THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CAIRO UNIV., VOL., 41, NO.1, 1997

Test for Exponential Better than Renewal in Expectation class of life distribution

By
I. A. Alwasel
Department of Statistics, College of Science,
King Saud University, P. O. Box 2455,
Riyadh 11451, Saudi Arabia

Abstract

Based on comparisons of the survival of the renewal variable at age $t \geq 0$ with the survival of its parent variable at $t = 0$ a class of life distributions namely Exponential better than renewal in expectation (EBRE) is defined. The corresponding dual class (EWRE) is also introduced. In this paper, we construct test statistic based on the scaled total time on test (TTT)-transform, to test exponentiality versus EBRE class or its dual EWRE. The distribution of the test statistic is investigated via simulation for small samples. Power of the test is also estimated by simulation. An example of 40 patients of blood cancer shows the possibility of applying the derived methodology on real data situation.

1 Introduction and Definitions

Statisticians and reliability analysts have shown growing interest in modeling survival data using classification of life distribution based on different aspects of ageing. Various classes of life distributions and their dual have been introduced in literature to describe several types of deterioration or improvement that accompany ageing.

Let T be a non-negative random variable representing the lifetime of a unit with a (continuous) life distribution $F(t)$, where $F(t) = P(T \leq t)$ for which $F(t) = 0, t \leq 0$, density function (asssumed to exist) $f(t) = \frac{d}{dt}F(t)$ and survival (survivorship) function of a new system $\bar{F}(t) = P(T > t) = 1 - F(t)$, for $t \geq 0$.
Suppose that a device (system or component) with lifetime T and a continuous life distribution $F(t)$, is put in operation. The device is replaced uppon a failure by a sequence of mutually independent devices. Also, these devices are independent of the first device and identically distributed with the same life distribution $F(t)$. In the long run, the remaining life distribution of the system under operation at time t is given by stationary renewal (or equilibrium) distribution as follows:

$$W_F(t) = \mu_F^{-1} \int_0^t \bar{F}(u)du, \qquad for\ 0 \leq t < \infty$$

where $\mu_F = \int_0^\infty \bar{F}(x)dx < \infty$ is the mean life of the random variable T, see for example Barlow and Proschan (1981), Abouammoh and Ahmed (1992).
The corresponding renewal survival function is given by,

$$\bar{W}_F(t) = 1 - W_F(t) = \mu_F^{-1} \int_t^\infty \bar{F}(u)du, \qquad 0 \leq t < \infty$$

The problem of testing exponentiality against various classes of life distributions (IFR, IFRA, DMRL, NBUE, ... ETC) has seen a good deal of attention in the literature. For example Klefsjo (1983), Proschan and Pyke (1967), Ahmad (1975,1994, 1995), Hollander and Proschan (1972, 1975), Abouammoh et al. (1987, 1989, 1993, 1994 and 1996), Kanjo (1993) and Hendi et al.(1996) among others.

Definition 1.1. A non-negative random variable X with distribution F is said to be exponential beter than renewal in expectation (EBRE) if

$$\int_t^\infty \int_x^\infty \bar{F}(u)dudx \leq \mu_F \mu_W \exp(-t/\mu_W); \qquad x > 0,\ t > 0. \qquad (1.1)$$

where $\mu_W = \mu_F^{-1} \int_0^\infty \int_x^\infty \bar{F}(u)dudx$. This class of life distribution was introduced by Alwasel and AL-Nachawati (1996).

The main theme of this paper is dealing with the problem of testing $H_0 : \bar{F} = \exp(-\lambda t), \lambda > 0, t > 0$ versus $H_1 : F =$ EBRE (EWRE). In section 2, we give a brief account of the TTT-transform and TTT-plot and present a characterization for the class of life distributions EBRE (EWRE) via the scaled total time on test. In section 3, we derive the empirical test statistic for the EBRE (EWRE) based on the scaled TTT-transforms. In section 4, a small sample study of this test statistic is performed through simulation. The power estimates of this test statistic are given in section 5, with respect to some commonly used distributions in reliability. We conclude this paper with an application in section 6.

2 The Concept of Total Time on Test (TTT-test)

Let F be the life distribution with survival function $\bar{F} = 1 - F$ and finite mean $\mu = \int_0^\infty \bar{F}(u)du$. We present the following definition of Barlow and Campo (1975).

Definition 2.1.

(i) The function

$$H^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(u)du \qquad (2.1)$$

is called the TTT-transform. It is easy to verify that the mean μ, of F is given by

$$H^{-1}(1) = \int_0^{F^{-1}(1)} \bar{F}(u)du \qquad (2.2)$$

(ii) The function

$$\phi_F(t) = H^{-1}(t)/H^{-1}(1) \qquad (2.3)$$

is called the scaled TTT-transform.

Here we consider $F^{-1}(t)$ to be $\inf[x : F(x) \geq t]$. Note that if F is the exponential distribution then the scaled TTT-transform is given by

$$\phi_F(t) = t, \quad 0 \leq t \leq 1.$$

Now let $t_{(0)} < t_{(1)} < t_{(2)} < \ldots t_{(n)}$ to be an ordered sample from a life distribution F, where $t_{(0)} = 0$ and let

$$D_j = (n - j + 1)(t_{(j)} - t_{(j-1)}) \text{ for } j = 1, 2, \ldots, n \tag{2.4}$$

then

$$S_j = \sum_{k=1}^{j} D_k \text{ for } j = 1, 2, \ldots, n \tag{2.5}$$

denote the TTT-transform at $t_{(j)}$, where $S_0 = 0$. The value $W_j = S_j/S_n$ is an estimate of the scaled TTT-transform.

An estimate of the scaled TTT-transform is known as empirical scaled TTT-transform and is obtained as

$$\phi_F\left(\frac{j-1}{n}\right) = W_{j-1}, \text{ for } j = 1, 2, \ldots, n \tag{2.6}$$

The TTT-plot is obtained by plotting W_{j-1} against $\frac{(j-1)}{n}$, for $j = 1, 2, \ldots, n$, i.e. $(W_{j-1}, \frac{j-1}{n})$, for $j = 1, 2, \ldots, n$ and connecting the plotted points by straight lines.

It has been shown, Barlow et al (1972) (p. 237), using Glivenko- Cantelli Lemma, that for strictly increasing F, W_j converges to $\phi_F(t)$ with probability one and uniformly in [0,1] as n tends to ∞, and j/n converges to t.

Scaled TTT-transforms for some families of life distributions are given in Barlow and Compo (1975), Barlow (1979), Bergman (1979) and Klefsjo (1982, 1983). Here we present the following theorem for which proof can be found in those references.

Theorem 2.1. Let F be a life distribution and $\phi_F(t)$, for $0 \leq t \leq 1$, be the corresponding TTT-transform as defined in (2.3). Then we have

(i) F is IFR (DFR) if and only if (iff) $\phi_F(t)$ is concave (convex) for $0 \le t \le 1$.

(ii) F is IFRA (DFRA) iff $\phi_F(t)/t$ is decreasing (increasing) for $0 < t < 1$.

(iii) F is HNBUE (HNWUE) iff $t^{-1}\log(1-\phi_F(t)) \le -\mu^{-1}, 0 < t \le 1$.

(iv) F is DMRL (IMRL) iff $[1-\phi_F(t)]/(1-t)$ is decreasing (increasing) for $0 \le t < 1$.

Theorem 2.2. Let F and $\phi_F(t)$ (or simply $\phi(t)$) be as in Theorem 2.1, then we have the following:

F is EBRE (EWRE) iff

$$\int_{F(t)}^{1} [1-\phi_F(s)]\,\phi\prime_F(s)(1-s)^{-1}ds \le (\ge) \int_0^1 [1-\phi_F(s)]\,\phi\prime_F(s)(1-s)^{-1}ds$$
$$\times \exp\left(-F(t)/\mu_W\right) \quad \text{for} \quad 0 \le s \le 1,\ 0 \le t \le 1 \qquad (2.7)$$

Proof. We prove the theorem for the basic class, while the proof for the dual class can be carried out in parallel steps. From equation (1.1) the life distribution F has EBRE property, if

$$\int_t^\infty \left[\mu_F^{-1}\int_x^\infty \bar{F}(u)du\right]dx \le \int_0^\infty \left[\mu_F^{-1}\int_x^\infty \bar{F}(u)du\right]dx\,\exp\left(-t/\mu_W\right); \qquad x \ge 0, t \ge 0.$$

Rearranging the terms and using the transformation $x = F^{-1}(s)$ (i.e.,$f(x)dx = ds$ or $dx = ds/\left[f(F^{-1}(s))\right]$, yields

$$\mu_F \int_{F(t)}^{1} \left[1-\mu_F^{-1}(s)\bar{F}(u)du\right]\left[f(F^{-1}(s))\right]^{-1} ds \le \mu_F \int_0^1 \left[1-\mu_F^{-1}\int_0^{F^{-1}(t)} \bar{F}(u)du\right]$$
$$\times \left[f(F^{-1}(s))\right]^{-1} ds\,\exp(-F(t)/\mu_W), \qquad 0 \le s \le 1, 0 \le t \le 1.$$

Since $[f(F^{-1}(s))] = \mu_F \frac{\phi\prime_{F(s)}}{1-s}$, then one has

$$\int_{F(t)}^{1} [1-\phi_F(s)]\,\phi\prime_F(s)(1-s)^{-1}ds \le (\ge) \int_0^1 [1-\phi_F(s)]\,\phi\prime_F(s)(1-s)^{-1}ds$$
$$\times \exp\left(-F(t)/\mu_W\right) \text{ for } \quad 0 \le s \le 1, 0 \le t \le 1 \qquad (2.8)$$

this completes the proof. □

3 Test Statistic Based on the Scaled TTT-Transform

In this section we present a test statistic using the scaled TTT-transform for testing exponentiality or H_0 against H_1 or EBRE (EWRE) class (not exponential) based on a sample $t_1, \ldots, t_n$ from F.

Now, since the TTT-plot W_{j-1} converges to the scaled TTT-transform $\phi_F(t)$, for $0 \le t \le 1$ as $n \to \infty$ and $\frac{i-1}{n} \to t$, then, the TTT-plot based on an ordered sample

$$0 = t_{(0)} \le t_{(1)} \le \ldots \le t_{(n)},$$

behave as $\phi_F(t)$ does. This suggests the following test statistic based on the scaled TTT-transform. Since F is EBRE (EWRE), then from equation (2.7), let

$$U(F(t)) = \int_{F(t)}^{1} [1 - \phi_F(s)]\, \phi\prime_F(s)(1-s)^{-1} ds - \int_0^1 [1 - \phi_F(s)]\, \phi\prime_F(s)(1-s)^{-1} ds$$

$$\times \exp\left(-F(t)/\mu_W\right) \le (\ge) 0 \quad \text{for} 0 \le s \le 1, 0 \le t \le 1 \tag{3.1}$$

Integration of both sides of the above equation over $[0,1]$ with respect to $z = F(t)$, we get

$$\Delta_F = \int_0^1 \int_z^1 (1 - \phi_F(s))\phi\prime_F(s)(1 - s)^{-1} ds -$$

$$\int_0^1 (1 - \phi_F(s))\phi\prime_F(s)(1 - s)^{-1} ds \times \exp(-z/\mu_W) dz \le (\ge) 0, \tag{3.2}$$

$$\text{for } 0 \le s \le 1, \qquad 0 \le z \le 1$$

By using equation (3.2) we estimate Δ_F at a specified time t, as follow,

let $$T = \sum_{j=1}^{n} (1 - w_{j-1})(n - j + 1)^{-1} (w_{j-1} - w_{j-2})(t_{(j)} - t_{(j-1)}),$$

then,

$$\hat{\Delta}_{F_n} = n \sum_{i=1}^{n} \left\{ \sum_{j=i}^{n} (n - j + 1)^{-1} (w_{j-1} - w_{j-2})(t_{(j)} - t_{(j-1)}) - T \exp\left(-i/T\right) \right\} (t_{(i)} - t_{(i-1)}) \tag{3.3}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ are the ordered statistics of the independent random sample $x_1, x_2, \ldots, x_n, t_{(0)} = 0$.

To reduce the size of the test statistic we consider the version

$$\hat{\Delta}_n^* = \hat{\Delta}_n / n \qquad (3.4)$$

Note that

$H_0 : \Delta_F = 0$ if F is exponential

$H_1 : \Delta_F < (>) 0$ if F is EBRE (EWRE)

4 Simulation of small samples

It is difficult to find the exact distribution for $\hat{\Delta}_n^*$ statistic. Hence we would study the performance of our test statistic for small samples, which are commonly used by applied statisticians and reliability analysts. We have simulated the lower and upper percentile points for $\alpha = 0.01$, 0.05 and 0.10. Table (3.1) gives these percentile points of the statistic $\hat{\Delta}_n^*$ and the calculations are based on 10000 simulated samples of sizes $n = 5(1)20(5)60$.

5 The Power Estimates

The power of the test statistic $\hat{\Delta}_n^*$ is considered for the significance level $\alpha = 0.05$ and for commonly used distributions in reliability modeling. These distributions are

(i)	Linear failure-rate	: $\bar{F}_1(t) = \exp(-t - \frac{1}{2}\theta t^2)$	$\theta \geq 0, t \geq 0$
(ii)	Makeham	: $\bar{F}_2(t) = \exp[-\{t + \theta(t + e^{-t} - 1)\}]$	$\theta > 0, t \geq 0$
(iii)	Weibull	: $\bar{F}_3(t) = \exp(-t^{\theta})$	$\theta \geq 0, t \geq 0$
(iv)	Pareto	: $\bar{F}_4(t) = (1 + \theta t)^{-1/\theta}$	$\theta > 0, t \geq 0$

All these distributions IFR (for an appropriate restriction on θ), hence they all belong to a wider class. Table 4.1 contains the power estimates for the $\hat{\Delta}_n^*$ test statistic with respect to these distributions. The estimates are based on 10000 simulated samples of sizes $n = 10$, 20, 30 and significance level $\alpha = 0.05$. Note that the distributions F_1 and F_2 with $\theta = 0$ and the distribution

F_3 with $\theta = 1$ coincide with the (negative) exponential distribution $F(t) = exp(-t)$, for $t \geq 0$. These distributions become more EBRE as θ increases. Similarly, the distribution F_4 becomes more EWRE as θ decrease. From table (4.1) we note that the power estimates of EBRE statistic increase as θ or sample size increase. In otherwords, we notice clearly the departure from exponentiality towards (EBRE) properties as θ increases.

ISSR, Cairo Univ., Vol.,41, No.1, 1997, PP. 69-81

Table (3.1) Critical Values for the $\hat{\Delta}_n^*$-statistic

sample size	Significance level α					
	Lower Percentile			Upper Percentile		
n	0.01	0.05	0.10	0.10	0.05	0.01
5	-0.0048	-0.0169	-0.0295	0.9128	1.3036	2.2328
6	-0.0075	-0.0216	-0.0351	0.9973	1.4723	2.6029
7	-0.0109	-0.0270	-0.0410	1.0207	1.5762	3.1488
8	-0.0121	-0.0282	-0.0444	1.1010	1.7145	3.4289
9	-0.0131	-0.0301	-0.0474	1.1345	1.7869	3.6169
10	-0.0153	-0.0325	-0.0471	1.0572	1.7226	3.8196
11	-0.0172	-0.0342	-0.0501	1.0889	1.7282	3.9173
12	-0.0169	-0.0343	-0.0506	1.0744	1.7360	4.2219
13	-0.0188	-0.0347	-0.0490	1.0172	1.7306	4.1232
14	-0.0193	-0.0373	-0.0513	1.0395	1.7036	4.5375
15	-0.0204	-0.0378	-0.0525	1.0577	1.7125	3.7329
16	-0.0204	-0.0372	-0.0536	1.0170	1.7035	4.0948
17	-0.0209	-0.0379	-0.0526	0.9601	1.6248	3.7058
18	-0.0212	-0.0390	-0.0529	0.9747	1.5474	3.6784
19	-0.0209	-0.0373	-0.0512	0.9720	1.5075	3.8175
20	-0.0205	-0.0383	-0.0527	0.9585	1.6080	3.7270
25	-0.0221	-0.0374	-0.0507	0.8992	1.4404	3.7437
30	-0.0235	-0.0385	-0.0509	0.8059	1.2915	3.3898
35	-0.0233	-0.0374	-0.0491	0.7424	1.2058	3.0138
40	-0.0227	-0.0364	-0.0479	0.7131	1.1438	2.8321
45	-0.0221	-0.0353	-0.0458	0.6369	1.0139	2.5914
50	-0.0226	-0.0345	-0.0447	0.6143	0.9975	2.5364
55	-0.0222	-0.0341	-0.0439	0.5887	0.9582	2.3048
60	-0.0212	-0.0333	-0.0437	0.5389	0.8915	2.3084

Table (4.1) Power estimate for $\hat{\Delta}_n^*$-statistic

Distribution	Parameter	Sample size		
	θ	n=10	n=20	n=30
F_1	2	0.560	0.879	0.948
(Linear failure)	3	0.584	0.936	0.981
rate	4	0.828	0.970	0.994
F_2	2	0.452	0.623	0.698
(Makeham)	3	0.604	0.757	0.999
	4	0.700	0.843	1.000
F_3	2	0.351	0.875	0.967
(Weibull)	3	0.661	0.997	0.999
	4	0.769	0.996	1.000
F_4	2	0.769	0.895	0.903
(Pareto)	3	0.938	0.981	0.994
	4	0.979	0.998	0.999

6 Application

In this section we calculate the $\hat{\Delta}_n^*$ test statistic for the data representing the lifetimes of 40 patients suffering from blood cancer at one of the Hospitals of the ministry of Health in Saudi Arabia (see Abouammoh et al (1994)). The ordered life times (in days) are

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

Using equation (3.4), we find that the value of the statistic $\hat{\Delta}_n^*$ for this set of data is 5475.5.

It is clear from the computed value of the test statistic that we accept H_1 which states that this set of data has EWRE property under significant level $\alpha = 0.01$.

Acknowledgement The author is grateful to Dr. M. Hendi and Dr. H. Al-Nachawati for their valuable discussions when preparing the paper.

REFERENCES

Alwasel, I. A. and Al-Nachawati, H. M. (1996). Shock models with exponential better than renewal in expectation property. Pak. J. Statist. 12, 139-150.

Abouammoh, A. M. (1988). "On the criteria of the mean remaining life". Statistics & Probability Letters, 6, 207-211.

Abouammoh, A. M. (1994). "Partial orderings and testing for renewal failure rate classes of life distributions". Dirasat, 21B, 175-187.

Abouammoh, A. M., Abdulghani, S. A. and Qamber, I. S. (1994). "On partial orderings and testing of new better than renewal used classes". Reliability Eng. Syst. Safety 43, 37-41.

Abouammoh, A. M. and Ahmed, A. N. (1989). "Two notations of positive ageing based on stochastic dominance". Statische Hefte, 30, 213-219.

Abouammoh, A. M. and A. N. Ahmad (1992). "On renewal failure rate classes of life distributions, Statist. Prob. Lett., 14, 211-217.

Abouammoh, A. M., Ahmed, A. N. and Barry, A. M. (1993). "Shock models and testing for the renewal remaining life classes". Mocroelectron Reliab., 33, 729-740.

Abouammoh, A. M. and Al-Sadi, M. H. (1996). "On test of new better than renewal used classes". Arab J. Math. Sc., 2, 99-120.

Abouammoh, A. M. and Khalique, A. (1987). "Some tests for mean residual life criteria based on total time on test transform". Reliability Engineering 19, 85-101.

Abouammoh, A. M. and Newby, Martin J. (1989). "On partial ordering and tests of generalized new better than used class of life distributions".

Reliability Eng. Syst. Safety, 25, 207-217.

Ahmad, I. A. (1975). "A nonparametric test for monotonicity of a failure rate function". Communications in Statistics, 4, 967-974.

Ahmad, I. A. (1994). "A class of Statistics useful in testing increasing failure rate average and new better than used life distribution". J. Statist. Plant. Inf., 41, 141-149.

Ahmad, I. A. (1995). "Nonparametric testing of classes of life distributions derived from a variability ordering". Parisankhyan Samikkha, 2, 13-18.

Barlow, R. E. (1979). "Geometry of the total time on test transform". Naval Res. Logist. Quart., 26, 393-402.

Barlow, R. E. and Campo, R. (1975). "Total time on test processes and applications to failure data analysis". Reliab. & Fault Tree Analysis, SIAM, Philadelphia, 451-481.

Barlow, R. E. and Doksum, K. A. (1972). "Isotonic tests for convex ordering". Proc. 6^{th} Berkeley Symp, 1, 293-323.

Barlow, R. E. and Proschan F. (1981). "Statistical Theory of Reliability and life testing, Probability Models. (To Begin With, Silver Spring, MD)

Bergman, G. (1979). "On age replacement and the total time on test concept". Scand. J. Statist. 6, 161-168.

Deshpande, J. V., Kochar, S. C. and Singh, H. (1986). "Aspects of positive ageing". J. Applied Probability, 23, 748-758.

Hendi, M. I., Al-Nachawati, H., Montasser, M. and Alwasel, I. A. (1996). "An exact test for HNBUE class of life distributions". Tech. Report No. STR1417-03, Department of Statistics & O. R., Sci. College, King Saud University.

Hendi, M. I. and Rady, E. A. (1994). "On NBUA class of life distributions". J. King Saud Univ., Science (1), 6, 109-122.

Hollander, M. and Proschan, F. (1972). Testing weather new is better than used. Ann. Math. Stat., 43, 1136-1146.

Hollander, M. and Proschan, F. (1975). Test for the mean residual life. Biometrika, 62, 585-593.

Kanjo, A. I. (1993). "An exact test for NBUE class of survival functions". Commun. Statist. Theory and Methods, 22(3), 787-795.

Klefsjo, B. (1982). "Some aging properties and the total time on test transform". Res. Rep., 1979, Dept. of Math. Statist. Univ. of Umea, Sweden.

Klefsjo, B. (1983). "Some tests against ageing based on the total time on test transform". Communications in Statistics, Theory and Methods, 12, 907-927. Proschan, F. and Pyke, R. (1967). "Tests for monotone failure rate". Proc 5^{th} Berkeley Symp, 3, 293-312.

I. A. ALlwasel
Department of Statistics, College of Science,
King Saud University, P. O. Box 2455,
Riyadh 11451, Saudi Arabia.