THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CAIRO UNIV., VOL., 41, NO.1, 1997

LOCAL POWER OF TESTS IN THE PRESENCE OF NUISANCE PARAMETERS WITH AN APPLICATION

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Keywords and phrases: local power, nuisance parameters, bivariate normal distribution, combination methods

Abstract

In the presence of nuisance parameters the local power of tests for testing a simple hypothesis against one sided alternatives will be defined. In the case of a bivariate normal distribution with known correlation coefficient the local power will be derived for any symmetric test. In particular, it will be obtained for several combination methods.

1. Introduction

There are different methods for combining several tests into one overall test. Many authors studied the combination methods from the viewpoint of admissibility. If H_0 is simple, Birnbaum (1954) showed that given any nonparametric combination method with an acceptance region A which is monotone increasing in the p-values, there exists a problem for which this method is most powerful against some alternative. Brown, Cohen and Straderman (1976) have shown that such tests form a complete class. Oosterhoff (1969) applied Birnbaum's ideas to find a MCC (Monotone Complete Class) of invariant tests in non-exponential problem involving the combination of dependent noncentral t-test. Marden and Perlman (1981) found the MCC of tests for combining independent noncentral F tests. Marden (1982) looked at the problems of noncentral chi squared and noncentral F distributions. He gave necessary and sufficient conditions for a test when it is admissible in terms of the monotonocity and convexity of the acceptance region and also he determined the admissibility or inadmissibility of several combination methods. Marden (1985) determined the

MCC's of tests for the normal and noncentral t distributions.

Koziol, Perlman and Rasmussen (1988) compared the power of several combination methods in the case of noncentral F distribution.

In this paper, we extend the definition of the local power of a tests. In the presence of nuisance parameters we define the local power of tests for testing a simple hypothesis against one sided alternatives. The local power will be derived for any symmetric test in the case of a bivariate normal distribution of known correlation coefficient. In particular, several combination methods will be compared via local power.

2. The specific problem

Suppose that we have two testing problems where the i-th problem tests:

$$H_0^{(i)}: \theta_i = 0 \quad \text{vs} \quad H_1^{(i)}: \theta_i > 0 \qquad (2.1)$$

based on a random sample $X_{i1}, \ldots, X_{in_i}$ from $N(\theta_i, 1)$ for i = 1, 2. For testing $H_0^{(i)}$ there exists a UMP test which rejects $H_0^{(i)}$ for large value of $\overline{X}_i$ (the sample mean of the i-th sample). By sufficiency we can assume that $n_1 = n_2 = 1$ and without loss of generality we can assume that the i-th problem is based on X_i where $X_i \sim N(\theta_i, 1)$ for i = 1, 2.

We want to test the combined hypothesis;

$$H_0: \theta_1 = \theta_2 = 0 \text{ vs } H_1: \theta_1 \geq 0,\ \theta_2 \geq 0 \text{ and } \theta_1 + \theta_2 > 0. \qquad (2.2)$$

We will assume that $(X_1, X_2) \sim BVN(\theta_1, \theta_2, 1, 1, \rho)$ where ρ is known. In fact we will consider the special case $\theta_i = \gamma\eta_i$, for i = 1, 2 where η_1 and η_2 are nuisance parameters and η_1 and $\eta_2 \geq 0$. Moreover, we will assume that $\eta_1^2 + \eta_2^2 = 1$ because this will subject only to changing γ by a scale factor. Therefore, the hypothesis (2.2) is equivalent to

$$H_0: \gamma = 0 \quad \text{vs} \quad H_1: \gamma > 0. \qquad (2.3)$$

We will study six combinations methods. These methods are based on the p-values of the individual test statistics X_i, i = 1, 2. The i-th p-value is given by

$$P_i(x_i) = P_{\gamma=0} [\ X_i \geq x_i\ ] = 1 - \Phi(x_i) \qquad (2.4)$$

when X_i is observed to be x_i and where $\Phi(.)$ is the distribution function of standard normal distribution. We will study the Fisher, logistic, sum of the p-values, inverse normal, Tippitt and maximum of the p-values methods which reject H_0 for large values of

$$-2\sum_{i=1}^{2} \ln\left[1-\Phi(X_i)\right],\ -\sum_{i=1}^{2} \ln\left[\frac{1-\Phi(X_i)}{\Phi(X_i)}\right],\ -\sum_{i=1}^{2}\left[1-\Phi(X_i)\right],$$

$$\sum_{i=1}^{2} X_i,\ -\min_{1\le i\le 2}\left[1-\Phi(X_i)\right] \text{ and } -\max_{1\le i\le 2}\left[1-\Phi(X_i)\right]$$

respectively. We will denote the above tests by ψ_F, ψ_L, ψ_S, ψ_N, ψ_T and ψ_{max} respectively.

Note that the tests that have been combined are dependent unless $\rho = 0$.

3. The local power

We will define the local power $L_p(\psi,\psi^*)$ of a given test ψ relative to another test ψ^* in the presence of nuisance parameters η_1 and η_2 as follows:

$$L_p(\psi,\psi^*) = \inf_{\eta_1^2+\eta_2^2=1} \left[\frac{\frac{\partial}{\partial\gamma} E_{\gamma\eta}\, \psi\,(X_1,X_2)\Big|_{\gamma=0}}{\frac{\partial}{\partial\gamma} E_{\gamma\eta}\, \psi^*(X_1,X_2)\Big|_{\gamma=0}} \right] \tag{3.1}$$

where $\eta = (\eta_1,\eta_2)$ and η_1, $\eta_2 \ge 0$ and $\eta_1^2+\eta_2^2 = 1$. It is clear that (3.1) is a generalization of the definition of local power found in the literature. One should be careful in choosing the test ψ^* which is preferred to have relatively high power in a neighbourhood of zero. If the UMP test or the locally most power test exists, then it can be taken as ψ^*.

For testing (2.3), the UMP test ψ_U exists and is given by

$$\psi_U = \begin{cases} 1 & \text{if } (\eta_1-\rho\eta_2)x_1+(\eta_2-\rho\eta_1)x_2>Z_{1-\alpha}\sqrt{1-\rho^2-2\rho\eta_1\eta_2+2\rho^3\eta_1\eta_2} \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Therefore

$$L_P(\psi,\psi_U) = \inf_{\eta_1^2+\eta_2^2=1} \left[\frac{\frac{\partial}{\partial\gamma} E_{\gamma\underline{\eta}}\, \psi \Big|_{\gamma=0}}{\frac{\phi(Z_{1-\alpha})\,(1-2\rho\eta_1\eta_2)}{\sqrt{1-\rho^2-2\rho\eta_1\eta_2+2\rho^3\eta_1\eta_2}}} \right] \tag{3.3}$$

where ϕ is the pdf of standard normal distribution.

The following theorem will give us $\frac{\partial}{\partial\gamma} E_{\gamma\underline{\eta}}\, \psi \Big|_{\gamma=0}$.

Theorem 1:

(i) For any test ψ:

$$\frac{\partial}{\partial\gamma} E_{\gamma\underline{\eta}}\, \psi \Big|_{\gamma=0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\psi(x_1,x_2)\,\left[\eta_1(x_1-\rho x_1)+\eta_2(x_2-\rho x_1)\right]}{2\pi\,\left(1-\rho^2\right)^{3/2}} \cdot e^{-\frac{1}{2}\left[x_1^2+x_2^2-2\rho x_1x_2\right]}\, dx_2\, dx_1$$

(ii) For any test ψ which is symmetric in x_2 and x_2':

$$\frac{\partial}{\partial\gamma} E_{\gamma\underline{\eta}}\, \psi \Big|_{\gamma=0} = K_\psi \sum_{i=1}^{2} \eta_i\,(1-\rho) \tag{3.4}$$

where

$$K_\psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_1\,\psi(x_1,x_2)}{2\pi\,\left(1-\rho^2\right)^{3/2}}\, e^{-\frac{1}{2}\left[x_1^2+x_2^2-2\rho x_1x_2\right]} dx_2dx_1 \tag{3.5}$$

Proof: The proof of (i) follows by Lehmann (1986) and (ii) follows from the symmetry of ψ and the symmetry of the distribution of X_1 and X_2 under $\gamma = 0$.

Then it follows by (3.2), (3.4) and (3.5) that for any symmetric test ψ:

$$L_p(\psi,\psi_U)=\frac{K_\psi}{\phi(Z_{1-\alpha})}\inf_{\eta_1^2+\eta_2^2=1}\frac{\sqrt{1-\rho^2-2\rho\eta_1\eta_2+2\rho^3\eta_1\eta_2}}{\left[1-2\rho\eta_1\eta_2\right]}\sum_{i=1}^{2}\eta_i(1-\rho) \quad (3.6)$$

Thus comparing any 2 symmetric tests via local power is equivalent to comparing their corresponding K_ψ's. Therefore we will find the K_ψ's for the tests ψ_F, ψ_L, ψ_S, ψ_N, ψ_T, ψ_{max} and for the test:

$$\psi_0 = \begin{cases} 1 & \text{if } x_1^{+2} + x_2^{+2} > d \\ 0 & \text{otherwise}. \end{cases} \quad (3.7)$$

To compare two tests ψ_1 and ψ_2 via local power, i. e., via (3.6) we say that ψ_1 is better than ψ_2 if $L_P(\psi_1, \psi_U) > L_P(\psi_2, \psi_U)$ or equivaleutly if $K_{\psi_1} > K_{\psi_2}$. This indicates that the power of ψ_1 is greater than the power of ψ_2 for some values in a neighbourhood of the origin.

Computations and conculsions

Notice that all of the tests under study reject H_0 if $h(X_1, X_2) > c$ for different functions h where

$$P_{H_0}[h(X_1, X_2) > c] = 1 - \alpha \quad (3.8)$$

Table 1 gives the functions h(. , .) for the tests under study . To find K_ψ we have first to find c in (3.8) and then we put the value of c in K_ψ in (3.5) and then evaluate K_ψ. We used IMSL to find the values of the integrals .

Although we calculated K_ψ for α = .01, .025 and .05 and for ρ = -.9 , -.8, -.7, . . ., .8, .9 we report only the results for some representative values of ρ, namely ρ = -08, -.4, 0, 04, .8 to save space. The results for the other values of ρ are similar .

The results of the computations are summarized in Tables 2, 3, . . . 7. In Tables 2, 3 and 4 we give the value of c in (3.8) for each test under study and Tables 5,6 and 7 give the value of the K_ψ's for such tests.

The tests are compared in terms of the local power. We conclude from Tables 5,6 and 7 that K_ψ increases as α increases for fixed ρ for all tests under study . Also we see that Ψ'_N has the highest local power in case $\rho \le 0$ and has the lowest local power in case $\rho > 0$. On the other hand, we see that ψ_0 has low local power for $\rho \le 0$ and has high local power for $\rho > 0$. Finally, we see that ψ_L behaves reasonably well in all cases with regard to local power, while ψ_S is better than ψ_T, ψ_{max} and ψ_F in all cases.

Table 1

Test	$h(X_1, X_2)$
ψ_F	$-2 \sum_{i=1}^{2} \ln\left[1 - \Phi(X_i)\right]$
ψ_L	$- \sum_{i=1}^{2} \ln\left[\frac{1-\Phi(X_i)}{\Phi(X_i)}\right]$
ψ_S	$- \sum_{i=1}^{2} \left[1 - \Phi(X_i)\right]$
ψ_N	$\sum_{i=1}^{2} X_i$
ψ_T	$- \min_{1 \le i \le 2} \left[1 - \Phi(X_i)\right]$
ψ_{max}	$- \max_{1 \le i \le 2} \left[1 - \Phi(X_i)\right]$
ψ_0	$\sum_{i=1}^{2} X_i^{+2}$

Table 2: $\alpha = 0.01$

ρ	C_F	C_L	C_N	C_S	C_T	C_{max}	C_{ψ_0}
-.8	10.6300	2.7583	1.4713	-0.5324	-0.0050	-0.3850	6.6183
-.4	11.5390	4.8145	2.5484	-0.2676	-0.8185	-0.1815	6.7178
0	13.2765	6.2716	3.2900	-0.1414	-0.0050	-0.1000	7.2863
.4	15.2501	7.5083	3.8927	-0.0724	-0.0052	-0.0520	8.3974
.8	17.3960	8.6593	4.4139	-0.0333	-0.0061	-0.0231	9.9528

Table 3: $\alpha = 0.025$

ρ	C_F	C_L	C_N	C_S	C_T	C_{max}	C_{ψ_0}
-.8	8.8949	2.2764	1.2396	-0.6103	-0.0125	-0.3311	5.0230
-.4	9.8367	3.9537	2.0407	-0.3622	-0.7527	-0.2473	5.1262
0	11.1434	5.1176	2.7718	-0.2236	-0.0126	-0.1581	5.53618
.4	12.5533	6.0768	3.2796	-0.1343	-0.0132	-0.0958	6.1948
.8	14.0105	6.9248	3.7188	-0.0738	-0.0157	-0.0504	7.1303

Table 4: $\alpha = 0.05$

ρ	C_F	C_L	C_N	C_S	C_T	C_{max}	C_{ψ_0}
-.8	7.6174	1.8839	1.0403	-0.6796	-0.0250	-0.4352	3.8419
-.4	8.5092	3.2561	1.8018	-0.4566	-0.6862	-0.3138	3.9521
0	9.4878	4.1914	2.3262	-0.3162	-0.0253	-0.2236	4.2306
.4	10.4713	4.9442	2.7524	-0.2138	-0.0269	-0.1516	4.5895
.8	11.4770	5.5922	3.1209	-0.1345	-0.0325	-0.0906	5.0874

Table 5: $\alpha = 0.01$

ρ	K_F	K_L	K_N	K_S	K_T	K_{max}	K_{ψ_0}
-.8	0.0088	0.0222	0.0253	0.0218	0.0080	0.0191	0.0082
-.4	0.0135	0.0169	0.0223	0.0165	0.0155	0.0155	0.0109
0	0.0179	0.0186	0.0188	0.0182	0.0144	0.0175	0.0161
.4	0.0265	0.0265	0.0146	0.0260	0.0237	0.0255	0.0260
.8	0.0696	0.0697	0.0084	0.0699	0.0684	0.0692	0.0702

Table 6: $\alpha = 0.025$

ρ	K_F	K_L	K_N	K_S	K_T	K_{max}	K_{ψ_0}
-.8	0.0203	0.0496	0.0554	0.0476	0.0180	0.0404	0.0181
-.4	0.0303	0.0374	0.0489	0.0361	0.0335	0.0335	0.0247
0	0.0392	0.0410	0.0413	0.0398	0.0321	0.0382	0.0360
.4	0.0577	0.0580	0.0320	0.0571	0.0522	0.0557	0.0568
.8	0.1541	0.1541	0.0185	0.1533	0.1502	0.1516	0.1538

Table 7: $\alpha = 0.05$

ρ	K_F	K_L	K_N	K_S	K_T	K_{max}	K_{ψ_0}
-.8	0.0387	0.0806	0.0978	0.0839	0.0325	0.0683	0.0326
-.4	0.0549	0.0664	0.0863	0.0636	0.0581	0.0581	0.0449
0	0.0693	0.0724	0.0729	0.0704	0.0516	0.0668	0.0644
.4	0.1018	0.1025	0.0565	0.1008	0.0928	0.0979	0.0996
.8	0.2716	0.2716	0.0326	0.2707	0.2654	0.2675	0.2714

REFERENCES

Birnbaum. A. (1954). "Combining Independent Tests of Significance". JASA, 49, pp. 559-574.

Brown, L., Cohen, A., and Straderman, W. T. (1976). "A Complete Class Theorem for Strict Monotone Likelihood Ratio with Applications", Ann. Math. Statist., Vol. 4, pp. 712-722.

Koziol, J. A., Perlmann, M. D., and Rasmussen, U. A. (1988). "Combining Independent F Tests". Technical Report, No. 129, Department of Statistics, University of Washington.

Lehmann, E. L. (1986). "Testing Statistical Hypotheses, 2nd ed., John Wiley and Sons Inc., New York.

Marden, J. I., (1982). "Combining Independent Noncentral Chi-Squared or F Tests", Ann. Statist., Vol. 10, pp. 266-277.

Marden, J. I., (1985). "Combining Independent One-Sided Non-central t or Normal Mean Tests", Ann. Statist., Vol. 13, pp. 1535-1553.

Marden, J. I. and Perlman, M. D. (1982). "The Minimal Complete Class of Procedures for Combining Independent Noncentral F-Tests, "Decision Theory and Related Topics III, Vol. 2, pp. 139-181.

Oosterhoff, J. (1969). "Combination of One-Sided Statistical Tests", Mathematisch Centrum, Amsterdam.