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THE COMPOUND GOMPERTZ DISTRIBUTION

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ABSTRACT

In this study the properties of the compound Gompertz distribution have been studied. Parameters have been estimated for non-censored, for singly censored and for progressively censored data. The distribution has been applied to a set of data previously modelled using the Gamma distribution. It was found that the compound Gompertz gives a better fit.

1. INTRODUCTION

The notion of compound and generalized distributions has been formulated and studied by Feller (1943), and since then applied by many authors. The most common way of arriving at a Compound distribution is by taking one of the well known distributions and assuming that one (or more) of its parameters is (are) a random variable(s), taking only finite number of possible values, or following a certain distribution. The objective of this paper is to present new characteristics of the compound Gompertz distribution that has been suggested by Osman (1987).

2. REVIEW

Osman (1987) developed the compound Gompertz distribution by assuming a conditional random variable following the Gompertz distributon with parameters α and δ (2.1), where α is gamma distributed with parameters β and μ (2.2). Using the formula (2.3) the resulting unconditional distribution can be derived (2.4).

$$f(t/\alpha) = \alpha e^{\delta t} \exp[-\alpha(e^{\delta t} - 1)/\delta], \quad t, \alpha, \delta > 0 \quad (2.1)$$
$$= 0, \text{ otherwise.}$$

$$g(\alpha; \beta, \mu) = \beta^{\mu} \alpha^{\mu-1} e^{\beta\alpha} / \Gamma(\mu), \quad \alpha, \beta, \mu > 0$$
$$= 0, \text{ otherwise.} \quad (2.2)$$

$$f(t) = \int_{\alpha} f(t/\alpha)\, g(\alpha)\, d\alpha \quad (2.3)$$

$$f(t) = \frac{\mu}{\beta} e^{\delta t} \left(1 + \frac{e^{\delta t} - 1}{\delta\beta}\right)^{-\mu-1}, \quad t \geq 0$$
$$\mu, \beta, \delta > 0 \quad (2.4)$$
$$= 0, \text{ otherwise.}$$

when $\delta\beta = 1$ the compound Gompertz distribution reduces to the exponential distribution with p.d.f.

$$f(t) = \delta\mu\, e^{-\delta\mu t}, \quad t \geq 0 \quad \delta, \mu > 0$$
$$= 0, \text{ otherwise.}$$

Osman (1987) derived the survival, the hazard and the median residual lifetime function (2.5) - (2.7) as well as transformations from the compound Gompertz to other well known distributions. He also showed that the hazard function has an interesting property. If the conditional hazard function is increasing, then the unconditional hazard can be decreasing, constant or increasing those findings are summarized in Table I.

$$S(t) = \left(1 + \frac{e^{\delta t} - 1}{\delta\beta}\right)^{-\mu} \quad (2.5)$$

$$h(t) = \delta\mu\, e^{\delta t} \left(\delta\beta + e^{\delta t} - 1\right)^{-1} \quad (2.6)$$

$$M(R/T) = \frac{1}{\delta} \ln \left[2^{1/\mu} e^{\delta t} - (1 - \delta\beta)(2^{1/\mu} - 1)\right] - t \quad (2.7)$$

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TABLE I

BEHAVIOUR OF THE HAZARD FUNCTION OF THE CONDITIONAL AND COMPOUND GOMPERTZ DISTRIBUTION

	Conditional	Compound
$\delta = 0$	constant	decreasing
$0 < \delta < \frac{1}{\beta}$	increasing	decreasing
$\delta = \frac{1}{\beta}$	increasing	constant
$\delta > \frac{1}{\beta}$	increasing	increasing

3. NEW PROPERTIES OF THE COMPOUND GOMPERTZ DISTRIBUTION

3.1 The Density Function

The first derivative of f (t) is found to be

$$f'(t) = \mu\, \beta^{\mu}\, \delta^{\mu+2}\, e^{\delta t} (\delta\beta + e^{\delta t} - 1)^{-\mu-2} (\delta\beta - 1 - \mu e^{\delta t}) \tag{3.1}$$

When $\delta\beta > \mu+1$, f(t) increases reaching its maximum at $t^* = \frac{1}{\delta} \ln \left(\frac{\delta\beta-1}{\mu}\right)$

and then decreases, and $f(t^*) = \left(\frac{\delta\mu}{\mu+1}\right) \left[\frac{(\mu+1)(\beta\delta-1)}{\mu\beta\delta}\right]^{-\mu}$.

When $\delta\beta \leq \mu+1$, f(t) is a monotone decreasing function.

Some typical compound Gompertz density functions are shown in Figure I. The Figure illustrates that δ is a scale parameter and that the shape of f(t) depends on the value of $\delta\beta$.

3.2 The Hazard Function

The hazard function of the compound Gompertz distribution is given as

$$h(t) = \delta \mu e^{\delta t} (\delta\beta + e^{\delta t} - 1)^{-1} \qquad (3.2)$$

studying the above equation it is found that

(i) $h(0) = \mu / \beta$

(ii) $\lim_{t \to \infty} h(t) = \delta \mu$

(iii) Osman (1987) showed that h(t) is increasing if $(\delta\beta - 1) > 0$ decreasing if $(\delta\beta - 1) < 0$ and constant if $(\delta\beta - 1) = 0$.

Then from (i), (ii) and (iii), it is concluded that the hazard function of the compound Gompertz distribution starts from the point μ/β at $t = 0$, then increases or decreases, depending on the value of $\delta\beta$, to reach some limit that equal to $\delta\mu$ as t approaches infinity. In other words, h (0) is larger than h (∞) if and only if $\delta\beta < 1$, is equal to h (∞) if and only if $\delta\beta = 1$, and is smaller than h (∞) if and only if $\delta\beta > 1$. Figure II indicates that the shape of h (t) depends on the value of $\delta\beta$, and that the level of h (t) depends on the value of μ.

3.3 The Survival Function

The concavity of the survival function of the distribution depends on the value of $\delta\beta$ and the greater the value of μ the more rapid the decline of S (t) towards the zero point is.

3.4 The Moment Generating Function

Let T be a r.v. that follows the compound Gompertz distribution with parameters δ, β, and μ, then the moment generating function of t when $\delta\beta > 1$ and $\mu > s/\delta$ takes the form.

$$M_t(s) = \mu\,(\delta\beta-1)^{s/\delta-\mu}\,(\delta\beta)^{\mu}\,\beta\left(\frac{s}{\delta}+1,\mu-\frac{s}{\delta}\right)\left[1-I_{1/\delta\beta}\left(\frac{s}{\delta}+1;\mu-\frac{s}{\delta}\right)\right] \quad (3.4)$$

where, $$I_k(p,q) = \int_0^k \frac{x^{p-1}(1-x)^{q-1}dx}{\beta(p,q)}, \qquad k<1,\quad p\ q>0$$

is the incomplete beta function ratio, and $\beta(.,.)$ is the beta function.

Proof:

$$M_t(s) = E(e^{st}) = \int_{-\infty}^{\infty} e^{st} f(t)\,dt \quad (3.5)$$

$$= \int_0^{\infty} e^{st}\left[\frac{\mu}{\beta} e^{\delta t}(\delta\beta)^{\mu+1}(\delta\beta-1+e^{st})^{-\mu-1}\right]dt \quad (3.6)$$

Let $x = e^{st}/(\delta\beta-1+e^{st})$. (3.7)

Substituting x in (3.6),

$$M_t(s) = \mu(\delta\beta)^{\mu}(\delta\beta-1)^{s/\delta-\mu}\int_{\frac{1}{\delta\beta}}^{1} x^{s/\delta}(1-x)^{(\mu-s/\delta-1)}dx$$

$$= \mu(\delta\beta)^{\mu}(\delta\beta-1)^{s/\delta-\mu}\left[\beta\left(\frac{s}{\delta}+1,\mu-\frac{s}{\delta}\right)-\int_0^{\frac{1}{\delta\beta}} x^{s/\delta}(1-x)^{(\mu-s/\delta-1)}dx\right]$$

$$= \mu(\delta\beta)^{\mu}(\delta\beta-1)^{s/\delta-\mu}\,\beta\left(\frac{s}{\delta}+1,\mu-\frac{s}{\delta}\right)\left[1-I_{1/\delta\beta}\left(\frac{s}{\delta}+1;\mu-\frac{s}{\delta}\right)\right],$$

provided that $\delta\beta > 1$ and $\mu > \frac{s}{\delta}$.

An explicit expression for $M_t(s)$ can not be derived so the compound Gompertz distribution is an example of continuous distributions for which the characteristic function cannot be used to determine moments and direct evaluation is needed to compute them.

3.5 The Mean

The mean of the compound Gompertz distribution is derived for the following cases:

a. If $\delta\beta = 1$, then

$$E(t) = 1/\delta\mu = \beta/\mu \qquad (3.8)$$

b. If $\delta\beta \neq 1$, and $\mu = i + f$

when i is an integer ≥ 1, and $0 \leq f < 1$, then

$$E(t) = \beta \left[\sum_{r=1}^{i=1} \frac{(-1)^{r-1} (\delta\beta)^{r-1}}{(\mu-r)(1-\delta\beta)^{r}} + \frac{(-1)^{i-1} (\delta\beta)^{i-1}}{(1-\delta\beta)^{i-1}} I \right] \qquad (3.9)$$

where $I = \frac{-1}{1-\delta\beta} \ln \delta\beta,$ if $f = 0$,

$I = \int_0^1 \frac{z^f}{\delta\beta + (1-\delta\beta) z} dz.$ if $0 < f < 1$,

which can be calculated numerically.

c. If $\delta\beta \neq 1$ and $0 < \mu < 1$, then

$$E(t) = \beta \int_0^1 \frac{z^{\mu-1}}{\delta\beta + (1-\delta\beta) z} dz \qquad (3.10)$$

That depends on the value of μ, and can be calculated numerically also.

Proof:

$$E(t) = \int_{-\infty}^{\infty} t f(t) dt = \int_0^{\infty} t \frac{\mu}{\beta} e^{\delta t} \left(1 + \frac{e^{\delta t} - 1}{\delta\beta}\right)^{-\mu-1} dt \qquad (3.11a)$$

using integration by parts, then (3.11a) reduces to

$$E(t) = \int_0^{\infty} \left(1 + \frac{e^{\delta t} - 1}{\delta\beta}\right)^{-\mu} dt \qquad (3.11b)$$

Let $Z = \delta\beta / (\delta\beta + e^{\delta t} - 1)$.

Substituting Z in (3.11b),

$$E(t) = \beta \int_0^1 \frac{z^{\mu-1}}{\delta\beta + (1-\delta\beta)z} dz \qquad (3.12)$$

From (3.12),

(i) when $\delta\beta = 1$, E(t) reduces to

$$E(t) = \beta \int_0^1 z^{\mu-1} dz = \frac{\beta}{\delta} = \frac{1}{\delta\mu}$$

and that is the mean of the exponential distribution that the compound Gompertz reduces to when $\delta\beta = 1$.

Hence the proof of (3.8).

(ii) When $\delta\beta \neq 1$, and $\mu = i + f$, where i is integer ≥ 1, and $0 \leq f < 1$, then. (3.12) can be written as

$$E(t) = \beta \int_0^1 \left[\sum_{r=1}^{i=1} \frac{(-1)^{r-1} (\delta\beta)^{r-1}}{(1-\delta\beta)^r} z^{i-(r+1)} + \frac{(-1)^{i-1} (\delta\beta)^{i-1}}{(1-\delta\beta)^{i-1}} \frac{z^f}{\delta\beta + (1-\delta\beta)z} \right] dz \quad (3.13)$$

which yields

$$E(t) = \beta \left[\sum_{r=1}^{i=1} \frac{(-1)^{r-1} (\delta\beta)^{r-1}}{(i-r)(1-\delta\beta)^r} + \frac{(-1)^{i-1} (\delta\beta)^{i-1}}{(1-\delta\beta)^{i-1}} I \right]$$

where, $I = \int_0^1 \frac{z^f}{\delta\beta + (1-\delta\beta)z} dz$ when $0 < f < 1$, and

$$I = \int_0^1 \frac{dz}{\delta\beta + (1-\delta\beta)z} \text{ when } f = 0$$

$$= \frac{1}{1-\delta\beta} \ln \delta\beta$$

and hence the proof of (3.9)

(iii) when $\delta\beta \neq 1$, and $0 < \mu < 1$, directly from (3.12)

$$E(t) = \beta \int_0^1 \frac{z^{\mu-1}}{\delta\beta + (1-\delta\beta)z} dz$$

Hence the proof of (3.10).

In Table II the mean of the compound Gompertz distribution is tabulated for 140 sets of parameters.

4. PARAMETRIC ESTIMATION

The method of moments and the method of maximum likelihood are used for estimating the parameters of the compound Gompertz distribution.

4.1 Method of Moments

For the compound Gompertz distribution, let θ denote the vector of parameters $\underline{\theta} = (\delta, \mu, \beta)$ and let $G_r(\underline{\theta})$ denotes the r^{th} moment about the origin

$$G_r(\underline{\theta}) = E(t^r) = \int_0^\infty t \frac{\mu}{\beta} e^{\delta t} (1 + \frac{e^{\delta t} - 1}{\delta\beta})^{-\mu-1} dt$$

and let $Q_r(\underline{\theta}) = \frac{1}{n} \sum_{i=1}^{n} t_i^r$ denote the r^{th} sample moment about origin.

By equating $G_r(\underline{\theta})$ to $Q_r(\underline{\theta})$ for $r = 1, 2, 3$, it is not possible to obtain the moment estimate $\underline{\theta}$ explicitly. However, when $\delta = 1$ and $\mu > 2$, it is possible to get explicit estimates.

Dubey (1968) has shown that the r^{th} moment of the compound weibull distribution exists for $\delta\mu > r$ and is given by:

$$\theta_r(\theta) = \mu\, \beta^{r/\delta}\, \beta(\mu - \frac{r}{\delta}, \frac{r}{\delta} + 1) \tag{4.1}$$

where $\beta(.,.)$ is the beta function.

Parekh (1972) has used the above formula to derive explicit estimates for μ, β given $\delta = 1$ as

$$\hat{\mu} = \frac{2\hat{\phi}_2 - 2\hat{\phi}_1^2}{\hat{\phi}_2 - 2\hat{\phi}_1^2}, \tag{4.2}$$

$$= \frac{2(\hat{\phi}_2 - \hat{\phi}_1^2)}{(\hat{\phi}_2 - \hat{\phi}_1^2) - \hat{\phi}_1^2}$$

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TABLE II
MEAN OF THE COMPOUND GOMPERTZ DISTRIBUTION

		δβ						
δ	μ	0.1	0.5	1	5	10	15	20
	1	25.8428	69.3147	100.0000	201.1797	255.8400	290.1482	315.3402
	2	8.2684	30.6853	50.0000	126.4747	173.1586	203.7302	226.6738
δ=0.01	3	4.6368	19.3147	33.3333	95.5933	136.8429	164.7110	185.9724
	4	3.2057	14.0186	25.0000	77.8251	115.0106	140.7617	160.6728
	5	3.0408	10.9814	20.0000	66.0313	100.0118	124.0304	133.4150
	1	2.5843	6.9315	10.0000	20.1180	25.5840	29.0148	31.5340
	2	0.8268	3.0685	5.0000	12.6475	17.3158	20.3730	22.6674
δ=0.1	3	0.4637	1.9315	3.3333	9.5593	13.6843	16.4711	18.5972
	4	0.3206	1.4019	2.5000	7.7825	11.5011	14.0762	16.0673
	5	0.3041	1.0981	2.0000	6.6031	10.0012	12.4030	13.3415
	1	0.2584	0.6932	1.0000	2.0118	2.5584	2.9015	3.1534
	2	0.0827	0.3069	0.5000	1.2648	1.7316	2.0373	2.2667
δ=1	3	0.0464	0.1992	0.3333	0.9559	1.3684	1.6171	1.8597
	4	0.0321	0.1401	0.2500	0.7783	1.1501	1.4076	1.6067
	5	0.0304	0.1098	0.2000	0.6603	1.0001	1.2403	1.3342
	1	0.0258	0.0693	0.1000	0.2012	0.2558	0.2902	0.3153
	2	0.0083	0.0307	0.0500	0.1265	0.1732	0.2037	0.2267
δ=10	3	0.0046	0.0193	0.0333	0.0956	0.1368	0.1647	0.1860
	4	0.0032	0.0140	0.0250	0.0778	0.1150	0.1408	0.1607
	5	0.0030	0.0110	0.0200	0.0660	0.1000	0.1240	0.1334

$$= 2 + \frac{2\hat{\phi}_1^2}{\hat{\phi}_2 - \hat{\phi}_1^2}$$

$$\hat{\beta} = \hat{\phi}_1 / \hat{\mu}\, \beta(\hat{\mu}-1,2) \tag{4.3}$$

when $\hat{\phi}_r(\underline{\theta})$ is the r^{th} sample moment about the origin and Osman (1987) has shown that:

If the r.v.t. follows the compound Gompertz distribution with parameters μ, β and δ, then the r.v.c. where $c = (\frac{e^{\delta t}-1}{\delta})^{1/\delta}$ follows the compound Weibull distribution with parameters δ, β and μ with p.d.f.

$$g(c) = \frac{\mu}{\beta}\, \delta\, c^{\delta-1} \left(1 + \frac{c^{\delta}}{\beta}\right)^{-\mu-1} \tag{4.4}$$

Hence from the above results, for the compound Gompertz distribution given $\delta = 1$.

Let $c = e^t - 1$, then c follows the compound weibull distribution with parameters 1, β and μ and hence (4.2), (4.3) can be used as an estimate for μ, β given $\delta = 1$ and $\mu > 2$ where,

$$\hat{\phi}_1 = \frac{1}{n} \sum_{i=1}^{n} c_i = \frac{1}{n} \sum_{i=1}^{n} (e^t - 1),$$

$$\hat{\phi}_2 = \frac{1}{n} \sum_{i=1}^{n} c_i^2 = \frac{1}{n} \sum_{i=1}^{n} (e^{t_i} - 1)^2.$$

When the sample standard deviation of c is greater than its sample mean, $\hat{\mu}$ is larger than 2. Then, any permissible estimator for μ will be greater than 2.

From equation (4.3) it is also seen that

$$\hat{\beta} = \hat{\phi}_1 / \left[\hat{\mu}\, B(\hat{\mu}-1,2)\right]$$

$$= \hat{\phi}_1 \Gamma(\hat{\mu}+1) / [\hat{\mu} \Gamma(\hat{\mu}-1) \Gamma(2)]$$

$$= \hat{\mu}(\hat{\mu}-1) \hat{\phi}_1 / \hat{\mu} = (\hat{\mu}-1) \hat{\phi}_1 \tag{4.5}$$

Hence $\hat{\beta} > 0$ for $\hat{\mu} > 1$. Therefore, wherever $\hat{\mu}$ is permissible, $\hat{\beta}$ will be permissible.

4.2 Maximum Likelihood Estimation

The method of maximum likelihood remains one of the most important methods of estimation due to several reasons. The method is intuitively appealing and the likelihood equations can be written quite easily. In most cases, it may not be possible to obtain explicit estimates from the equations, but their numerical solution can always be computed. Therefore, the application of the method of maximum likelihood will be investigated for estimating the parameters of the compound Gompertz distribution.

Let $t_1, t_2, \ldots, t_n$ denote a random sample of size n from the compound Gompertz distribution, the likelihood equations can be written as

$$\frac{\partial \ln L}{\partial \delta} = n(\mu+1)/\delta + \Sigma t_i - (\mu+1) \Sigma (\beta + t_i e^{\delta t_i})(\delta \beta + e^{\delta t_i} - 1)^{-1} \tag{4.6}$$

$$\frac{\partial \ln L}{\partial \mu} = \frac{n}{\mu} + n \ln \beta + n \ln \delta - \sum_{i=1}^{n} \ln (\delta \beta + e^{\delta t_i} - 1) \tag{4.7}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n\mu}{\beta} - \delta(\mu+1) \sum_{i=1}^{n} (\delta \beta + e^{\delta t_i} - 1)^{-1} \tag{4.8}$$

equating these expressions to zero, it is not possible to obtain explicit expressions for δ, μ and β simultaneously. Hence, numerical iterative techniques have to be applied.

4.3 MLE under Various Types of Censoring

The likelihood equations under type I, type II and progressive censoring are considered.

4.3.1 Singly Censored Samples

In this case N items are placed on test and at each failure the time of failure is noted, at some pre-determined fixed time t_0 or after some pre-determined fixed number of sample specimen fail, the test is terminated. In both these cases the data collected consists of observations $t_1, t_2, ..., t_n$ and the information that (N - n) items survived beyond the time of termination t_0 in the former case and t_n in the latter.

When t_0 is fixed and n is thus a random variable, censoring is said to be of type I. When n is fixed and time of termination t_n is a random variable, censoring is said to be of type II. The likelihood function may be written as,

$$L = \frac{N!}{(N-n)} \left[\prod_{i=1}^{n} f(t) \right] [1 - F(T)]^{(N-n)} \tag{4.9}$$

where $T = t_0$ in type I censoring, and $T = t_n$ in type II censoring.

Therefore, for a censored sample from the compound Gompertz distribution the likelihood equations can be written as

$$\frac{\partial \ln L}{\partial \delta} = \frac{n+N\mu}{\delta} + \sum t_i - (\mu+1) \sum_{i=1}^{n} (\beta + t_i e^{\delta t_i} - 1)(\delta\beta + e^{\delta t_i} - 1)^{-1}$$

$$- \mu (N-n)(\beta + T e^{\delta T})(\delta\beta + e^{\delta T} - 1)^{-1} . \tag{4.10}$$

$$\frac{\partial \ln L}{\partial \mu} = \frac{n}{\mu} + N \ln \beta + N \ln \delta - \sum_{i=1}^{n} \ln (\delta\beta + e^{\delta t_i} - 1)$$

$$- (N-n) \ln (\delta\beta + e^{\delta T} - 1). \tag{4.11}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{N\mu}{\beta} - \delta(\mu+1) \sum_{i=1}^{n} (\delta\beta + e^{\delta t_i} - 1)^{-1} - \delta\mu(N-n)(\delta\beta\, e^{\delta T} - 1)^{-1} \tag{4.12}$$

4.3.2 Progressively Censores Samples

Suppose the censoring occurs in k stages at times T_i where $T_i > T_{i-1}$, i = 1, 2, ..., k and that at the i^{th} stage of censoring r_i sample specimens selected randomly from the survivors at time T_i are removed (censored) from further observations. Let N be the total sample size and n the number of specimens which fail and therefore provide completely determined life spans, it follows that

$$N = n + \sum_{i=1}^{k} r_i$$

For type 1 censoring, where the T_i are fixed the likelihood function may be written as

$$L = c \prod_{i=1}^{n} f(t_i) \prod_{i=1}^{k} \left[1 - F(T_i)\right]^{r_i} \qquad (4.13)$$

where c is a constant.

Hence, for the compound Gompertz distribution, the likelihood equations under the above mentioned type of progressive censoring is given by,

$$\frac{\partial L}{\partial \delta} = (n(\mu+1) + \mu \Sigma r_i) / \delta - (\mu+1) \sum_{i=1}^{n} (\beta + t_i e^{\delta t_i}) (\delta\beta + e^{\delta t_i} - 1)^{-1}$$

$$- \mu \sum_{i=1}^{k} r_i (\beta + T_i e^{\delta T_i}) (\delta\beta + e^{\delta T_i} - 1)^{-1} + \sum_{i=1}^{n} t_i \qquad (4.14)$$

$$\frac{\partial L}{\partial \mu} = n/\mu + (n + \sum_{i=1}^{k} r_i) \ln \beta + (n + \Sigma r_i) \ln \delta - \sum_{i=1}^{n} \ln (\delta\beta + e^{\delta t_i} - 1)$$

$$- \sum_{i=1}^{k} r_i \ln (\delta\beta + e^{\delta T_i} - 1) \qquad (4.15)$$

$$\frac{\partial L}{\partial \beta} = \mu(n + \Sigma r_i) / \beta - (\mu+1) \delta \sum_{i=1}^{n} (\delta\beta + e^{\delta t_i} - 1)^{-1}$$

$$- \delta\mu \sum_{i=1}^{k} r_i (\delta\beta + e^{\delta T_i} - 1)^{-1} \qquad (4.16)$$

5. APPLICATION

In a study by Birnbaum and Saunders (1958) seventeen sets of six strips were placed in a specially designed machine. Periodic loading was applied to the strips with a frequency of 18 cycles per second, and a maximum stress of 21,000 psi. The 102 strips were run until all of them failed. One of the 102 strips were run until all of them failed. One of the 102 strips tested had to be discarded for an extraneous reason, yielding 101 observations. Bimbaum and Saunders (1958) used the Gamma distribution to model strips failure time and estimated its two parameters. The goodness of fit test yielded a Chi-square value of 4.46 (d.f. = 6).

The compound Gompertz distribution has been used to model the same data. The goodness of fit test indicates that the compound Gompertz distribution provides a good fit (χ^2=4.31, d.f.=5), i.e. both theoretical distributions can fit the data.

6. DISCUSSION

When compared to the Gamma distribution, the compound Gompertz distribution provided a slightly better fit in modelling lifetime of aluminum coupons. The compound Gompertz distribution has two advantages over the Gamma distribution. First, its hazard function has explicit solution. Moreover, it can be seen as an infinite mixture of lifetimes following a Gompertz distribution reflecting the one-hit model described by Elandt-Johnson and Johnson (1980). The one-hit model assumes that it is sufficient to change one type of cell or a single site in a cell to initiate the disease. The model assumes that during the course of the interaction between the carcinogenic agent and cell constituents, the following events can occur: the defense mechanisms eliminate part of the agent (at rate δ_e), or some cells adapt the agent (at rate δ_a) but some others undergo multiplication (at rate δ_g). To initiate the disease we must have $\delta_g > \delta_e + \delta_a$ and the rate of cancer growth is $h(t) = \lambda \exp(\delta t)$, where $\delta = \delta_g - \delta_e - \delta_a$, and λ is a constant depending on concentrations of various cell constituents and toxicity of the agent.

The art of modeling is not only to find a mathematical formula that provides the best fit. Finding the mathematical formulation that possesses a meaningful explanation of the process at hand and provides a good fit to the observed data is certainly a goal that should be fulfilled in modeling. By using a conceptually meaningful underlying distribution and allowing for heterogenity to exist, the compound Gompertz distribution is a model that can help in achieving that goal.

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FIGURE I
Density function of the compound Gompertz distribution.

δ=.25 δ=.125

δB=.25

δB=.5

δB=1

δB=2

δB=4

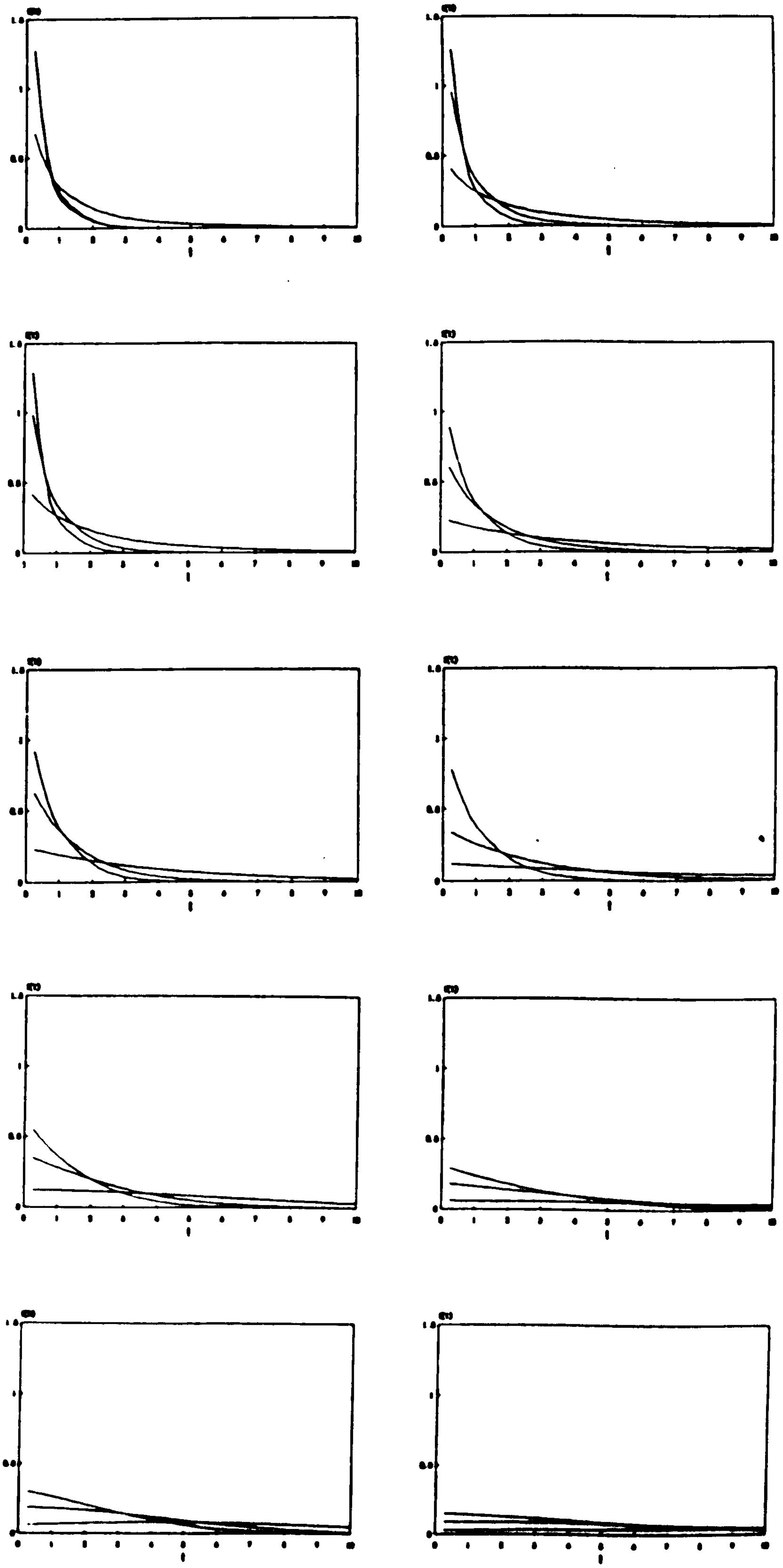

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FIGURE II
Hazard function of the compound Gompertz distribution.