above equation will be given later)

and substituting (A1.2), (A1.3), (A1.4) and (A1.5) in (A1.1) we obtain equation (2.13) .

A2: **Derivation of Equation (2.14)**

Using the transformation $u = \frac{s}{h}$ it can be shown that :

$$\int_{s=0}^{m_0} \frac{h^{g}}{\Gamma(g)}\frac{s^{r-1}\Gamma(g+r)}{\Gamma(r)(h+s)^{g+r}}\,ds = \frac{1}{\beta(r,g)}\int_{u=0}^{a}\frac{u^{r-1}}{(1+u)^{g+r}}\,du$$

Note that :

$$\int_{s=0}^{m_0} \frac{h^{g}}{\Gamma(g)}\frac{s^{r-1}}{\Gamma(r)}\frac{\Gamma(g+r)}{(h+s)^{g+r}}\,ds = \int_{s=0}^{m_0} s^{r-1}\frac{h^{g}}{\Gamma(g)}\left[\int_{\lambda=0}^{\infty}\frac{e^{-\lambda(h+s)}\lambda^{g+r-1}}{\Gamma(r)}\,d\lambda\right]ds$$

$$= \int_{\lambda=0}^{\infty}\frac{h^{g}}{\Gamma(g)}\lambda^{g-1}e^{-\lambda h}\left[\int_{s=0}^{m_0}\frac{\lambda^{r}s^{r-1}e^{-\lambda s}}{\Gamma(r)}\,ds\right]d\lambda$$

$$= \int_{\lambda=0}^{\infty}\frac{h^{g}}{\Gamma(g)}\lambda^{g-1}e^{-\lambda h}\left[\int_{x=0}^{m_0\lambda}\frac{x^{r-1}e^{-x}}{\Gamma(r)}\,dx\right]d\lambda$$

$$= \int_{\lambda=0}^{\tau} \frac{h^g}{\Gamma(g)} \lambda^{g-1} e^{-\lambda h} \left[1 - \sum_{d=0}^{r-1} e^{-mt_0\lambda} \frac{(mt_0\lambda)^d}{d!} \right] d\lambda$$

$$= 1 - \frac{h^g}{\Gamma(g)} \sum_{d=0}^{r-1} \frac{\Gamma(g+d)}{(mt_0+h)^{g+d}} \frac{(mt_0)^d}{d!}$$

$$= 1 - \frac{1}{\Gamma(g)} \sum_{d=0}^{r-1} \frac{\Gamma(g+d) a^d}{(1+a)^{g+d} d!}$$

A3: **Derivation of Equation (2.15)**

Consider $I_1 = \frac{1}{\beta(r,g)} \int_{z=0}^{\frac{a}{1+a}} \{-ln(1-z)\} z^{r-1} (1-z)^{g-1} dz$.

Using the transformation $w=1-z$, we obtain

$$I_1 = \frac{1}{\beta(r,g)} \int_{w=\frac{1}{1+a}}^{w=1} \{-ln\, w\} (1-w)^{r-1} w^{g-1} dw$$

Using the expansion

$$(1-w)^{r-1} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} w^i$$

and integration by parts we obtain equation (2.15)

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