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The Statistical Curvature of Seemingly Unrelated Unrestricted Regression Equations

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We study the finite sample properties of an asymptotically efficient estimator for coefficients of seemingly unrelated unrestricted regression (*SUUR*) equations. *Zellner*(1963) derived the exact probability density function of the *SUUR* estimator. The new form of the probability density function, the r th moment, the characteristic function and the asymptotic expansion distribution of *SUUR* equations up to order n^{-r} are derived by Youssef, A.(1996). Youssef et. al.(1995) studied the statistical curvature for *SUUR* estimator up to order n^{-1}. In this paper, we study the statistical curvature for *SUUR* estimator when the asymptotic expansion distribution of SUUR equations is of order n^{-2} , because it is hard to deal with order higher than two, to examine how close the density function of *Zellner's* estimator is to the normal distribution.

Key words: Seemingly unrelated unrestricted regression equation, statistical curvature.

1. The Model and Some Results For SUUR Estimators

The basic model that we are concerned with consists of two multiple regression equations as :

$$y_i = x_i \beta_i + u_i, \tag{1}$$

where y_i is a Tx1 vector of observations on the i th dependent variable, x_i is a T x k_i matrix, each column of which comprises the T observations on a regressor in the i th equation of the model, β_i is a k_i x 1 vector of coefficients in the i th equation, u_i is a T x 1 disturbance vector and $i = 1,2$.

By writing (1) as :

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

the model may be expressed in compact form as

$$Y = X\beta + U \tag{2}$$

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where Y is $2T \times 1$, X is $2T \times K$, β is $K \times 1$, U is $2T \times 1$, and $K = k_1 + k_2$. Treating each of the two equations as classical linear regression relationships, we make the conventional assumptions about the regressors:

$$x_i \text{ is fixed with rank } \Re(x_i) = k_i \,, \tag{3}$$

and

$$\lim_{T \to \infty} \frac{1}{T} x_i' x_j = Q_{ij}, \tag{4}$$

further, we assume that the elements of the disturbance vector u_i, follow a multivariate probability distribution with

$$E(u_i) = 0, \tag{5}$$

$$E(u_i u_j') = \begin{cases} \sigma_{ii} I & \text{i} = \text{j}, \\ \sigma_{ij} I & \text{i} \neq \text{j}, \end{cases} \tag{6}$$

where Q_{ii} (i= 1,2) is non-singular with fixed and finite elements, and σ_{ij} represents the covariance between the disturbances of the i th and j th equations for each observation in the sample. Writing (5), and (6) more compactly, we have

$$E(U) = 0, \tag{7}$$

and

$$\begin{aligned} E(UU') &= \begin{pmatrix} \sigma_{11} I & \sigma_{12} I \\ \sigma_{21} I & \sigma_{22} I \end{pmatrix} \\ &= (\Sigma \otimes I) \\ &= \Omega . \end{aligned} \tag{8}$$

Where $\otimes$ denotes the usual Kronecker product, so that Ω is $2T$ x $2T$, and $\Sigma = [\sigma_{ij}]$ is a 2 x 2 positive definite symmetric matrix. The definiteness of the Σ precludes the possibility of any linear dependencies among the contemporaneous disturbances in the two equations of the model.

Zellner(1963) derived the seemingly unrelated unrestricted residuals, because the restrictions on the coefficients of the seemingly unrelated regression equation (SURE) model which distinguishes it from the multivariate regression model are ignored when obtaining the residuals to be used for constructing the s_{ij}'s , as follows:

$$\tilde{\beta}_{SU} = \left(X' \left(\tilde{S}^{-1} \otimes I_T \right) X \right)^{-1} X' \left(\tilde{S}^{-1} \otimes I_T \right) Y , \tag{9}$$

where

$$\tilde{s}_{ij} = \frac{1}{T}\tilde{u}_i{}'\tilde{u}_j$$
$$= \frac{1}{T}y_i{}'\overline{M}_Z y_j, \tag{10}$$

$$\tilde{u}_i = \overline{M}_Z y_i, \tag{11}$$

and

$$\overline{M}_Z = I_T - Z(Z'Z)^{-1}Z', \tag{12}$$

where $i, j = 1,2$. Because x_i is a sub-matrix of Z, we have

$$x_i = Z J_i, \tag{13}$$

where J_i is a selection matrix of order $K \times K_i$, $i=1,2$ with elements taking the value zero or one, as appropriate. It is easy to see that

$$\overline{M}_Z x_i = 0. \tag{14}$$

Using this result, we have

$$y_i{}'\overline{M}_Z y_j = u_i{}'\overline{M}_Z u_j. \tag{15}$$

So that

$$E(\tilde{s}_{ij}) = \frac{1}{T}\sigma_{ij} Tr\overline{M}_Z$$
$$= (1 - \frac{K}{T})\sigma_{ij}. \tag{16}$$

From which it follows that an unbiased estimator of σ_{ij} is obtained by replacing $1/T$ by $1/(T-K)$ in (10) .Assuming that the regressors in the two equations are orthogonal, we get

$$\tilde{\beta}_{(1)SU} = (x_1{}'x_1)^{-1}x_1{}'y_1 - \frac{\tilde{s}_{12}}{\tilde{s}_{22}}(x_1{}'x_1)^{-1}x_1{}'y_2, \tag{17}$$

and

$$\tilde{\beta}_{(2)SU} = (x_2{}'x_2)^{-1}x_2{}'y_2 - \frac{\tilde{s}_{12}}{\tilde{s}_{11}}(x_2{}'x_2)^{-1}x_2{}'y_1. \tag{18}$$

In the sequel, we shall confine our study to the *SUUR* estimator of β_1. The results that will be obtained for $\tilde{\beta}_{(1)SU}$ can be exactly derived for $\tilde{\beta}_{(2)SU}$ and thus will not be presented. Zellner(1963) derived the sampling distribution of the *SUUR* estimator $\tilde{\beta}_{(1)SU}$ as follows:

$$f(w)=\frac{1}{\sqrt{2\Pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\frac{\Gamma\left(t+\frac{n+1}{2}\right)}{\Gamma\left(t+\frac{n+2}{2}\right)}\left[\frac{w^2}{2}\right]^t,$$

$$-\infty\leq w\leq+\infty,\qquad (19)$$

where

$$w=\frac{\tilde{\beta}_{(1)SU}-\beta_{(1)}}{\sqrt{\sigma_{11}(1-\rho^2)}}.\qquad (20)$$

In the non-orthogonal case, the sampling distribution of the *SUUR* estimator will be the same as (19) except that w in (20) will be multiplied by $(x_1'x_1)^{-1/2}$, (see Srivastava and Giles,1987).

Youssef et. al.(1995) represent the probability density function of w in terms of the confluent hypergeometric function as follows:

$$f(w)=\frac{1}{\sqrt{2\Pi}}\frac{\left(\Gamma\frac{n+1}{2}\right)^2}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}e^{-\frac{w^2}{2}}\,{}_1F_1\left(\frac{1}{2};\frac{n+2}{2};\frac{w^2}{2}\right).\qquad (21)$$

Youssef (1996) derived the asymptotic expansion of the density function of *SUUR* estimator up to any order in terms of Pochhammer symbol, $(.)_n$, and the Hermite Polynomial, $H_{e_n}(x)$, as follows:

$$f(w)=\sqrt{\frac{1}{2\Pi}}e^{-\frac{w^2}{2}}\sum_{r=0}^{\infty}\frac{(-1)^r}{2^r r!}\frac{\left(\frac{1}{2}\right)_r}{\left(-\frac{n-2}{2}\right)_r}H_{e_{2r}}(w),\qquad (22)$$

where

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\qquad n=0,1,2,\ldots,$$

and

$$H_{e_n}(x)=n!\sum_{m=0}^{\frac{n}{2}}\frac{(-1)^m(x)^{n-2m}}{2^m m!(n-2m)!}.$$

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2. The statistical Curvature of SUUR Estimator

Zellner(1963) studied the closeness of the density function of the SUUR estimator to the normal distribution. We know that the normal distribution is a member of the exponential family and according to *Efron* (1975, 1978), the statistical curvature of an exponential family is zero. Youssef (1996) studied the density function only to the first order. Now, we are going to study the statistical curvature of the density function to the second order, because it is hard to deal with the exact probability density function of *SUUR* estimator, to see how close this density is to the exponential family.

From (22), we have the following probability density function of w up to order n^{-2}.

$$f(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}\left[1 + a_1 + a_2 w^2 + a_3 w^4\right], \tag{23}$$

where

$$a_1 = -\frac{4n-25}{8(n-2)(n-4)}, \tag{24}$$

$$a_2 = \frac{4n-34}{8(n-2)(n-4)}, \tag{24}$$

and

$$a_3 = \frac{3}{8(n-2)(n-4)}. \tag{25}$$

The first and second derivatives of the log of both sides of (23) are:

$$h' = \frac{d}{dw}\log f(w) = -w + \frac{(2a_2 w + 4a_3 w^3)}{\left[1 + a_1 + a_2 w^2 + a_3 w^4\right]}, \tag{26}$$

and

$$h'' = \frac{d^2}{dw^2}\log f(w) = -1 + \frac{(2a_2 + 12a_3 w^2)}{\left[1 + a_1 + a_2 w^2 + a_3 w^4\right]} - \frac{(2a_2 w + 4a_3 w^3)^2}{\left[1 + a_1 + a_2 w^2 + a_3 w^4\right]^2}. \tag{27}$$

To calculate the statistical curvature[2], we need to find $E(h'^2)$, $E(h''^2)$ and $E(h'h'')$. From (26) and (27), we find that

[2] See Efron, B. (1975).

$$E\left(h'^2\right)=E(w^2)+E\left\{\frac{2a_2w+4a_3w^3}{[1+a_1+a_2w^2+a_3w^4]}\right\}^2-2E\left\{\frac{2a_2w^2+4a_3w^4}{[1+a_1+a_2w^2+a_3w^4]}\right\},\tag{28}$$

$$\begin{aligned}E\left(h''^2\right)=1&+E\left\{\frac{2a_2+12a_3w^2}{[1+a_1+a_2w^2+a_3w^4]}\right\}^2+E\left\{\frac{2a_2w+4a_3w^3}{[1+a_1+a_2w^2+a_3w^4]}\right\}^4\\&-2E\left\{\frac{(2a_2+12a_3w^2)(2a_2w+4a_3w^3)^2}{[1+a_1+a_2w^2+a_3w^4]^3}\right\}-2E\left\{\frac{2a_2+12a_3w^2}{[1+a_1+a_2w^2+a_3w^4]}\right\}\\&+2E\left\{\frac{2a_2w+4a_3w^3}{[1+a_1+a_2w^2+a_3w^4]}\right\}^2,\end{aligned}\tag{29}$$

and

$$E(h'h'')=0.\tag{30}$$

So, we need the following expectations:

$$E(w^2)=1+\frac{1}{n}+\frac{2}{n^2}+\frac{4}{n^3}+\frac{8}{n^4},\tag{31}$$

$$E\left(\frac{w^{2r}}{\left[1+a_1+a_2w^2+a_3w^4\right]}\right)=2^r\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)},\qquad r=0,1,2,\ldots,\tag{32}$$

$$E\left(\frac{w^{2r}}{\left[1+a_1+a_2w^2+a_3w^4\right]^2}\right)=\sum_{i=0,\,j=0,\,k=0}^{\infty,\,i,\,j}(-1)^i2^{-(r+j+k)}\binom{i}{j}\binom{j}{k}\frac{2(r+j+k)!}{(r+j+k)!}a_1^{i-j}a_2^{j-k}a_3^k,\quad r=1,2,\ldots,\tag{33}$$

$$E\left(\frac{w^{2r}}{\left[1+a_1+a_2w^2+a_3w^4\right]^3}\right)=\sum_{i=1,\,j=0,\,k=0}^{\infty,\,i-1,\,j}i(-1)^{i-1}2^{-(r+j+k)}\binom{i-1}{j}\binom{j}{k}\frac{2(r+j+k)!}{(r+j+k)!}a_1^{i-j-1}a_2^{j-k}a_3^k,\quad r=1,2,\ldots,\tag{34}$$

and

$$E\left(\frac{w^{2r}}{\left[1+a_1+a_2w^2+a_3w^4\right]^4}\right)=\sum_{i=2,\,j=0,\,k=0}^{\infty,\,i-2,\,j} i(i-1)\,(-1)^{i-2}\,2^{-(r+j+k+1)}\binom{i-2}{j}\binom{j}{k}\frac{2(r+j+k)!}{(r+j+k)!}a_1^{i-j-2}a_2^{j-k}a_3^{k},$$

$$r=1,2,\ldots$$

(35)

Using the above expectations in (28) and (29), we get

$$E\left(h'^2\right)=\left(1-\frac{1}{n}-\frac{1}{n^2}-\frac{1}{n^3}-\frac{1}{2n^4}\right),$$

(36)

and

$$E\left(h''^2\right)=\left(1-\frac{2}{n}-\frac{1}{n^2}+\frac{16}{n^4}\right).$$

(37)

Then, we can calculate the statistical curvature to the order o (n^{-4}), as defined by Efron' s, as follows:

$$\gamma^2=\frac{E(h''^2)}{E(h'^2)^2}-1$$

$$=\frac{\left(1-\frac{2}{n}-\frac{1}{n^2}+\frac{16}{n^4}\right)}{\left(1-\frac{1}{n}-\frac{1}{n^2}-\frac{1}{n^3}-\frac{1}{2n^4}\right)^2}-1. \qquad (38)$$

We now calculate the value of the statistical curvature as given in (38) for various values of n.

Table 1: The Statistical Curvature For SUUR Estimator

n	Curvature Youssef (1996)	Curvature Using Eqn.(38)
5	0.020400	0.0377432
10	0.002290	0.0017310
15	0.000648	0.0003159
20	0.000551	0.0000964
25	0.000141	0.0000397
30	0.000078	0.0000395
40	0.000032	0.0000056
50	0.000016	0.0000022
∞	0.0	0.0

A comparison between the statistical curvature of the density function, to the first and second order, of the SUUR estimator are given in table (1). From table (1), we see that the curvature tends to zero when the sample size increases and is close to zero when $n = 15$, and the density function of *SUUR* estimator to the order n^{-2} is going faster to zero than that of order n^{-1} which was derived by Youssef (1996). Then our density function of the *SUUR* estimator tends to the normal distribution when the sample size is large.

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