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# ON MOMENTS OF ORDER STATISTICS FOR SAMPLE RANGE AND SAMPLE MIDRANGE FROM UNIFORM POPULATION.

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## ABSTRACT

*This paper deals with some recurrence relations satisfied by the single and the product moments of order statistics for the sample range and sample midrange from n independent and identically distributed uniform random variables. The negative moments and some applications are also obtained.*

***Keywords and Phrases*** : *Order Statistics; Range; Midrange; Uniform population; Negative Moments; Product and Quotient Moments .*

## 1- INTRODUCTION

Let $\{X_n\}$ be a sequence of mutually independent random variables (r.v's) all having the same continuous distribution function (d.f.) $F(x) = P(X_n \le x)$. The order statistics of $X_1, X_2, \ldots, X_n$ are denoted respectively by

$$X_1^{(n)} \le X_2^{(n)} \le \ldots \le X_n^{(n)}$$

The probability density function (p.d.f.) of the sample range $W_{1,n}^{(n)} = X_n^{(n)} - X_1^{(n)}$ from the standard uniform distribution (S.U.D) is

$$f(w) = n(n-1)w^{n-2}(1-w) \qquad 0 < w < 1 \tag{1.1}$$

It is easy from Equation (1.1) to show that;

$$(n-1)w^{-1}(F(w) - w^{n}) = f(w), \tag{1.2}$$

where $F(w)$ is the d.f. of $W_{1,n}^{(n)}$ from the (S.U.D).

We write the k-th moment of $W_{1,n}^{(n)}$ as

$$\mu_{r:n}^{(k)} = C_{r:n} \int_{0}^{1} w^{k}(F(w))^{r-1}(1-F(w))^{n-r} f(w)dw, \tag{1.3}$$

and the product moments of $W_{1,n}^{(n)}$ as

$$\mu_{r,s:n}^{(j,k)} = C_{r,s:n} \int_{0}^{1}\int_{0}^{1} w_1^{j} w_2^{k}(F(w_1))^{r-1}(F(w_2) - F(w_1))^{s-r-1}$$

$$(1-F(w_2))^{n-s} f(w_1)f(w_2)dw_1dw_2, \tag{1.4}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!} \quad \text{and} \quad C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Also, the p.d.f. of the sample midrange $V_{1,n}^{(n)} = \left(\frac{X_n^{(n)} + X_1^{(n)}}{2}\right)$ from (S.U.D)

is given by:

$$f(v) = \begin{cases} 2^{n-1} n v^{n-1}, & 0 \le v \le \frac{1}{2}, \\ 2^{n-1} n(1-v)^{n-1}, & \frac{1}{2} \le v \le 1. \end{cases} \tag{1.5}$$

It is easy from Equation (1.5) to show that:

$$f(v) = \begin{cases} nv^{-1}F(v), & 0 \le v \le \frac{1}{2}, \\ n(1-v)^{-1}(1-F(v)), & \frac{1}{2} \le v \le 1. \end{cases} \tag{1.6}$$

where $F(v)$ is the d.f. of $V_{1,n}^{(n)}$ from the (S.U.D).

The moments of order statistics have been assumed considerable interest in recent years and have been tabulated quite extensively for several distributions. Several recurrence relations and identities among the moments are also available, see Govindarajulu (1963), Downton (1966), Krishnaiah and Rizvi (1966), Joshi (1971, 1973) and David (1981). Joshi and Balakrishnan (1982) obtained several new recurrence relations linking the product moments with single moments and moments in samples of sizes n-1 and less. Some recurrence relations for mixed moments of order statistics in random samples of size n from an exponential and right truncated exponential distributions were derived by Joshi (1982). He has shown that one can calculate all of these moments without evaluating any single or double integral. Recurrence relations for negative and fractional moments of single order statistics and product and quotient moments of two order statistics drown from log-logistic distribution have been obtained by Masoon and Khan (1987). Balakrishnan (1987), used a basic result due to David and Joshi (1968) and showed that these identities for the moments also hold when the order statistics arise from exchangeable variables. A duality principle for order statistics in the arbitrary case, using which many known dual results on order statistics was established by Balasubramanian and Balakrishnan (1993). By considering order statistics arising from n independent non-identically distributed right-truncated exponential random variables, Balakrishnan (1994), derived several recurrence relations for the single and the product moments of order statistics. These recurrence relations will enable one to compute all the single and the product moments of all order statistics in a simple recursive manner. The results for the multiple-outlier model were deduced as a special cases. The results were further generalized to the case of a right-truncated exponential population. Balakrishnan and Balasubramanian (1995), investigated the above results from non-identical power function random variables. Nigm and El-Hawary (1996) obtained some recurrence relations for the negative and fractional moments of single and product of two order statistics drawn from Weibul distribution. Also, all variances and covariances were computed by using the Chebyshev approximation method. In this paper, some recurrence relations satisfied by the single and the product moments of order statistics for the sample range and sample midrange from n independent and identically distributed uniform function random variables are obtained. A relation between negative and positive moments of the single and the product moments of order statistics for the sample range and sample midrange from n independent and identically distributed uniform random variables is given.

## 2. RECURRENCE RELATIONS BETWEEN SINGLE MOMENTS OF ORDER STATISTICS.

### 2.1. FOR THE SAMPLE RANGE FROM S.U.D.

**Theorem 2.1.1.** For $2 < r \leq n$, $n > 2$,

$$\mu_{r:n}^{(k)} = C_1 \mu_{r-1:n}^{(k)} + C_2 \left( \mu_{r-1:n-1}^{(k+n)} - \mu_{r-2:n-1}^{(k+n)} \right),$$

where

$$C_1 = \frac{k+r-1}{(n-1)(r-1)} \text{ and } C_1 = \frac{nk}{(k+n)(n-1)(r-1)}.$$

**Proof.** From Equations (1.2) and (1.3), we have

$$\mu_{r:n}^{(k)} = (n-1)C_{r:n}\left[ \int_0^1 w^{k-1}(F(w))^r (1-F(w))^{n-r} dw - \int_0^1 w^{k+n-1}(F(w))^{r-1}(1-F(w))^{n-r} dw \right] \quad (2.1.1)$$

ISSR, Cairo Univ., Vol.,41, No.1, 1997, PP.28-42



Upon integrating Equation (2.1.1) by parts, treating $w^{k-1}$ for integration and the rest of the integrand for differentiation in the first integral and treating $w^{k+n-1}$ for integration and the rest of the integrand for differentiation in second integral, we get,

$$\mu_{r:n}^{(k)} = (n-1)C_{r:n}\left[\frac{(n-r)}{k}\int_0^1 w^k(F(w))^r(1-F(w))^{n-r-1}f(w)dw\right.$$

$$-\frac{r}{k}\int_0^1 w^k(F(w))^{r-1}(1-F(w))^{n-r}f(w)dw$$

$$-\frac{(n-r)}{k+n}\int_0^1 w^{k+n}(F(w))^{r-1}(1-F(w))^{n-r-1}f(w)dw$$

$$\left.+\frac{(r-1)}{k+n}\int_0^1 w^{k+n}(F(w))^{r-2}(1-F(w))^{n-r}f(w)dw\right].$$

Hence,

$$\mu_{r:n}^{(k)} = \frac{(n-1)r}{k}\mu_{r+1:n}^{(k)} - \frac{r}{k}\mu_{r:n}^{(k)} - \frac{n}{k+n}\mu_{r:n-1}^{(k+n)} + \frac{n}{k+n}\mu_{r-1:n-1}^{(k+n)}. \qquad (2.1.2)$$

Replacing $r+1$ by $r$ in Equation (2.1.2), we have the result.

The negative moments of single order statistics for the sample range from uniform distribution have been evaluated by using the following corollary.

**Corollary 2.1.1.** From Theorem 2.1.1., with $k+n=\alpha$, $\alpha<n$, we have

$$\mu_{r:n}^{(\alpha-n)} = \frac{(\alpha-n)+(r-1)}{(n-1)(r-1)}\mu_{r-1:n}^{(\alpha-n)} + \frac{n(\alpha-n)}{\alpha(n-1)(r-1)}\left(\mu_{r-1:n-1}^{(\alpha)} - \mu_{r-2:n-1}^{(\alpha)}\right).$$

**Corollary 2.1.2.** $\mu_{1:1}^{(k)} = 0.$

**Proof.** It is obvious.

**Theorem 2.1.2.** For $n \geq 2$,

$$\mu_{1:n+1}^{(k)} = \frac{n(n+1)}{k+n+1}\left(\mu_{1:n}^{(k)} - \mu_{1:n}^{(k+n+1)}\right).$$

**Proof.** From Equations (1.2) and (1.3), with $r = 1$ we get,

$$\mu_{r:n}^{(k)} = (n-1)C_{1:n}\left[\int_0^1 w^{k-1} F(w)(1-F(w))^{n-1} dw - \int_0^1 w^{k+n-1}(1-F(w))^{n-1} dw\right]$$

Put $F(w) = 1-(1-F(w))$, then we have,

$$\mu_{r:n}^{(k)} = (n-1)C_{1:n}\left[\int_0^1 w^{k-1}(1-F(w))^{n-1} dw - \int_0^1 w^{k-1}(1-F(w))^{n} dw \right.$$
$$\left. - \int_0^1 w^{k+n-1}(1-F(w))^{n-1} dw\right]. \quad (2.1.3)$$

Upon integrating the three integrals in Equation (2.1.3) by parts similar to technique Theorem 2.1.1., we get the result.

**Corollary 2.1.3.** From Theorem 2.1.2., with $k+n=\alpha$, $\alpha < n$, we have

$$\mu_{1:n+1}^{(\alpha-n)} = \frac{n(n+1)}{\alpha+1}\left(\mu_{1:n}^{(\alpha-n)} - (\alpha-n)\mu_{1:n}^{(\alpha+1)}\right).$$

**Corollary 2.1.4.** For $n \geq 2$,

$$\mu_{n:n}^{(k)} = \left(1 - \frac{k}{k+n(n-1)}\right)\left(1 - \frac{k}{k+n}\left(1 - \frac{1}{n}\mu_{1:n}^{(k)}\right)\right).$$

**Proof of Corollary 2.1.4.** From Equation (1.3) with $r=n$ and from Equation (1.2), we have,

$$\mu_{n:n}^{(k)} = n(n-1)\left[\int_0^1 w^{k-1}(F(w))^n f(w)dw - \int_0^1 w^{k+n-1}(F(w))^{n-1} f(w)dw\right].$$

Using the same manner of Theorem 2.1.1., the proof is complete.

## 2.2. FOR THE SAMPLE MIDRANGE FROM S.U.D.

The Equation (1.3) can be written as the form for the sample midrange,

$$\mu_{r:n}^{(k)} = C_{r:n}\left[\int_0^{1/2} v^k(F(v))^r(1-F(v))^{n-r} f(v)dv + \int_{1/2}^1 v^k(F(v))^{r-1}(1-F(v))^{n-r} f(v)dv\right] = \xi_{r:n}^{(k)} + \eta_{r:n}^{(k)} \qquad (2.2.1)$$

**Theorem 2.2.1.** For $1 \le r < n$,

$$\mu_{r:n}^{(k)} = e_1 + e_2\left[(n-2r)\xi_{r+1:n}^{(k)} + n\xi_{r:n-1}^{(k)}\right] + e_3\left[n\,\eta_{r:n-1}^{(k)} - r\,\eta_{r+1:n}^{(k)}\right],$$

where

$$e_1 = \frac{n}{k}\left(\tfrac{1}{2}\right)^{k+n} C_{r:n}, \quad e_2 = \frac{nr}{k(n-r)} \text{ and } e_3 = \frac{1}{n-r}.$$

**Proof.** From Equations (1.6) and (2.2.1), taking the same manner of Theorem 2.1.1., the proof is complete.

**Corollary 2.2.1.** With $r=1$ in Theorem 2.2.1. and for $n>1$, we have

$$\mu_{1:n}^{(k)} = e_4 + e_5\left[(n-2)\xi_{2:n}^{(k)} + n\xi_{1:n-1}^{(k)}\right] + e_6\left[n\,\eta_{1:n-1}^{(k)} - \eta_{2:n}^{(k)}\right],$$

where

$$e_4 = \frac{n^2}{k}\left(\tfrac{1}{2}\right)^{k+n}, \qquad e_5 = \frac{n}{k(n-1)} \text{ and } e_6 = \frac{1}{n-1}.$$

**Proof.** It is obvious.

**Theorem 2.2.2.** For $n \geq 2$,

$$\mu_{n:n}^{(k)} = \frac{n}{k}\left(\tfrac{1}{2}\right)^{k+n} - \frac{n^2}{k}\xi_{n:n}^{(k)} + \eta_{n-1:n-1}^{(k)} - \eta_{n-1:n}^{(k)}.$$

**Proof.** It is obvious.

**Corollary 2.2.2.** $\mu_{1:1}^{(k)} = \dfrac{1}{k+1}.$

**Proof.** From Equations (1.6) and (2.2.1) with $r = n = 1$ and using the same manner of Theorem 2.1.1., the proof is complete.

## 3. RECURRENCE RELATIONS BETWEEN PRODUCT MOMENTS OF ORDER STATISTICS.

### 3.1. FOR THE SAMPLE RANGE FROM S.U.D.

**Theorem 3.1.1.** For $1 \leq r < s \leq n, \quad n \geq 2,$

$$\mu_{r,s:n}^{(j,k)} = a_1\left[\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)} + a_2\left(\mu_{r,s:n-1}^{(j,k+n)} + \mu_{r,s-1:n-1}^{(j,k+n)}\right) + a_3\,\mu_{r,s-1:n}^{(j,k)}\right],$$

where

$$a_1 = \frac{n(n-1)}{k+(n-1)(n-s+1)}, \qquad a_2 = \frac{k}{k+n} \text{ and } a_3 = \left(1 - \frac{s-1}{n}\right).$$

**Proof.** From Equations (1.2) and (1.4), we have,

$$\mu_{r,s:n}^{(j,k)}=(n-1)C_{r,s:n}\left[\int_0^1\int_{w_1}^1 w_1^j w_2^{k-1}\left(F(w_1)\right)^{r-1}\left(F(w_2)-F(w_1)\right)^{s-r-1}\right.$$

$$\cdot\ \left(1-F(w_2)\right)^{n-s} f(w_1)F(w_2)\,dw_2\,dw_1$$

$$-\int_0^1\int_{w_1}^1 w_1^j w_2^{k+n-1}\left(F(w_1)\right)^{r-1}\left(F(w_2)-F(w_1)\right)^{s-r-1}$$

$$\left.\left(1-F(w_2)\right)^{n-s} f(w_1)\,dw_2\,dw_1\right]$$

$$=(n-1)C_{r,s:n}\left[\int_0^1\int_{w_1}^1 w_1^j w_2^{k-1}\left(F(w_1)\right)^{r-1}\left(F(w_2)-F(w_1)\right)^{s-r-1}\right.$$

$$\left(1-F(w_2)\right)^{n-s} f(w_1)\,dw_2\,dw_1$$

$$-\int_0^1\int_{w_1}^1 w_1^j w_2^{k-1}\left(F(w_1)\right)^{r-1}\left(F(w_2)-F(w_1)\right)^{s-r-1}$$

$$\left(1-F(w_2)\right)^{n-s+1} f(w_1)\,dw_2\,dw_1$$

$$-\int_0^1\int_{w_1}^1 w_1^j w_2^{k+n-1}\left(F(w_1)\right)^{r-1}\left(F(w_2)-F(w_1)\right)^{s-r-1}$$

$$\left.\left(1-F(w_2)\right)^{n-s} f(w_1)\,dw_2\,dw_1\right]. \tag{3.1.1}$$

Upon integrating Equation (3.1.1) by parts, treating $w^{k-1}$ for integration and the rest of the integrand for differentiation in the first and second integral and treating $w^{k+n-1}$ for integration and the rest of the integrand for differentiation in the third integral, we get,

$$\mu_{r,s:n}^{(j,k)}=(n-1)C_{r,s:n}\Bigg[\frac{(n-s)}{k}\int_0^1\int_{w_1}^1 w_1^j w_2^k\big(F(w_1)\big)^{r-1}\big(F(w_2)-F(w_1)\big)^{s-r-1}$$

$$\big(1-F(w_2)\big)^{n-s-1}f(w_1)f(w_2)\,dw_2\,dw_1$$

$$-\frac{(s-r-1)}{k}\int_0^1\int_{w_1}^1 w_1^j w_2^k\big(F(w_1)\big)^{r-1}\big(F(w_2)-F(w_1)\big)^{s-r-2}$$

$$\big(1-F(w_2)\big)^{n-s}f(w_1)f(w_2)\,dw_2\,dw_1$$

$$-\frac{(n-s+1)}{k}\int_0^1\int_{w_1}^1 w_1^j w_2^k\big(F(w_1)\big)^{r-1}\big(F(w_2)-F(w_1)\big)^{s-r-1}$$

$$\big(1-F(w_2)\big)^{n-s}f(w_1)f(w_2)\,dw_2\,dw_1$$

$$+\frac{(s-r-1)}{k}\int_0^1\int_{w_1}^1 w_1^j w_2^k\big(F(w_1)\big)^{r-1}\big(F(w_2)-F(w_1)\big)^{s-r-2}$$

$$\big(1-F(w_2)\big)^{n-s+1}f(w_1)f(w_2)\,dw_2\,dw_1$$

$$-\frac{(n-s)}{k+n}\int_0^1\int_{w_1}^1 w_1^j w_2^{k+n}\big(F(w_1)\big)^{r-1}\big(F(w_2)-F(w_1)\big)^{s-r-1}$$

$$\cdot\left(1-F(w_2)\right)^{n-s-1} f(w_1) f(w_2)\, dw_2\, dw_1$$

$$+\frac{(s-r-1)}{k+n}\int_0^1\int_{w_1}^1 w_1^j w_2^{k+n}\left(F(w_1)\right)^{r-1}\left(F(w_2)-F(w_1)\right)^{s-r-2}$$

$$\left.\left(1-F(w_2)\right)^{n-s} f(w_1) f(w_2)\, dw_2\, dw_1\right]. \qquad (3.1.2)$$

From Equation (1.4) then Equation (3.1.2) becomes

$$\mu_{r,s:n}^{(j,k)}=\frac{n(n-1)}{k}\left[\mu_{r,s:n-1}^{(j,k)}-\mu_{r,s-1:n-1}^{(j,k)}\right]+\frac{(n-1)(n-s+1)}{k}\left[\mu_{r,s-1:n}^{(j,k)}-\mu_{r,s:n}^{(j,k)}\right]$$
$$+\frac{n(n-1)}{k}\left[\mu_{r,s-1:n-1}^{(j,k+n)}-\mu_{r,s:n-1}^{(j,k+n)}\right],$$

which simply implies the result.

**Corollary 3.1.1.** From Theorem 3.1.1., with $k+n=\alpha$, $\alpha<n$, we have

$$\mu_{r,s:n}^{(j,\alpha-n)}=b_1\left[\mu_{r,s:n-1}^{(j,\alpha-n)}-\mu_{r,s-1:n-1}^{(j,\alpha-n)}+b_2\left(\mu_{r,s:n-1}^{(j,\alpha)}+\mu_{r,s-1:n-1}^{(j,\alpha)}\right)+a_3\,\mu_{r,s-1:n}^{(j,\alpha-n)}\right],$$

where

$$b_1=\frac{n(n-1)}{\alpha-n+(n-1)(n-s+1)},\quad b_2=\left(1-\frac{n}{\alpha}\right)\text{ and } a_3=\left(1-\frac{s-1}{n}\right).$$

**Corollary 3.1.2.** With $s=n$ in Theorem 3.1.1., we have

$$\mu_{r,n:n}^{(j,k)}=c_1\left[\mu_{r,n:n-1}^{(j,k)}-\mu_{r,n-1:n-1}^{(j,k)}+c_2\left(\mu_{r,n:n-1}^{(j,k+n)}+\mu_{r,n-1:n-1}^{(j,k+n)}\right)+c_3\,\mu_{r,n-1:n}^{(j,k)}\right],$$

where,

$$c_1=\frac{n(n-1)}{k+n-1},\quad c_2=\left(1-\frac{n}{k+n}\right)\text{ and } c_3=\frac{1}{n}.$$

**Theorem 3.1.2.** For $1 \le r < s \le n, \quad n \ge 2,$

$$\sigma_{r,s:n} = d_1\Big[\sigma_{r,s:n-1} - \sigma_{r,s-1,n-1} + d_2\Big(\sigma^{(1,1+n)}_{r,s:n-1} + \sigma^{(1,1+n)}_{r,s-1:n-1}\Big)$$

$$+\mu_{r:n}\Big[\big(\mu_{s:n-1} - \mu_{s-1:n-1}\big) + d_3\Big(\mu^{(n+1)}_{s:n-1} + \mu^{(n+1)}_{s-1:n-1}\Big)\Big]\Big]$$

where

$$d_1 = \frac{n(n-1)}{k+(n-1)(n-s+1)}, \qquad d_2 = d_3 = \frac{1}{n+1}.$$

**Proof.** Since

$$\sigma_{r,s:n} = \mu_{r,s:n} - \mu_{r:n}\,\mu_{s:n}.$$

The proof is complete after using Theorems 3.1.1 and 2.1.1.

## 3.2. FOR THE SAMPLE MIDRANGE FROM S.U.D.

The Equation (1.4) can be written in the following form:

$$\mu^{(j,k)}_{r,s:n} = C_{r,s:n}\Bigg[\int_0^{1/2}\int_{v_1}^{1/2} v_1^j v_2^k \big(F(v_1)\big)^{r-1}\big(F(v_2)-F(v_1)\big)^{s-r-1}$$

$$\big(1-F(v_2)\big)^{n-s} f(v_1) f(v_2)\, dv_2\, dv_1 +$$

$$+\int_{1/2}^{1}\int_{v_1}^{1} v_1^j v_2^k \big(F(v_1)\big)^{r-1}\big(F(v_2)-F(v_1)\big)^{s-r-1}$$

$$\big(1-F(v_2)\big)^{n-s} f(v_1) f(v_2)\, dv_2\, dv_1\Bigg] = \xi^{(j,k)}_{r,s} + \eta^{(j,k)}_{r,s:n}. \qquad (3.2.1)$$

**Theorem 3.2.1.** For $1 \le r < s < n$,

$$\mu^{(j,k)}_{r,s:n} = \xi^{(j,k)}_{r,s:n} + \frac{1}{n-s}\left[\eta^{(j,k)}_{r,s:n-1} - (s-r)\eta^{(j,k)}_{r,s+1:n-1} - r\,\eta^{(j,k)}_{r+1,s+1:n}\right].$$

**Proof.** From Equations (1.6) and (3.2.1) and using the same manner of Theorem 2.2.1., the proof is complete.

## 4. FURTHER REMARKS

The results for the sample range from S.U.D. can be generalized to the spacing from S.U.D.

$$W^{(n)}_{i,j} = X^{(n)}_{j} - X^{(n)}_{i}, \qquad 1 \le i < j \le n.$$

It may be noted that the i-th quasi-range $W^{(n)}_{i} = X^{(n)}_{n-i+1} - X^{(n)}_{i}$ is a special case of $W^{(n)}_{i,j}$ and hence a spacing is sometimes called a generalized quasi-rang. The spacing $W^{(n)}_{i,j}$ has a Beta $(j-1, n-j+i+1)$ distribution which depends only on $j-i$ and not on $i$ and $j$ individually. Further, we note from Equation (1.1) that for the (S.U.D) the sample range is distributed exactly same as the (n-1) the order statistics in a sample of size n from the (S.U.D). The results for the sample midrange can be derived for the density function and the distribution function of the generalized quasi-midrange

$$W^{(n)}_{i,j} = \left(\frac{X^{(n)}_{j} + X^{(n)}_{i}}{2}\right), \qquad 1 \le i < j \le n.$$

which of course will include the i-th quasi-midrange as a special case. Theorems 2.1.1. and 2.1.2. and Corollary 2.1.2. are used to calculate $\mu^{(k)}_{r:n}$, $2 \le r \le n$. Once $\mu^{(k)}_{r:n}$ are know, Corollaries 2.1.1. and 2.1.3. may be used to obtain $\mu_{r:n}^{(\alpha-n)}$ . The beauty of the result is that from $\mu_{r:n}^{(\alpha-n)}$, we can find negative moments of order statistics with some constraints, viz. at $\alpha = 1$, $n = 2$, $\alpha - n = -1$..For calculating product moments matrix $\left(\left(\mu_{r,s:n}^{(j,k)}\right)\right)$, the diagonal elements $\mu_{r,r:n}^{(j,k)} = \mu_{r:n}^{(j+k)}$ can be filled up first. $\mu_{1,2:2}^{(j,k)}$ can be obtained easily by direct numerical integration. The elements $\mu_{r,r+\ell:n}^{(j,k)}$ , $2 \le r \le n-\ell-1$, , $\ell = 1,2,\ldots,n-r-1$ are obtained from Theorem 2.2.1. Once $\mu_{r,s:n}^{(j,k)}$ are known, $\mu_{r,s:n}^{(j,\alpha-n)}$ can be obtained by Corollary 2.2.1. Finally, from Theorem 2.2.2., we can obtain

$$\mathrm{Cov}\left(X_r^{(n)}, X_s^{(n)}\right) = \sigma_{r,s:n} = \mu_{r,s:n} - \mu_{r:n}\,\mu_{s:n}$$

and

$$\mathrm{Var}\left(\frac{X_r^{(n)}}{X_s^{(n)}}\right) = \mu_{r,s:n}^{(2,-2)} - \left(\mu_{r,s:n}^{(1,-1)}\right)^2 .$$

These moments may also be used to find best linear unbiased estimates of location and scale parameters of sample range for the (S.U.D). The above applications can be satisfied for the sample midrange.

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