THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CARIO, UNIV., VOL. (42) NO.2, 1998

Moments Of Order Statistics From Right Truncated Log-Logistic Distribution

S. E. Abu-Youssef
Department of Statistics and Operations Research
King Saud University
P. O. Box 2455, Riyadh 11451
Saudi Arabia

Abstract

The moments of order statistics from a right truncated log-logistic distribution are given based on hypergeometeric function. Some new recurrence relations between the moments of $X_{r:n}$ are obtained.

Key words:

Order statistics, moments of order statistics, right truncated log-logistic distribution, hypergeometeric function and recurrence relations.

1 Introduction

Masoom and Khan (1987) obtained recurrence relations for negative and fractional moments of single order statistics and product and quotient moments of two order statistics drawn from log-logistic distribution

Balakrishnan et al (1987) have established several recurrence relations for both single and product moments of order statistics from right truncated log-logistic distribution. They have generalized these results to the doubly truncated log-logistic distribution.

Khurana and Jha (1991) have derived moments of order statistics from a doubly truncated Pareto distribution in terms of hypergeometric function and, they have obtined fifteen recurrence relations between the moments of $X_{r:n}$. Mohie El-Din et al (1995) have obtained moment generating function of order statistics from a doubly truncated exponential distribution in terms of hypergeometric function. They have derived some recurrence relations between these moment generating functions. Mohie El-Din et al (1996) have obtained moments of order statistics from a doubly truncated power function distribution base in hypergeometric function. They have used the properties of hypergeometric function and its contignuous functions to obtain fifteen recurrence relations between the moments of $X_{r:n}$.

In this paper we derive the moments of order statistics from a right truncated log-logistic distribution in terms of hypergeometric function. Also, we obtain some new recurrence relations between moments of order statistics for a right truncated log-logistic distribution by using the properties of hypergeometric function, and its contignuous functions.

If $X_1, X_2, ..., X_n$ a random sample of size n drown from a right truncated log-logistic distribution. Then the p.d.f of right truncated log-logistic distribution is given by

$$f(x) = \frac{\beta x^{\beta-1}}{P(1+x^{\beta})^2}, \qquad 0 \le x \le P_1, \beta > 0. \tag{1.1}$$

where

$$P_1^{\beta} = \frac{P}{1-P}. \tag{1.2}$$

The commulative distribution function (c.d.f) is

$$F(x) = \frac{x^{\beta}}{P(1+x^{\beta})}, \beta > 0 \tag{1.3}$$

The hypergeometric function defined as follows :

$$F(a,b,c,z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}. \tag{1.4}$$

where

$$(\alpha)_m = \alpha(\alpha+1)(\alpha+2).......(\alpha+m-1) \ , m \geq 1 \, , \alpha \neq 0.$$

Note that the hypergeometric function (1.4) is known to be convergent for $|z| < 1$ so long as $c > 0$ [Rainville (1960)].

2 Moments of order statistics

Let $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq ... \leq X_{n:n}$ be order statistics from a random sample of size n with p.d.f $f(x)$ and c.d.f $F(x)$. The p.d.f of $X_{r:n}$ and rth moment $\mu_{r:n}^{(k)}$ are

$$f(x_{r:n}) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \qquad 1 \leq r \leq n. \tag{2.1}$$

and

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k) = \int_0^{P_1} x f(x_{r:n}) dx. \tag{2.2}$$

From (1.3), we have

$$x^k = P^{\frac{k}{\beta}}[F(x)]^{\frac{k}{\beta}}[1-F(x)]^{-\frac{k}{\beta}}. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we have

$$\mu_{r:n}^{(k)} = \frac{n!P^{\frac{k}{\beta}}}{(r-1)!(n-r)!}\int_0^{P_1}[F(x)]^{\frac{k}{\beta}+r-1}[1-PF(x)]^{-\frac{k}{\beta}}[1-F(x)]^{n-r}dF(x). \tag{2.4}$$

Putting $u = F(x)$ in (2.4), we have

$$\mu_{r:n}^{(k)} = \frac{n!P^{\frac{k}{\beta}}}{(r-1)!}\frac{(r+\frac{k}{\beta}-1)!}{(n+\frac{k}{\beta})!}\sum_{m=0}^{\infty}\frac{(\frac{k}{\beta})_m(r+\frac{k}{\beta})_m}{(n+\frac{k}{\beta}+1)_m}\frac{z^m}{m!}. \tag{2.5}$$

From (1.4) and (2.5), we get

$$\mu_{r:n}^{(k)} = \frac{n!P^{\frac{k}{\beta}}}{(r-1)!}\frac{(r+\frac{k}{\beta}-1)!}{(n+\frac{k}{\beta})!}F(\frac{k}{\beta}, r+\frac{k}{\beta}, n+\frac{k}{\beta}+1, P), |P| < 1. \tag{2.6}$$

Putting $P = 1$ in (2.6), the moments of order statistics for untruncated log-logistic distribution is given by

$$\mu_{r:n}^{(k)} = \frac{n!}{(r-1)!} \frac{(r+\frac{k}{\beta}-1)!}{(n+\frac{k}{\beta})!} F(\frac{k}{\beta}, r+\frac{k}{\beta}, n+\frac{k}{\beta}+1, 1), \qquad (2.7)$$

For $r = 1$, $n = 1$ and using the property that [Rainville 1960]

$$F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

The equation (2.7) becomes

$$\mu_{1:1}^{(k)} = \Gamma\left(1+\frac{k}{\beta}\right)\Gamma\left(1-\frac{k}{\beta}\right), \qquad k < \beta. \qquad (2.8)$$

[See Khan, A. H. and Masoom, ALI, M. (1987) theorem (2-2) p.105].

3 Recurrence relations

In this section we obtain fifteen recurrence relations between moments of order statistics for a right truncated log-logistic distribution by using fifteen recurrence relation between $F(a, b, c, z)$ and its contignuous functions [Rainville (1960)].

It is well known that only five of these fifteen relation are independent and the other relations can be deduced from them. This reflected in our new results.

Our new recurrence relations are represented as follow

Relation 1

$$\mu_{r:n+1}^{(k)} + P_1^{-\beta}\mu_{r:n}^{(k)} = \frac{\beta r + k - \beta}{(r-1)P}\mu_{r-1:n}^{(k)}, 1 < r \leq n. \qquad (3.1)$$

Proof:

Let

$$\begin{aligned} \alpha &= \frac{k}{\beta} \\ b &= r+\frac{k}{\beta} \\ c &= n+\frac{k}{\beta}+1 \\ \text{and} z &= P. \end{aligned} \tag{3.2}$$

Substituting from (3.2) in the relation

$$(1-z)F(a,b,c,z) = F(a,b-1,c,z) - \frac{(c-a)z}{c}F(a,b,c+1,z)$$

we get

$$\mu_{r:n+1}^{(k)} + \frac{1-P}{P}\mu_{r:n}^{(k)} = \frac{r+\frac{k}{\beta}+1}{(r-1)P}\mu_{r-1:n}^{(k)}.$$

Using (1.2) in a bove, we obtain the relation 1. Note that, from (3.1), it is clear that

$$\mu_{r+1:n+1}^{(k)} + P_1^{-\beta}\mu_{r+1:n}^{(k)} = \frac{1}{P}(1+\frac{k}{rP})\mu_{r:n}^{(k)}. \tag{3.3}$$

[See Balakrishnan et al (1987) relation (2-2) p. 253].

Relation 2

$$\mu_{r+1:n}^{(k)} - \mu_{r:n}^{(k)} = \frac{(k)}{r\beta P}\frac{n\beta+k+\beta}{r\beta+k}\mu_{r:n}^{(k+\beta)}. \tag{3.4}$$

Proof:

Using a, b, c and z as above in relation

$$(a-b)F(a,b,c,z) = aF(a+1,b,c,z) - bF(a,b+1,c,z)$$

we obtain

$$-\mu_{r:n}^{(k)} = \frac{k}{r\beta P}\frac{n+\frac{k}{\beta}+1}{r+\frac{k}{\beta}}\mu_{r:n}^{(k+\beta)}.$$

The relation (3.4) is proved.

Relation 3

$$\mu^{(k)}_{r:n-1} - \mu^{(k)}_{r:n} = \frac{(k)}{n\beta P}\frac{n\beta + k + \beta}{r\beta + k}\mu^{(k+\beta)}_{r:n}. \tag{3.5}$$

Proof:

We use the same steps in the a bove in the contignuous function relation

$$(a - c + 1)F(a, b, c, z) = aF(a + 1, b, c, z) - (c - 1)F(a, b, c - 1, z)$$

we obtain

$$-n\mu^{(k)}_{r:n} = \frac{k}{\beta P}\frac{n + \frac{k}{\beta} + 1}{r + \frac{k}{\beta}}\mu^{k+\beta}_{r:n} - n\mu^{(k)}_{r:n-1}$$

The relation (3.5) is proved.

Relation 4

$$(n+1-r)P\mu^{(k)}_{r:n+1} + [(n+1-r)P - \frac{k}{\beta}]\mu^{(k)}_{r:n} = \frac{k(n\beta + k + \beta}{\beta(\beta r + k)}P_1^{-\beta}\mu^{(k+\beta)}_{r:n}. \tag{3.6}$$

Proof:

In the contignuous function

$$[a+(b-c)z]F(a, b, c, z) = a(1-z)F(a+1, b, c, z) - \frac{(c-a)(c-b)z}{c}F(a, b, c+1, z).$$

Use the same steps in relation 1, we obtian

$$[\frac{k}{\beta} - (n - r + 1)P]\mu^{(k)}_{r:n} = \frac{k}{\beta}P_1^{-\beta}\frac{n\beta + k + \beta}{r\beta + k}\mu^{(k+\beta)}_{r:n} + (n + 1 - r)P\mu^{(k)}_{r:n+1}$$

The relation 4 is obtained.

Relation 5

$$P_1^{-\beta}\mu^{(k)}_{r:n} - \frac{n + 1 - r}{n + 1}\mu^{(k)}_{r:n+1} = \frac{k + \beta r - \beta}{k + n\beta}\mu^{(k-\beta)}_{r:n}. \tag{3.7}$$

Proof:

In the contignuous functions

$$(1-z)F(a,b,c,z) = F(a-1,b,c,z) - \frac{(c-b)z}{c}F(a,b,c+1,z)$$

we use the same steps as above to obtain (3.7).

Using the remaining contignuous functions relations of $F(a,b,c,z)$ [Rainville (1960)] to obtain the following recurrence relations between moments of order statistics from a right truncated log-logistic distribution.

$$[n+\frac{k}{\beta}-(2n-r+1)P]\mu_{r:n}^{(k)}+(n+1-r)\mu_{r:n+1}^{(k)} = n(1-P)\mu_{r:n-1}^{(k)}. \quad (3.8)$$

$$\frac{(n+1)(\beta r+k-\beta)P}{n\beta+k}\mu_{r:n}^{(k-\beta)} - \frac{k}{\beta}P_1^{-\beta}\frac{n\beta+k+\beta}{\beta r+k}\mu_{r:n}^{(k+\beta)} = $$
$$= \left(n+1-\frac{k}{\beta}-rP\right)\mu_{r:n}^{(k)}. \quad (3.9)$$

$$\frac{(n+1-r)}{(r-1)P}\mu_{r-1:n}^{(k)} + \frac{\beta r P_1^{-\beta}}{\beta r+k-\beta}\mu_{r:n}^{(k)} = \frac{(n+1)\beta}{n\beta+k}\mu_{r:n}^{(k-\beta)}. \quad (3.10)$$

$$[\frac{k}{\beta}-(n+1-r)P]\mu_{r:n}^{(k)}+(n+1-r)P\mu_{r:n+1}^{(k)} = \frac{k}{\beta}P_1^{-\beta}\frac{n\beta+k+\beta}{\beta r+k}\mu_{r:n}^{(k)}. \quad (3.11)$$

$$n(1-P)\mu_{r:n}^{(k)} - [(n-r)P-\frac{k}{\beta}+1]\mu_{r:n}^{(k)} = \frac{(n+1)P(\beta r+k-\beta)}{n\beta+k}\mu_{r:n}^{(k-\beta)}. \quad (3.12)$$

$$(n-r)\mu_{r:n}^{(k)} + r\mu_{r+1:n}^{(k)} = n\mu_{r:n-1}^{(k)}. \quad (3.13)$$

$$[r+\frac{k}{\beta}-(n+1)P]\mu_{r:n}^{(k)}+(n+1-r)P\mu_{r:n+1}^{(k)} = r(1-P)\mu_{r+1:n}^{(k)}. \quad (3.14)$$

$$\frac{(n+1-r)(\beta r+k-\beta)}{(r-1)\beta}\mu_{r-1:n}^{(k)} - r(1-P)\mu_{r+1:n}^{(k)} = \left(n+1-rP-2r-\frac{(k)}{\beta}\right)\mu_{r:n}^{(k)}. \quad (3.15)$$

$$\left[n+1-r-\frac{(k)}{\beta}\right]\mu_{r:n}^{(k)} + r(1-P)\mu_{r+1:n}^{(k)} = \frac{(n+1)P(\beta r+k-\beta)}{n\beta+k}\mu_{r:n}^{(k-\beta)}. \quad (3.16)$$

$$\left[n+1-, -\frac{(k)}{\beta}\right]\mu_{r:n}^{(k)} + \frac{k}{\beta}P_1^{-\beta}\frac{n\beta+k+\beta}{\beta r+k}\mu_{r:n}^{(k+\beta)} = \frac{(n+1-r)(\beta r+k-\beta)}{\beta(r-1)}\mu_{r-1:n}^{(k)}. \quad (3.17)$$

4 Recurrence relations for untruncated case

Recurrence relations between moments of order statistics for untruncated log-logistic distribution are directly obtained from (3.1), (3.4), (3.5), ... and (3.17) by putting $P = 1$. The results in this case are

$$\mu_{r:n+1}^{(k)} = \frac{\beta r + k - \beta}{(r-1)\beta} \mu_{r-1:n}^{(k)}. \tag{4.1}$$

Note that, equation (4.1) leads to the following

$$\mu_{r+1:n+1}^{(k)} = \left(1 + \frac{k}{rP}\right) \mu_{r:n}^{(k)}$$

and

$$\mu_{r:n}^{(k)} = \left(1 + \frac{k}{(r-1)P}\right) \mu_{r-1:n-1}^{(k)}$$

[See Khan, A. H. and Masoom ALI, M. (1987), theorem (2-1) p. 104]

$$\mu_{r+1:n}^{(k)} - \mu_{r:n}^{(k)} = \frac{(k)}{r\beta} \frac{n\beta + k + \beta}{r\beta + k} \mu_{r:n}^{(k+\beta)}. \tag{4.2}$$

$$\mu_{r:n-1}^{(k)} - \mu_{r:n}^{(k)} = \frac{(k)}{n\beta} \frac{n\beta + k + \beta}{r\beta + k} \mu_{r:n}^{(k+\beta)}. \tag{4.3}$$

$$\mu_{r:n+1}^{(k)} = \left(\frac{k}{(n+1-r)\beta} - 1\right) \mu_{r:n}^{(k)}. \tag{4.4}$$

$$\frac{k + \beta r - \beta}{k + n\beta} \mu_{r:n}^{(k-\beta)} + \frac{n+1-r}{n+1} \mu_{r:n+1}^{(k)} = 0. \tag{4.5}$$

$$\frac{(n+1)(\beta r + k - \beta)}{n\beta + k} \mu_{r:n}^{(k-\beta)} = \left(n + 1 - \frac{k}{\beta} - r\right) \mu_{r:n}^{(k)}. \tag{4.6}$$

$$\frac{n+1-r}{r-1} \mu_{r-1:n}^{(k)} = \frac{(n+1)\beta}{n\beta + k} \mu_{r:n}^{k-\beta}. \tag{4.7}$$

$$\left(n + 1 - \frac{k}{\beta} - 3r\right) \mu_{r:n}^{(k)} = \frac{(n+1-r)(\beta r + k - \beta)}{(r-1)\beta} \mu_{r-1:n}^{(k)}. \tag{4.8}$$

$$(n - r)\mu_{r:n}^{(k)} + r\mu_{r+1:n}^{(k)} = n\mu_{r:n-1}^{(k)}. \tag{4.9}$$

$$\left(n+1-r-\frac{(k)}{\beta}\right)\mu_{r:n}^{(k)}=\frac{(n+1-r)(\beta r+k-\beta)}{(r-1)\beta}\mu_{r-1:n}^{(k)}. \tag{4.10}$$

$$\left(n+1-r-\frac{k}{\beta}\right)\mu_{r:n}^{(k)}=\frac{(n+1)(\beta r+k-\beta)}{n\beta+k}\mu_{r:n}^{(k-\beta)}. \tag{4.11}$$

Note that, the relations (4.1), (4.2), (4.3), (4.4) and (4.5) are independent and the remaining are deducible from them.

Remarks:

(1) Recurrence relations (3.6), (3.8), (3.11) and (3.14) in the truncated case lead to the recurrece relation (4.4) in the untruncated case.

(2) The recurrence relations (3.12) and (3.16) in the truncated case lead to the reurrence relation (4.10) in the untruncated case.

(3) Recurrence relation (3.13) in the truncated case and recurrence relation (4.9) are identitical, also it is valid for any aribitrary continuous distribution.

REFERENCES

(1) Balakrishnan, N. and Malik, H. J. (1987): Moments of order statistics from truncated log-logistic distribution. J. Statist. Plann. and Inference 17, 251-267.

(2) Khurana, A. P. and Jha, V. D. (1991): Recurrence relations between moments of order statistics from a doubly truncated Pareto distribution. Sankhya 53B, 11-16.

(3) Masoom, ALI, M. and Khan, A. H. (1987): On order statistics from the log-logistic distribution. J. Statist. Plann. and Inference 17, 103-105.

(4) Mohie El-Din, M. M., Mahmoud M. A. W. and Sultan K. S. (1995): On moments generation function of order statistics for a doubly truncated exponential distribution. Metron, L III, n. 1-2, 171-183.

(5) Mohie El-Din, M. M., Mahmoud. M. A. W. and Sultan K. S. (1996): On order statistics of doubly truncated power function distribution. Metron L IV n. 1-2, 83-93.

(6) Rainville, E. D. (1960): Special Functions. The Macmi. Company, New-York.