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ESTIMATION OF PARAMETERS OF COMPLETE EVEN POWER EXPONENTIAL DISTRIBUTION

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Mostafa and Mazloum (1995) have presented geometric and analytical characteristics of the complete even power exponential distribution. In this paper, the problem of estimating the parameters of this distribution using both methods of moments and maximum likelihood is considered.

1. Introduction

The general form of the complete even power exponential density is given by:

$$f_{b,m,\mu,\sigma}(x) = \frac{m}{\sigma b^{1/2m}\Gamma(1/2m)} e^{-(1/b)((x-\mu)/\sigma)^{2m}} \qquad (1\text{-}1)$$

where $-\infty < x < \infty$; $-\infty < \mu < \infty$; $\sigma > 0$, m is a positive integer; $b > 0$ and m and b are chosen arbitrarily.

The form (1-1) produces families of distributions depending upon the values of m and b chosen. For example, $m = 1$, and $b = 2$ yield the normal distribution. Also, $m = 2$ and $b = 1$ give the special form of the 4th power exponential distribution termed complete fourth power exponential distribution. The problem of estimating μ and σ for this later distribution has been considered by Ibrahim, A.R. (1993).

The density function $f_{b,m,\mu,\sigma}(\cdot)$ given by (1-1) is unimodal and symmetric about its mean μ, i.e.

$$E(X) = \mu \qquad (1\text{-}2)$$

and

$$\mu_{2r+1} = E(X-\mu)^{2r+1} = 0 \qquad r = 1,2,3,\ldots \qquad (1\text{-}3)$$

The even central moments are:

$$\mu_{2r}(m) = E(X-\mu)^{2r} = \frac{\sigma^{2r}\ b^{r/m}\Gamma((2r+1)/2m)}{\Gamma(1/2m)} \quad r = 1,2,3,\ldots \qquad (1\text{-}4)$$

In particular,

$$\mu_2(m) = \frac{\sigma^2\ b^{1/m}\ \Gamma(3/2m)}{\Gamma(1/2m)} \qquad (1\text{-}5)$$

and

$$\mu_4(m) = \frac{\sigma^4\ b^{2/m}\Gamma(5/2m)}{\Gamma(1/2m)} \qquad (1\text{-}6)$$

The class of densities $\{f_{b,m,\mu,\sigma}(\cdot)$: $b > 0$ and m is a positive integer$\}$ has been studied geometrically and analytically by Mostafa and Mazloum (1995).

In section 2, the method of moment's estimators, for the location and scale parameters of such distributions, as well as their associated properties are obtained.

In section 3, the maximum likelihood estimators of those parameters along with some of their properties are discussed.

In section 4, an iterative method for solving a likelihood equation for estimating the location parameter μ is suggested.

2. Moment's Estimators of μ and $\sigma^2(\sigma)$

Let $X_1, X_2, \ldots, X_n$ be a random sample from the density $f_{b,m,\mu,\sigma}(\cdot)$ defined in (1-1). The method of moments estimators $\hat{\mu}$ and $\hat{\sigma}^2$ of μ and σ^2 are the solutions of the equations:

$$E(X^r) = \frac{1}{n}\sum_{i=1}^{n} X_i^r \qquad r = 1,2 \qquad (2\text{-}1)$$

Now, substituting from (1-2) and (1-5) into (2-1), we get the moments estimators:

$$\hat{\mu} = \bar{X}_n \tag{2-2}$$

and

$$\hat{\sigma}^2 = \frac{\Gamma(1/2m)}{b^{1/m}\,\Gamma(3/2m)}\, S_n^2 \quad \text{where } S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 \tag{2-3}$$

Also,

$$\hat{\sigma} = \sqrt{\frac{\Gamma(1/2m)}{b^{1/m}\,\Gamma(3/2m)}}\; S_n \tag{2-4}$$

Properties of the Moments Estimators $\hat{\mu}$ and $\hat{\sigma}^2(\hat{\sigma})$

i) $\hat{\mu}$ is an unbiased and consistent estimator of μ. Moreover, $\hat{\mu}$ is asymptotically normally distributed (AN), i.e.

$$\hat{\mu} \sim AN\left(\mu\ ,\ \frac{\sigma^2 b^{1/m}\,\Gamma(3/2m)}{n\,\Gamma(1/2m)}\right) \tag{2-5}$$

ii) $\hat{\sigma}^2(\hat{\sigma})$ is biased, consistent and asymptotically normally distributed, i.e.

$$\hat{\sigma}^2 \sim AN\left(\sigma^2, \frac{\sigma^4}{n}\left[\frac{\Gamma(1/2m)\,\Gamma(5/2m)}{(\Gamma(3/2m))^2} - 1\right]\right) \tag{2-6}$$

and

$$\hat{\sigma} \sim AN\left(\sigma, \frac{\sigma^2}{4n}\left[\frac{\Gamma(1/2m)\,\Gamma(5/2m)}{(\Gamma(3/2m))^2} - 1\right]\right) \tag{2-7}$$

iii) $\hat{\mu}$ and $\hat{\sigma}^2$ are uncorrelated.

Proof of part (i) is easy and therefore is omitted.

Proof of part (ii):

a) $$E(\hat{\sigma}^2) = \frac{\Gamma(1/2m)}{b^{1/m}\,\Gamma(3/2m)}\, E(S_n^2) \tag{2-8}$$

It is well known [1] that

$$E(S_n^2) = \frac{n-1}{n}\,\mu_2(m) \tag{2-9}$$

Hence, (1-5), (2-8) and (2-9) give:

$$E(\hat{\sigma}^2) = \frac{n-1}{n}\sigma^2 \qquad (2\text{-}10)$$

i.e., $\hat{\sigma}^2$ is negatively biased. However, it is asymptotically unbiased.

Now, to show that $\hat{\sigma}$ is biased, we proceed as follows:

Since $\hat{\sigma}$ is a nondegenerate random variable, then $v(\hat{\sigma}) > 0$, i.e., $E(\hat{\sigma}^2) > E^2(\hat{\sigma})$. Substituting from (2-10), we get:

$$\frac{(n-1)}{n}\sigma^2 > E^2(\hat{\sigma})$$

Hence,

$$E(\hat{\sigma}) < \sigma$$

i.e., $\hat{\sigma}$ is negatively biased.

b) To prove the asymptotic normality of $\hat{\sigma}^2$ and $\hat{\sigma}$, given by (2-6) and (2-7), we need the following theorem cited in [6]:

<u>Theorem</u>

Let X_n be $AN(\mu,\sigma_n^2)$ with $\sigma_n \to 0$ and $g(.)$ be a real valued function. If $g(.)$ is differentiable at $X = \mu$ with $g'(\mu) \neq 0$, then

$$g(x_n) \text{ is } AN(g(\mu),\{g'(\mu)\}^2\,\sigma_n^2).$$

It is well known [1] that:

$$S_n^2 \sim AN(\mu_2, \frac{\mu_4 - \mu_2^2}{n}) \qquad (2\text{-}11)$$

From (1-5), (1-6) and (2-11), we notice that $\sqrt{((\mu_4 - \mu_2^2)/n)} \to 0$ as $n\to\infty$.

Now, to prove the asymptotic normality of $\hat{\sigma}^2$, we take

$$g(S_n^2) = \hat{\sigma}^2 = \frac{\Gamma(1/2m)}{b^{1/m}\ \Gamma(3/2m)} S_n^2 \qquad (2\text{-}12)$$

Note that

$$g'(\mu_2) = \frac{\Gamma(1/2m)}{b^{1/m}\ \Gamma(3/2m)} \neq 0 \qquad (2\text{-}13)$$

Hence, by the previous theorem, we get:

$$\hat{\sigma}^2 \sim AN(g(\mu_2),\{g'(\mu_2)\}^2\ \sigma_n^2) \tag{2-14}$$

where

$$g(\mu_2) = \frac{\Gamma(1/2m)}{b^{1/m}\ \Gamma(3/2m)}\ \mu_2 \tag{2-15}$$

and

$$\{g'(\mu_2)\}^2\ \sigma_n^2 = [\frac{\Gamma(1/2m)}{b^{1/m}\ \Gamma(3/2m)}]^2(\frac{\mu_4 - \mu_2^2}{n}) \tag{2-16}$$

Substituting for μ_2 and μ_4 from (1-5) and (1-6) into (2-15) and (2-16), we get:

$$g(\mu_2) = \sigma^2 \tag{2-17}$$

and

$$\{g'(\mu_2)\}^2\ \sigma_n^2 = \frac{\sigma^4}{n}\ [\frac{\Gamma(1/2m)\ \Gamma(5/2m)}{\{\Gamma(3/2m)\}^2} - 1] \tag{2-18}$$

Hence, (2-14), (2-17) and (2-18) give:

$$\hat{\sigma}^2 \sim AN(\sigma^2, \frac{\sigma^4}{n}\ [\frac{\Gamma(1/2m)\ \Gamma(5/2m)}{(\Gamma(3/2m))^2} - 1] \tag{2-19}$$

To prove the asymptotic normality of $\hat{\sigma}$, we take:

$$g(S_n^2) = \hat{\sigma} = \sqrt{\frac{\Gamma(1/2m)}{b^{1/m}\ \Gamma(3/2m)}}\ S_n \tag{2-20}$$

Note that:

$$g'(\mu_2) = \sqrt{\frac{\Gamma(1/2m)}{b^{1/m}\ \Gamma(3/2m)}} \cdot \frac{1}{2\mu_2^{\frac{1}{2}}}$$

Substituting for μ_2 from (1-5) we get:

$$g'(\mu_2) = \frac{\Gamma(1/2m)}{b^{1/m}\ \Gamma(3/2m)} \cdot \frac{1}{2\sigma} \neq 0 \tag{2-21}$$

Hence, by the previous theorem, we get:

$$\hat{\sigma} \sim AN(g(\mu_2),\{g'(\mu_2)\}^2\ \sigma_n^2) \tag{2-22}$$

where

$$g(\mu_2) = \sqrt{\frac{\Gamma(1/2m)}{b^{1/m}\,\Gamma(3/2m)}\,\mu_2} \tag{2-23}$$

and

$$\{g'(\mu_2)\}^2\,\sigma_n^2 = [\frac{\Gamma(1/2m)}{b^{1/m}\,\Gamma(3/2m)} \cdot \frac{1}{2\sigma}]^2 (\frac{\mu_4 - \mu_2^2}{n}) \tag{2-24}$$

Substituting for μ_2 and μ_4 from (1-5) and (1-6) into (2-23) and (2-24), we get:

$$g(\mu_2) = \sigma \tag{2-25}$$

and

$$\{g'(\mu_2)\}^2\,\sigma_n^2 = \frac{\sigma^2}{4n}\,[\frac{\Gamma(5/2m)\,\Gamma(1/2m)}{(\Gamma(3/2m))^2} - 1] \tag{2-26}$$

Consequently, (2-22), (2-25) and (2-26) imply

$$\hat{\sigma} \sim AN(\sigma, \frac{\sigma^2}{4n}\,[\frac{\Gamma(5/2m)\,\Gamma(1/2m)}{(\Gamma(3/2m))^2} - 1] \tag{2-27}$$

c) The mean squared error consistency of $\hat{\sigma}^2$ and that of $\hat{\sigma}$ follow part (b) since (2-19) and (2-27) show that both $\hat{\sigma}^2$ and $\hat{\sigma}$ are asymptotically unbiased and both $v(\hat{\sigma}^2)$ and $v(\hat{\sigma})$ tend to 0 as n tends to ∞.

<u>Proof of part (iii)</u>

To show that $\hat{\mu}$ and $\hat{\sigma}^2$ are uncorrelated, we appeal to the following well known result [2]:

"For a random sample of n from any symmetric parent distribution, an odd location statistic and an even location free statistic are uncorrelated where:

"A statistic $T_1(X_1, X_2, \ldots, X_n)$ is an odd location statistic if for all $x_1, x_2, \ldots, x_n$, and every h, $T_1(x_1+h, x_2+h, \ldots, x_n+h) = T_1(x_1, x_2, \ldots, x_n)$+ and $T_1(-x_1, -x_2, \ldots, -x_n) = -T_1(x_1, x_2, \ldots, x_n)$"
Likewise " a statistic $T_2(X_1, X_2, \ldots, X_n)$ is an even location free statistic if for all $x_1, x_2, \ldots, x_n$ and every h,

$$T_2(x_1+h, x_2+h, \ldots, x_n+h) = T_2(x_1,x_2,\ldots,x_n)$$

and

$$T_2(-x_1,-x_2,\ldots,-x_n) = T_2(x_1,x_2,\ldots,x_n)".$$

Now, it is easy to show that $\hat{\mu}$ is an odd location statistic, $\hat{\sigma}^2$ is an even location statistic and the parent distribution is symmetric. Hence, by the above result, $\hat{\mu}$ and $\hat{\sigma}^2$ are uncorrelated.

3. Maximum Likelihood (ML) Estimators of μ and σ

For a random sample of size n from the density $f_{b,m,\mu,\sigma}(\cdot)$ given by (1-1), the likelihood function is:

$$L(\mu,\sigma) = (K')^n \frac{1}{\sigma^n} e^{-\frac{1}{b}\sum_{i=1}^{n}((x_i-\mu)/\sigma)^{2m}} \qquad (3\text{-}1)$$

where $K' = \dfrac{m}{b^{1/2m}\,\Gamma(1/2m)}$.

The logarithm of the likelihood function is:

$$\ell n\, L(\mu,\sigma) = n\,\ell n\, K' - n\,\ell n\sigma - \frac{1}{b}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{2m}$$

Now, differentiating $\ell n\, L(\mu,\sigma)$ with respect to μ and σ and setting the partial derivatives equal to 0 yield the likelihood equations:

$$\frac{2m}{b\,\sigma^{2m}}\sum_{i=1}^{n}(x_i-\mu)^{2m-1} = 0 \qquad (3\text{-}2)$$

and

$$-\frac{n}{\sigma} + \frac{2m}{b\,\sigma^{2m+1}}\sum_{i=1}^{n}(x_i-\mu)^{2m} = 0 \qquad (3\text{-}3)$$

Note that for any simultaneous solution (μ,σ) of the equations (3-2) and (3-3) we have:

$$\frac{\partial^2 \ell n\, L}{\partial\mu^2} = \frac{-2m(2m-1)}{b\,\sigma^{2m}}\sum_{i=1}^{n}(x_i-\mu)^{2m-2} < 0, \qquad (3\text{-}4)$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{2m(2m+1)}{b\,\sigma^{2m+2}} \sum_{i=1}^{n} (x_i-\mu)^{2m}$$

$$= -\frac{2mn}{\sigma^2} < 0 \qquad (3\text{-}5)$$

and

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} = -\frac{(2m)^2}{b\,\sigma^{2m+1}} \sum_{i=1}^{n} (x_i-\mu)^{2m-1} = 0 \qquad (3\text{-}6)$$

Hence,

$$\left(\frac{\partial^2 \ln L}{\partial \mu^2}\right)\left(\frac{\partial^2 \ln L}{\partial \sigma^2}\right) - \left(\frac{\partial^2 \ln L}{\partial \mu\, \partial \sigma}\right)^2 > 0$$

Consequently, any simultaneous solution of the equations (3-2) and (3-3) maximizes the likelihood function $L(\mu,\sigma)$.

It is clear that equation (3-2) is of degree 2m-1 and hence, it has at most (2m-1) solutions. Consequently, the likelihood equations (3-2) and (3-3) have at most (2m-1) solutions. However, as we have seen, all these solutions maximize the likelihood function which is impossible unless they are all equal. Hence, the likelihood equations have only a unique solution.

In section 4, a suggested iterative method for solving equation (3-2) is presented. Suppose the solution of this equation is $\tilde{\mu}$, then substituting in (3-3), we get the estimator for σ, namely,

$$\tilde{\sigma} = \left[\frac{2m}{bn} \sum_{i=1}^{n} (x_i-\tilde{\mu})^{2m}\right]^{1/2m} \qquad (3\text{-}7)$$

or, alternatively,

$$\tilde{\sigma}^2 = \left[\frac{2m}{bn} \sum_{i=1}^{n} (x_i-\tilde{\mu})^{2m}\right]^{1/m} \qquad (3\text{-}8)$$

Properties of the Maximum Likelihood Estimators $\tilde{\mu}$ and $\tilde{\sigma}$

i) $\tilde{\mu}$ and $\tilde{\sigma}$ are joint asymptotically efficient and having asymptotically bivariate normal distribution with dispersion matrix whose inverse is given by:

$$V^{-1} = \begin{bmatrix} -E(\frac{\partial^2 \ln L}{\partial \mu^2}) & -E(\frac{\partial^2 \ln L}{\partial \mu \, \partial \sigma}) \\ -E(\frac{\partial^2 \ln L}{\partial \mu \, \partial \sigma}) & -E(\frac{\partial^2 \ln L}{\partial \sigma^2}) \end{bmatrix} \qquad (3\text{-}9)$$

Now, (1-4), (3-4), (3-5), (3-6) and (3-9) give:

$$V^{-1} = \begin{bmatrix} \frac{2m(2m-1)n \, \Gamma((2m-1)/2m)}{\sigma^2 \, b^{1/m} \, \Gamma(1/2m)} & 0 \\ 0 & \frac{2mn}{\sigma^2} \end{bmatrix}$$

so that

$$V = \begin{bmatrix} \frac{\sigma^2 \, b^{1/m} \, \Gamma(1/2m)}{2m(2m-1)n\Gamma((2m-1)/2m)} & 0 \\ 0 & \frac{\sigma^2}{2mn} \end{bmatrix} \qquad (3\text{-}10)$$

This property follows from the properties of the ML estimators [4].

ii) $\tilde{\mu}$ and $\tilde{\sigma}$ are asymptotically independent. This property is an immediate consequence of property (1).

4. A Suggested Iterative Method to Solve a Likelihood Equation

In this section, the following iterative method is suggested to solve the likelihood equation given by (3-2): Letting

$$m_o = 2m-1, \qquad (4\text{-}1)$$

the likelihood equation given by (3-2) becomes:

$$\sum_{i=1}^{n} (x_i-\mu)^{m_o} = 0 \qquad (4\text{-}2)$$

Adding and subtracting $\bar{x}$ inside the parenthesis, we get:

$$\sum_{i=1}^{n} [(x_i-\bar{x}) + (\bar{x} - \mu)]^{m_o} = 0$$

which can be written, using the binomial theorem, as:

$$\sum_{i=1}^{n} \sum_{r=0}^{m_o} \binom{m_o}{r}(\bar{x}-\mu)^r (x_i-\bar{x})^{m_o-r} = 0$$

This equation takes the form:

$$\sum_{r=0}^{m_o} \binom{m_o}{r}(\bar{x}-\mu)^r M_{m_o-r} = 0 \tag{4-3}$$

where

$$M_{m_o-r} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^{m_o-r} \tag{4-4}$$

i.e., M_{m_o-r} is the (m_o-r)-th central sample moment.

Equation (4-3) can be rewritten as:

$$M_{m_o} - m_o(\mu-\bar{x})M_{m_o-1} + \sum_{r=2}^{m_o} \binom{m_o}{r}(\bar{x}-\mu)^r M_{m_o-r} = 0 \tag{4-5}$$

Now, neglecting the 3rd term in (4-5) and solving the resulting equation, we get:

the first approximation $\hat{\mu}_{(1)}$ for μ as:

$$\hat{\mu}_{(1)} = \bar{x} + x_o \tag{4-6}$$

where

$$x_o = \frac{M_{m_o}}{m_o M_{m_o-1}} \tag{4-7}$$

To get the 2nd approximation $\hat{\mu}_{(2)}$ for μ, substitute the 1st approximation for μ, given by (4-6), into the 3rd term of (4-5), i.e.,

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$$M_{m_o} - m_o(\hat{\mu}_{(2)} - \bar{x})M_{m_o-1} + \sum_{r=2}^{m_o} \binom{m_o}{r}(-x_o)^r M_{m_o-r} = 0$$

From which we get:

$$\hat{\mu}_{(2)} - \bar{x} = \frac{1}{m_o M_{m_o-1}} [M_{m_o} + \sum_{r=0}^{m_o} \binom{m_o}{r}(-x_o)^r M_{m_o-r}]$$

$$= \frac{M_{m_o}}{m_o M_{m_o-1}} [1 - \frac{1}{m_o M_{m_o-1}} \sum_{r=2}^{m_o} \binom{m_o}{r}(-x_o)^{r-1} M_{m_o-r}]$$

Hence, the 2nd approximation $\hat{\mu}_{(2)}$ for μ would be:

$$\hat{\mu}_{(2)} = \bar{x} + x_o[1 - \sum_{r=2}^{m_o} y_r] \quad (4\text{-}8)$$

where

$$y_r = \frac{1}{m_o M_{m_o-1}} \binom{m_o}{r}(-x_o)^{r-1} M_{m_o-r} \quad (4\text{-}9)$$

For the 3rd approximation $\hat{\mu}_{(3)}$ of μ, we substitute the 2nd approximation $\hat{\mu}_{(2)}$ for μ given by (4-8) into the 3rd term of (4-5) to get:

$$M_{m_o} - m_o(\hat{\mu}_{(3)} - \bar{x})M_{m_o-1} + \sum_{r=2}^{m_o} (-1)^r \binom{m_o}{r} x_o^r (1 - \sum_{r=2}^{m_o} y_r)^r M_{m_o-r} = 0$$

From which, we get:

$$\hat{\mu}_{(3)} - \bar{x} = \frac{1}{m_o M_{m_o-1}} [M_{m_o} + \sum_{r=2}^{m_o} \binom{m_o}{r}(-x_o)^r (1 - \sum_{r=2}^{m_o} y_r)^r M_{m_o-r}]$$

$$= \frac{M_{m_o}}{m_o M_{m_o-1}} [1 - \frac{1}{m_o M_{m_o-1}} \sum_{r=2}^{m_o} \binom{m_o}{r}(-x_o)^{r-1} M_{m_o-r} (1 - \sum_{r=2}^{m_o} y_r)^r]$$

Hence, the 3rd approximation $\hat{\mu}_{(3)}$ for μ would be:

$$\hat{\mu}_{(3)} = \bar{x} + x_o[1 - \sum_{r=2}^{m_o} y_r (1 - \sum_{r=2}^{m_o} y_r)^r] \quad (4\text{-}10)$$

Continuing as above, the N-th ($N \geq 2$) approximation $\hat{\mu}_{(N)}$ for μ can be written as:

$$\hat{\mu}_{(N)} = \bar{x} + x_o\{1 - \sum_{r=2}^{m_o} y_r(1 - \sum_{r=2}^{m_o} y_r(1 - \sum_{r=2}^{m_o} y_r(\ldots(1 - \sum_{r=2}^{m_o} y_r)^r \ldots)^r)^r)^r\} \qquad (4\text{-}11)$$

where the number of parentheses with the exponent r is (N-2), $N \geq 2$. To verify that (4-11) is true for all N, we show that it is true for (N+1): substituting the N-th approximation for μ into the 3rd term of (4-5), we get:

$$M_{m_o} - m_o(\hat{\mu}_{(N+1)} - \bar{x})M_{m_o-1} + \sum_{r=2}^{m_o} \binom{m_o}{r} M_{m_o-r}(-x_o)^r[1 - \sum_{r=2}^{m_o} y_r(1 - \sum_{r=2}^{m_o} y_r$$

$$(1 - \sum_{r=2}^{m_o} y_r(\ldots(1 - \sum_{r=2}^{m_o} y_r)^r \ldots)^r)^r)^r]^r = 0$$

From which we get:

$$\hat{\mu}_{(N+1)} - \bar{x} = \frac{1}{m_o M_{m_o-1}}\{M_{m_o} + \sum_{r=2}^{m_o} \binom{m_o}{r} M_{m_o-r}(-x_o)^r[1 - \sum_{r=2}^{m_o} y_r(1 - \sum_{r=2}^{m_o} y_r$$

$$(1 - \sum_{r=2}^{m_o} y_r(\ldots(1 - \sum_{r=2}^{m_o} y_r)^r \ldots)^r)^r)^r]^r\}$$

$$= \frac{M_{m_o}}{m_o M_{m_o-1}}\{1 - \frac{1}{m_o M_{m_o-1}} \sum_{r=2}^{m_o} \binom{m_o}{r} M_{m_o-r}(-x_o)^{r-1}[1 - \sum_{r=2}^{m_o} y_r$$

$$(1 - \sum_{r=2}^{m_o} y_r(1 - \sum_{r=2}^{m_o} y_r(\ldots(1 - \sum_{r=2}^{m_o} y_r)^r \ldots)^r)^r)^r]^r\}$$

$$= x_o\{1 - \sum_{r=2}^{m_o} y_r[1 - \sum_{r=2}^{m_o} y_r(1 - \sum_{r=2}^{m_o} y_r(1 - \sum_{r=2}^{m_o} y_r(\ldots$$

$$(1 - \sum_{r=2}^{m_o} y_r)^r \ldots)^r)^r)^r]^r\}$$

Therefore, the (N+1)th approximation for μ is:

$$\hat{\mu}_{(N+1)} = \bar{x} + x_o\{1-\sum_{r=2}^{m_o} y_r(1-\sum_{r=2}^{m_o} y_r(1-\sum_{r=2}^{m_o} y_r(1-\sum_{r=2}^{m_o} y_r(\ldots(1-\sum_{r=2}^{m_o} y_r)^r\ldots)^r)^r)^r)^r\}$$

where the number of parentheses with the exponent r is (N-1). Therefore, the approximation given in (4-11) is true for all N ($N \geq 2$) noting that the 1st approximation is given by (4-6).

<u>Special Case</u>

We have shown that to solve the likelihood equation given by:

$$\sum_{i=1}^{n} (x_i - \mu)^{2m-1} = 0 , \qquad (4\text{-}12)$$

We may use the iterative method consisting of the 1st and the Nth ($N \geq 2$) approximations given by:

$$\hat{\mu}_{(1)} = \bar{x} + x_o \qquad (4\text{-}13)$$

and

$$\hat{\mu}_{(N)} = \bar{x} + x_o\{1-\sum_{r=2}^{m_o} y_r(1-\sum_{r=2}^{m_o} y_r(1-\sum_{r=2}^{m_o} y_r(\ldots(1-\sum_{r=2}^{m_o} y_r)^r\ldots)^r)^r)^r\} \qquad (4\text{-}14)$$

where: the number of parentheses with the exponent r is (N-2),

$$m_o = 2m-1, \quad M_{m_o-r} = \frac{1}{n}\sum_{i=1}^{n} (x_i-\bar{x})^{m_o-r} \quad r = 0,1,\ldots,m_o ,$$

$$x_o = \frac{M_{m_o}}{m_o\, M_{m_o-1}} , \quad \text{and} \quad y_r = \frac{1}{m_o\, M_{m_o-1}} \binom{m_o}{r}(-x_o)^{r-1} M_{m_o-r} .$$

We now consider the special case m = 2, i.e. the case of the 4th power exponential distribution. In this case, the likelihood equation is $\sum_{i=1}^{n} (x_i-\mu)^3 = 0$, and it is straightforward to show that the 1st and the Nth ($N \geq 2$) approximations given by (4-13) and (4-14) would be respectively:

$$\hat{\mu}_{(1)} = \bar{x} + \sqrt{\frac{M_2}{3}}\,(\alpha) \tag{4-15}$$

and

$$\hat{\mu}_{(N)} = \bar{x} + \frac{\sqrt{M_2}}{3}\,\alpha\{1-\frac{\alpha^2}{27}(1-\frac{\alpha^2}{27}(1-\frac{\alpha^2}{27}(\ldots(1-\frac{\alpha^2}{27})^3\ldots)^3)^3)^3\} \tag{4-16}$$

where

$M_2 = \frac{1}{n}\sum_{i=1}^{n} (x_i-\bar{x})^2$, α is the coefficient of skewness

and the number of parentheses with the exponent 3 in (4-16) is (N-2).

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