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CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION VIA MIXING DISTRIBUTIONS

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ABSTRACT

In this paper some characterization results, for the exponential distribution, are given using mixing distributions. The achieved results generalize some known results in this connection and have its relevance to some practical applications.

KEY WORDS : Exponential distribution, Mixing distribution, characterization , Laplace transform , generalized hypergeometric distribution.

1. INTRODUCTION

Applied statisticians, interested in adapting characterization theorems to practical problems, seek a good insight in the properties of various statistical distributions, since statistical and probabilistic models of real world phenomina depend on populations some or all of its specific parameters are random variables having their specific populations, that is models specified by mixtures of distributions. In fact, mixtures of distributions have been used extensively as models in a wide variety of practical applications, where data can be viewed as arising from two or more mixed populations (see [6], [9], [10], [13]) . For example, the distribution of height in a population of children might be expressed as :

The distribution of the r.v. Θ is, usually , called the mixing distribution .

THEOREM 1.

Given that the r.v. Θ has a gamma distribution with density :

$$g_1(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \qquad \alpha > 0 , \beta > 0, \theta > 0 \qquad (2.2)$$

Then the conditional distribution of the r.v. X is exponential with density:

$$f(x|\theta) = \theta e^{-\theta x} \qquad , x > 0, \theta > 0 \qquad (2.3)$$

if and only if the mixture distribution is Pareto of the second kind with density :

$$h1(x,\alpha,\beta) = \frac{\alpha \beta^{\alpha}}{(x+\beta)^{\alpha+1}} \quad , x > 0, \alpha > 0, \beta > 0 \qquad (2.4)$$

THEOREM 2 .

Given that the r.v. Θ^{-1} has the gamma distribution given by (2.2) ,Then the conditional distribution of the r.v. X is exponential with the density (2.3) if and only if the mixture distribution is a compound gamma distribution with parameters $(1 , \alpha , \beta^{-1})$, where the p.d.f $w(x , a , b , c)$ of the three parameters compound gamma distribution (see [12] , vol. 2 , p. 195) is defined by :

$$w(x , a , b , c) = 2 \left[c \Gamma(a) \Gamma(b)\right]^{-1} \left(\frac{x}{c}\right)^{\frac{1}{2}(a+b)-1} K_{b-a}(2\sqrt{x/c}),$$
$$x > 0 , a > 0 , b > 0 , \qquad (2.5)$$

Where , $K_v(.)$ is the modified Bessel function of the third kind of order v .

THEOREM 3 .

Given that the r.v. Θ has a beta distribution over the interval $[0, \alpha]$, $\alpha > 0$, with density :

$$g_3(\theta) = \frac{\alpha^{1-p-q}}{B(p,q)}\,\theta^{p-1}(\alpha-\theta)^{q-1}, \quad p>0,\ q>0,\ 0\le\theta\le\alpha \qquad (2.6)$$

Then the conditional distribution of the r.v. X is exponential with the density (2.3), if and only if the mixture distribution is the five parameters (a , c , d , β , r) generalized hypergeometric distribution with :

$$a=\alpha,\ c=d=1,\ \beta=p+1,\ \text{and } r=p+q+1+1.$$

Where the p.d.f $u(x, a, c, d, \beta, r)$ of this generalized hypergeometric distribution is obtained by Mathai and Saxena (1966) as :

$$u(x,a,c,d,\beta,r) = \frac{d\,a^{\frac{c}{d}}\,\Gamma(\beta)\,\Gamma\left(r-\frac{c}{d}\right)}{\Gamma\left(\frac{c}{d}\right)\Gamma(r)\,\Gamma\left(\beta-\frac{c}{d}\right)}\,x^{c-1}M(\beta, r, -a x^{d}),\quad x>0, c>0, \beta-\frac{c}{d}>0, \qquad (2.7)$$

Where , M (. , . , .) is the Kummer's confluent hypergeometric function .

THEOREM 4 .

Given that the r.v. Θ has the inverse-Gaussian distribution with density :

$$g_4(\theta) = (\lambda/2\Pi)^{1/2}\,\theta^{-2/3}\exp[-\lambda(\theta-\mu)^2/2\mu^2\theta],$$
$$\lambda>0,\ \mu>0,\ \theta>0 \qquad (2.8)$$

Then the r.v. X has the exponential distribution with the density (2.3) if and only if the mixture distribution is that of the quantity $\frac{1}{2}\lambda\,(Y^2 - \mu^{-2})$ where Y is an exponentially distributed r.v.(independent of X and Θ) with denisty :

$$\lambda\exp-\lambda(y-\mu^{-1}), \qquad y>\mu^{-1}, \lambda>0, \mu>0 \qquad (2.9)$$

3. PROOFS OF THE THEOREMS

The technique used for the proofs is based , mainly , on the Laplace-Stieltijes transform (L.T.) . It is well known (see , for example, [7], P.19) that this transform is a one - to - one mapping . Moreover it is used by some authors (see, for example, [8]) in proving characterization theorems but our use, here, is based on a different technique .

PROOF OF THEOREM 1

NECESSITY :

If X has the density (2.3) , and Θ has the distribution (2.2) , then according to (2.1), the density function h(x) of the mixture distribution is given by :

$$h(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \theta^{\alpha} e^{-(x+\beta)\theta} \, d\theta = \frac{\alpha \beta^{\alpha}}{(x+\beta)^{\alpha+1}} \quad , x > 0, \alpha > 0, \beta > 0 \tag{3.1}$$

The R.H.S of (3.1) is the density function of the pareto distribution of the second kind (see [12], vol.3) given by (2.4) .

SUFFICIENCY :

If The conditional distribution of the r.v X has the density $f(x \mid \Theta)$, Θ has the distribution (2.2) and the mixture distribution has the density (2.4), then we have

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} f(x \mid \theta)\, \theta^{\alpha-1} e^{-\beta\theta} \, d\theta = \frac{\alpha\beta^{\alpha}}{(x+\beta)^{\alpha+1}}, \quad x > 0, \alpha > 0, \beta > 0.$$

This can be written as

$$L_{\beta}\left\{\theta^{\alpha-1} f(x \mid \theta)\right\} = \frac{\Gamma(\alpha+1)}{(x+\beta)^{\alpha+1}}, \quad x > 0, \alpha > 0, \beta > 0. \tag{3.2}$$

where L_{β} denotes the L.T. with β as the parameter of the transformation (P.T.) . Now , the R.H.S of (3.2) is the L.T. (with β as the P.T.) of the function $\theta^{\alpha} e^{-\theta x}$ (see [7]). Hence , we must have

$$\theta^{\alpha-1} f(x \mid \theta) = \theta^{\alpha} e^{-\theta x}, \quad x > 0,\ \alpha > 0, \theta > 0 \quad .$$

Consequently ,

$$f(x \mid \theta) = \theta e^{-\theta x}, \quad x > 0, \theta > 0 .$$

Since the L.T. is a one - to - one mapping this completes the proof of theorem 1.

PROOF OF THEOREM 2

If the r.v. Θ^{-1} has the gamma distribution given by (2.2) ,Then Θ has the density

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{-\alpha-1} e^{-\beta/\theta}, \quad \alpha > 0,\ \beta > 0,\ \theta > 0 \tag{3.3}$$

NECESSITY :

If the conditional distribution of X has the density (2.3) , where Θ has the distribution (3.3) , then the p.d.f h(x) of the mixture distribution is given by

$$h(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \theta^{-\alpha} e^{-(x\theta + \beta\theta^{-1})} \, d\theta =$$

$$= \frac{2\beta^{(\alpha+1)/2}}{\Gamma(\alpha)} x^{(\alpha-1)/2} k_{\alpha-1}(2\sqrt{\beta x}), \quad x > 0\ \alpha > 0, \beta > 0. \tag{3.4}$$

(see [15] , vol. 1, p.344) .

The R.H.S of (3.4) is the p.d.f of the three parameters (a , b , c) compound gamma distribution , given by (2.5) , with $a = 1$, $b = \alpha$, and $c = \beta^{-1}$.

SUFFICIENCY :

If the conditional distribution of X has the density $f(x \mid \Theta)$, θ

has the density (3.3) and the p.d.f of the mixture distribution is given by the R.H.S of (3.4), then we must have

$$\frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} f(x \mid \theta)\, \theta^{-\alpha-1} e^{-\beta/\theta}\, d\theta = \frac{2}{\Gamma(\alpha)} \beta^{(\alpha+1)/2} x^{(\alpha-1)/2} K_{\alpha-1}(2\sqrt{\beta x}),$$

$$x > 0, \alpha > 0, \beta > 0.$$

This can be witten as :

$$\int_0^{\infty} [\theta^{-\alpha-1} e^{x\theta-\beta/\theta} f(x \mid \theta)]\, e^{-x\theta}\, d\theta = 2(x/\beta)^{(\alpha-1)/2} K_{\alpha-1}(2\sqrt{\beta x}),$$

or,

$$L_x\, [\theta^{-\alpha-1} e^{x\theta-\beta/\theta} f(x \mid \theta)] = 2(x/\beta)^{(\alpha-1)/2} K_{\alpha-1}(2\sqrt{\beta x}) \qquad (3.5)$$

Now using the L.T. inversion formula (see, for example, [7]) one can see that the R.H. S of (3.5) is the L.T. (with x as the P.T.) of the function

$$\theta^{-\alpha} e^{-\beta/\theta}, \ \theta > 0, \alpha > 0, \beta > 0. \qquad (3.6)$$

Thus , from (3.5) and (3.6) we have

$$\theta^{-\alpha-1} e^{x\theta-\beta/\theta} f(x \mid \theta) = \theta^{-\alpha} e^{-\beta/\theta}, \quad \theta > 0, \alpha > 0, \beta > 0.$$

from which we get

$$f(x \mid \theta) = \theta e^{-\theta x}, \quad x > 0, \theta > 0.$$

since the L.T. is a one - to - one mapping this completes the proof of theorem 2.

PROOF OF THEOREM 3

NECESSITY :

If the conditional distribution of X has the density (2.3) where Θ has the density (2.6) then, according to (2.1), the p.d.f h(x) of the mixture distribution is given by :

$$h(x) = \frac{1}{\alpha^{p+q-1}B(p,q)} \int_0^{\alpha} \theta^{p} (\alpha-\theta)^{q-1} e^{-\theta x}\, d\theta$$

$$= \frac{\alpha p}{p+q} M(p+1, p+q+1, -\alpha x), \quad x > 0, \alpha > 0, p > 0, q > 0 \quad (3.7)$$

(see [15] , vol.1, p.324) .

The R.H.S of (3.7) is the p.d.f given by (2.7) with $a = \alpha$, $c = d = 1$, $\beta = p + 1$, $r = p + q + 1$.

SUFFICIENCY :

If the conditional distribution of X has the density $f(x \mid \Theta)$, Θ has the density (2.6) , and the p.d.f of the mixture distribution is given by the R.H.S of (3.7), then we have

$$\frac{1}{\alpha^{p+q-1}B(p,q)} \int_0^{\alpha} \theta^{p-1}(\alpha-\theta)^{q-1} f(x \mid \theta)\, d\theta =$$

$$= \frac{\alpha p}{p+q} M(p+1, p+q+1, -\alpha x), x > 0, \alpha > 0, p > 0, q > 0.$$

This can be written as

$$\int_0^{\alpha} [\theta^{p-1}(\alpha-\theta)^{q-1} f(x \mid \theta)\, e^{\theta x}]\, e^{-x\theta}\, d\theta$$

$$= \alpha^{p+q} B(p+1, q)\, M(p+1, p+q+1, -\alpha x) .$$

Or ,

$$L_x [\theta^{p-1} (\alpha-\theta)^{q-1} f(x \mid \alpha)\, e^{\theta x}]$$

$$= \alpha^{p+q} B(p+1, q)\, M(p+1, p+q+1, -\alpha x) \quad (3.8)$$

One can see (using [15], Vol. 3, p.255) that the R.H.S of (3.8) is the L.T. (with x as the P.T.) of the function

$$\theta^{p} (\alpha - \theta)^{q -1} , \quad p > 0, \ q > 0, \ 0 \le \theta \le \alpha \tag{3.9}$$

(Through the derivation of (3.9) , Kummer transformation and the relation between Kummer and Whittaker functions (see [1] , P.505) are used) .

Finally , from (3.8) and (3.9) we get that

$$f(x \mid \theta) = \theta e^{-\theta x} , \quad x > 0, \theta > 0 .$$

Since the L.T. is a one - to - one mapping, this completes the proof of theorem 3 .

REMARK

If Θ^{-1}, instead of Θ, has the distribution (2.6), theorem 3 remains valid but the p.d.f of the mixture distribution will be

$$\frac{\Gamma(p+q)}{\alpha\,\Gamma(p)} e^{-x/\alpha} \psi (q , 2 - p , \frac{x}{\alpha}) , \quad x > 0, \alpha > 0, p > 0, q > 0, \tag{3.10}$$

Where ψ (.,.,.) is the Tricomi confluent hypergeometric function defined by :

$$\psi (a, b, z) = \frac{\Gamma(1 - b)}{\Gamma (a - b + 1)} M (a , b , z) + + \frac{\Gamma (b -1)}{\Gamma (a)} Z^{1 - b} M (a - b + 1 , 2 - b , z) ,$$

$$|Z| < \infty , \ b \neq 0 , \ \pm 1 , \pm 2 , \ ... \tag{3.11}$$

(see [15] , vol. 3 , P. 434) .

It is easy to see (using [15] , Vol 3, P.256) that (3.10) is a proper p.d.f. Moreover, in accordance with (3.11), one can see that the distribution (3.10), in its five parameters form, is more general than the five parameters generalized hypergeometric distributions defined by Mathai and Saxena (1966).

PROOF OF THEOREM 4

We first note that if Y is an exponentially distributed r.v. with the density (2.9), then it is easy to see that the distribution of the quantity $\frac{\lambda}{2}(Y^2 - \mu^{-2})$ has the density :

$$g(y, \lambda, \mu) = \mu\,[1+(2\mu^2/\lambda)\,y\,]^{-1/2} \exp(\lambda/\mu)\{1-[1+(2\mu^2/\lambda)y]^{1/2}\},$$

$$y > 0,\ \lambda > 0,\ \mu > 0, \tag{3.12}$$

NECESSITY :

If the conditonal distribution of X has the density (2.3) and Θ has the density (2.8), then, according to (2.1), the p.d.f h(x) of the mixture distribution is given by :

$$h(x) = \sqrt{\frac{\lambda}{2\Pi}}\; e^{\frac{\lambda}{\mu}} \int_0^{\infty} \theta^{-1/2} \exp - \{ [(\lambda/2\mu^2) + x]\,\theta + (\lambda/2\theta) \}\, d\theta =$$

$$= \frac{2}{\sqrt{\Pi}} (\lambda/2)^{3/4}\, e^{\lambda/\mu}\, [(\lambda/2\mu^2) + x]^{-1/4}\, K_{1/2}\left([(\lambda/\mu)^2 + 2\lambda x]^{1/2} \right),$$

$$x > 0,\ \lambda > 0,\ \mu > 0, \tag{3.13}$$

(see [15] , vol. 1 , p.344) .

Now, (3.12) and (3.13) are identical since ,

$$K_{1/2}(u) = (\Pi/2u)^{1/2}\, e^{-u} \qquad \text{(see [15] , vol. 2 , p. 730).}$$

SUFFICIENCY :

If the conditonal distribution of X has the density f (x I Θ) , where Θ has the density (2.8) and the mixture distribution is given by (3.12), then we have

$$\int_0^{\infty} \left\{ \theta^{-3/2} f(x\,|\,\theta) \exp [\, x\,\theta - [\, \lambda(\theta-\mu)^2/2\mu^2\theta\,]\,] \right\} e^{-x\theta}\, d\theta =$$

$$= (2\Pi\mu^2/\lambda)^{1/2}\, [1+(2\mu^2/\lambda)\,x]^{-1/2} \exp(\lambda/\mu) \left\{ 1 - [\,1 + (2\mu^2/\lambda)\,x\,]^{1/2} \right\}.$$

Or,

$$L_x\{\theta^{-3/2}\ f(x\mid\theta)\ \exp[[x-(\lambda/2\mu^2)]\theta-(\lambda/2\theta)]\}$$

$$=\sqrt{\Pi}\,[(\lambda/2\mu^2)+x]^{-1/2}\exp-[2\lambda[(\lambda/2\mu^2)+x]]^{1/2}, \qquad (3.14)$$

Now, using a well known rule (see [7], p. 24) concerning the L.T., one can get that the R.H.S. of (3.14) is the L.T. of the function

$$\theta^{-1/2}\ \exp-[(\lambda/2\mu^2)\theta+\lambda/2\theta],\ \theta>0,\lambda>0,\mu>0, \qquad (3.15)$$

Therefore, from (3.14) and (3.15), we have

$$\theta^{-3/2} f(x\mid\theta)\ \exp[(x-(\lambda/2\mu^2))\theta-(\lambda/2\theta)]=$$

$$=\theta^{-1/2}\ \exp-[(\lambda/2\mu^2)\theta+\lambda/2\theta],\ \theta>0,\lambda>0,\mu>0,$$

which gives

$$f(x\mid\theta)=\theta e^{-\theta x},\quad x>0,\theta>0.$$

Since the L.T. is a one - to - one mapping this completes the proof of theorem 4 .

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