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Bayesian Estimation of the Scale Parameter of the Complete Even Power Exponential Distribution

Reda I. Mazloum

Faculty of Economics and Political Science

Cairo University

This paper presents Bayes estimators of the scale parameter of complete even power exponential distribution under a variety of loss functions and using both the natural conjugate prior and the Jeffreys invariant prior.

1. Introduction

Let X be distributed according to the complete even power exponential distribution, i.e.,

$$f_{b,m,\mu,\sigma}(x) = \frac{m}{\sigma b^{1/(2m)}\,\Gamma(1/(2m))}\, e^{-\frac{1}{b}\left(\frac{x-\mu}{\sigma}\right)^{2m}} \qquad (1\text{-}1)$$

where $-\infty < x < \infty$ $-\infty < \mu < \infty$, $\sigma > 0$, m is a positive integer, $b > 0$ and m and b are chosen arbitrarily.

The class of densities $\{f_{b,m,\mu,\sigma}(.) : b > 0$ and m is a positive integer$\}$ given by (1-1) represents families of distributions depending upon the values of m and b chosen. For example, $m = 1$ and $b = 2$ yield the normal distribution. Also, $m = 2$ and $b = 1$ give the special form of the 4th power exponential distribution termed complete 4th power exponential distribution.

Geometric and analytical characteristics of this class of densities have been presented by Mostafa and Mazloum (1995). Estimators of μ and σ (σ^2) using both methods of moments and maximum likelihood have been obtained by Mazloum (1996).

In this paper, Bayesian point and interval estimators of σ^r (r is any positive integer) when the location parameter μ is known are derived under both the natural conjugate prior and the Jeffreys invariant prior. Different loss functions are considered namely: squared error loss, scale invariant loss, absolute error loss and zero-one loss.

2. Bayesian Point and Interval Estimation of σ^r:

Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from the probability density function (pdf) $f_{b,m,\mu,\sigma}(.)$ defined in (1-1) with μ known, say $\mu = \mu_o$. So, the joint density of the X's is given by:

$$f(x|\sigma) = \left(\frac{m}{\sigma b^{1/(2m)}\,\Gamma(1/(2m))}\right)^n e^{-\frac{1}{b}\sum_{i=1}^{n}\left(\frac{x_i-\mu_o}{\sigma}\right)^{2m}}$$

Letting

$$\tau = \frac{1}{b}\left(\frac{1}{\sigma}\right)^{2m}, \tag{2-1}$$

then the joint density $f(x|\sigma)$ can be rewritten as:

$$f(x|\tau) \propto \tau^{n/(2m)}\, e^{-\tau\sum_{i=1}^{n}(x_i-\mu_o)^{2m}} \tag{2-2}$$

where $\propto$ denotes proportionality.

As a natural conjugate prior (NCP) for τ, we take the gamma density with parameters α and β, i.e.:

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$$g(\tau) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\beta\tau} \qquad \tau > 0, \quad \alpha > 0, \quad \beta > 0 \tag{2-3}$$

We may write this briefly as $\tau \sim \Gamma(\alpha,\beta)$.

Hence, the posterior density of τ given the x's is:

$$g(\tau|x) \propto \tau^{\alpha'-1} e^{-\beta'\tau} \tag{2-4}$$

where $\alpha' = \alpha + (n/(2m))$ and $\beta' = \beta + \sum_{i=1}^{n} (x_i - \mu_o)^{2m}$, i.e., $\tau|x \sim \Gamma(\alpha',\beta')$.

In the presence of vague prior information about τ (or σ), we adopt the (improper) Jeffreys invariant prior (JIP) distribution, i.e.:

$$g_1(\tau) = 1/\tau \tag{2-5}$$

This distribution can be obtained from (2-3) by letting $\alpha \to 0$ and $\beta \to 0$. The resulting posterior distribution is given by (2-4) with $\alpha = \beta = 0$.

The following results give Bayesian point and interval estimators of σ^r. In particular, result (1) gives a $(1-\gamma)$ 100% Bayesian interval for σ^r while result (2) presents Bayesian point estimators of σ^r under both the squared error loss and a scale invariant loss. Bayes estimators of σ^r under the absolute error loss and the zero-one loss are given by results (3) and (4) respectively. Some special cases of those results are also given.

<u>Result (1)</u>

A $(1-\gamma)$ 100% Bayes interval for σ^r using the NCP is given by:

$$\left(\frac{2\beta'}{b\chi^2_{2\alpha',1-\frac{\gamma}{2}}}\right)^{r/2m} < \sigma^r < \left(\frac{2\beta'}{b\chi^2_{2\alpha',\frac{\gamma}{2}}}\right)^{r/2m} \qquad (2-6)$$

where α' and β' are defined in (2-4) and $\chi^2_{2\alpha',p}$ is the p^{th} quantile of the Chi square distribution with $2\alpha'$ degrees of freedom.

Proof

Since $\tau|x \sim \Gamma(\alpha',\beta')$ then $2\beta'\tau|x \sim \chi^2_{(2\alpha')}$. Hence a $(1-\gamma)$ 100% Bayes interval for τ is given by:

$$\frac{\chi^2_{2\alpha',\frac{\gamma}{2}}}{2\beta'} < \tau < \frac{\chi^2_{2\alpha',1-\frac{\gamma}{2}}}{2\beta'}$$

Substituting for τ from (2-1), a $(1-\gamma)$ 100% Bayes interval for σ^{2m} would be:

$$\frac{2\beta'}{b\chi^2_{2\alpha',1-\frac{\gamma}{2}}} < \sigma^{2m} < \frac{2\beta'}{b\chi^2_{2\alpha',\frac{\gamma}{2}}}$$

from which the interval given by (2-6) is easily obtained.

Result (2)

For estimating σ^r using the natural conjugate prior given by (2-3) and under the squared error loss and the scale invariant loss function given by:

$$L(\sigma,d) = \frac{(d-\sigma^r)^2}{\sigma^{2r}} \qquad (2-7)$$

The Bayes estimator $\hat{\sigma}^r$ is given by:

$$\hat{\sigma}^r = \left[\frac{\beta'}{b}\right]^{r/2m} \cdot \frac{\Gamma(\alpha' + (tr/2m))}{\Gamma(\alpha' + \frac{j}{2m})} \qquad (2-8)$$

where

$$(t,j) = \begin{cases} (-1,0) & \text{if the loss is squared error} \\ (1,2r) & \text{if the loss is (2-7)} \end{cases} \qquad (2\text{-}9)$$

and α' and β' are as defined by (2-4).

<u>Proof</u>

In general,

$$E[\sigma^k|x] = E[(b\tau)^{-k/(2m)}|x]$$

$$= b^{-k/(2m)} E[\tau^{-K/(2m)}|x]$$

$$= b^{-k/(2m)} \int_0^\infty \tau^{-k/(2m)} g(\tau|x)\,d\tau$$

$$= b^{-k/(2m)} \frac{(\beta')^{\alpha'}}{\Gamma(\alpha')} \int_0^\infty \tau^{\alpha'-k/(2m)-1} e^{-\beta'\tau} d\tau$$

Letting $u = \beta'\tau$, we get:

$$E[\sigma^k|x] = [\beta'/b]^{k/2m} \frac{\Gamma(\alpha' - k/(2m))}{\Gamma(\alpha')} \qquad (2\text{-}10)$$

(i) When the loss function is squared error, the Bayes estimator of σ^r would be:

$$\hat{\sigma}^r = E[\sigma^r|x]$$

Setting $k = r$ in (2-10), we get the Bayes estimator $\hat{\sigma}^r$ given by (2-8) with $(t,j) = (-1,0)$.

(ii) When the loss function is the scale invariant loss given by (2-7), the Bayes estimator of σ^r would be:

$$\hat{\sigma}^r = \frac{E[\sigma^r . \frac{1}{\sigma^{2r}} \mid x]}{E[\frac{1}{\sigma^{2r}} |x]} = \frac{E[\sigma^{-r}|x]}{E[\sigma^{-2r}|x]} \qquad (2\text{-}11)$$

Letting k = -r and -2r respectively in (2-10) to evaluate the numerator and the denominator of (2-11), we get the estimator $\hat{\sigma}^r$ given by (2-8) with (t,j) = (1,2r).

Result (3)

The Bayes estimator of σ^r with respect to the natural conjugate prior and using the absolute error loss is given by:

$$\hat{\hat{\sigma}}^r = \left(\frac{2\beta'}{b\,\chi^2_{2\alpha',\frac{1}{2}}}\right)^{r/2m} \qquad (2\text{-}12)$$

Proof

The Bayes estimator $\hat{\hat{\tau}}$ of τ with respect to the absolute loss $L(\tau,d) = |d-\tau|$ is such that $P(\tau < \hat{\hat{\tau}}|x) = \frac{1}{2}$.

Using the fact that $2\beta'\tau|x \sim \chi^2_{2\alpha'}$, we get:

$$\hat{\hat{\tau}} = \frac{1}{2\beta'}\,\chi^2_{2\alpha,\frac{1}{2}}$$

Now, $\sigma^r = (b\tau)^{-r/2m}$ is a strictly monotone function of τ. Hence, the absolute error loss Bayes estimator of σ^r is:

$$\hat{\hat{\sigma}}^r = (b\hat{\hat{\tau}})^{-r/2m} = \left(\frac{2\beta'}{b\,\chi^2_{2\alpha',\frac{1}{2}}}\right)^{r/2m}$$

Result (4)

The Bayes estimator of σ^r with respect to the natural conjugate prior and using the zero-one loss function is given by:

$$\tilde{\sigma}^r = \left[\frac{2m\beta'}{b(2m\alpha' + r)}\right]^{r/2m} \qquad (2\text{-}13)$$

Proof

For the zero-one loss function namely:

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$$L(\sigma^r,\tilde{\sigma}^r) = \begin{cases} 0 & \sigma^r - \varepsilon < \tilde{\sigma}^r < \sigma^r + \varepsilon; \quad \varepsilon > 0 \\ 1 & \text{otherwise} \end{cases}$$

and under the limiting case $\varepsilon \to 0$, the Bayes estimator $\tilde{\sigma}^r$ of σ^r is a mode of the posterior distribution $g(\sigma^r|x)$ which can easily be shown to have the form (2-13).

In the case of vague prior information, i.e. when τ (or σ) has the Jeffreys invariant prior given by (2-5), corresponding Bayes results could be easily obtained by simply setting $\alpha = \beta = 0$ in (2-6), (2-8), (2-12) and (2-13).

Some Special Cases

Case (1)

For m = 1 and b = 2, i.e., for $N(\mu_o,\sigma^2)$, the previous results give the following well known [1] Bayes answers for σ^r when using the natural conjugate prior:

(i) A$(1-\gamma)$ 100% Bayes interval for σ^r is:

$$\left(\frac{\beta + \sum_{i=1}^{n}(x_i - \mu_o)^2}{\chi^2_{2(\alpha+\frac{n}{2}),1-\frac{\gamma}{2}}}\right)^{r/2} < \sigma^r < \left(\frac{\beta + \sum_{i=1}^{n}(x_i-\mu_o)^2}{\chi^2_{2(\alpha+\frac{n}{2}),\frac{\gamma}{2}}}\right)^{r/2} \tag{2-14}$$

(ii) The Bayes estimator of σ^r with respect to the squared error loss and the scale invariant loss given by (2-7), is given by:

$$\hat{\sigma}^r = \left[\frac{\beta + \sum_{i=1}^{n}(x_i-\mu_o)^2}{2}\right]^{r/2} \cdot \frac{\Gamma(\alpha + \frac{n+tr}{2})}{\Gamma(\alpha+\frac{n+j}{2})} \tag{2-15}$$

where (t,j) is as defined in (2-9).

(iii) The absolute error loss Bayes estimator of σ^r is:

$$\hat{\sigma}^r = \left[\frac{\beta + \sum_{i=1}^{n} (x_i - \mu_o)^2}{\chi^2_{2(\alpha+\frac{n}{2}),\frac{1}{2}}}\right]^{r/2} \qquad (2\text{-}16)$$

(iv) The Bayes estimator of σ^r under the zero-one loss function is given by:

$$\tilde{\sigma}^r = \left[\frac{\beta + \sum_{i=1}^{n} (x_i - \mu_o)^2}{\alpha + \frac{n}{2} + r}\right]^{r/2} \qquad (2\text{-}17)$$

Corresponding Bayes results for the case of vague prior information could be obtained by setting $\alpha = \beta = 0$ in (2-14) through (2-17).

Case (2)

For the complete fourth power exponential distribution, corresponding Bayes results for σ^r could be obtained by setting $m = 2$ and $b = 1$.

3. Conclusion

The complete even power exponential distribution can be considered an extension of the normal distribution. In fact, it represents a class of distributions that includes as members: the normal distribution, the 4th power exponential distribution, the 6th power exponential distribution, ... etc.

Bayesian points and interval estimators of the scale parameter (more generally of σ^r; r is any positive integer) of this class of densities have been obtained using different loss functions and under the natural conjugate prior and the Jeffreys invariant prior.

Some special cases of these results have been given. In particular, special case (1) has been given to show, explicitly, that the results obtained for this more general class of densities reduce to the corresponding results, known in the literature, about the normal distribution.

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