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Asymptotic Distribution of Spectral Density Estimate of Continuous Time Series on Crossed - Intervals

BY

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Abstract. Let $X(t)$ $(-\infty < t < \infty)$ be a zero mean, r vector-valued, continuous-time strictly stationary process with spectral density function $\hat{f}_{XX}^{(T)}(\lambda)$, $(-\infty < \lambda < \infty)$. The problem of asymptotic distribution of estimate $\hat{f}_{XX}^{(T)}(\lambda)$ of $f_{XX}(\lambda)$ on crossed intervals based on the values of $X(t)$ $(t = 0, 1, \dots, T)$ are considered where the periodograms are calculated, using data window. The first, seconed-order moments and cumulant of $\hat{f}_{XX}^{(T)}(\lambda)$ are derived. Further, some results in this area are presented. Also, the asymptotic distribution of $\hat{f}_{XX}^{(T)}(\lambda)$ on crossed intervals is derived. Our purpose, is to indicate the appearance of the Wishart distribution as an exact limiting distribution of $\hat{f}_{XX}^{(T)}(\lambda)$.

Keywords. Spectral estimate of continuous-time processes; Data window; Finite Fourier transform; Periodograms; Asymptotic normality; Wishart distribution.

1. Introduction

Let $X(t)$ $(-\infty < t < \infty)$ be a real-valued stationary process with mean zero, continuous autocovariance function $C_{XX}(u)$ $(-\infty < u < \infty)$ and spectral density function $f_{XX}(\lambda)$ $(-\infty < \lambda < \infty)$. Many authors as e.g. Brillinger [2], Dahlhaus [3], Ghazal and Farag [6], Zurbenko [9] have studied the asymptotic expressions of the first, second-order moments and cumulant of estimates of the spectral density on crossed intervals. Brillinger [2], analysis the first two moments and cumulant in two cases, the first by using the tapered data (data window), and the other where the untapered data is used, and in these papers one can find the first two moments and cumulant by using data window. Our work mainly based on the properties of data window (Brillinger [2]; Ghazal and Trush [5]; Ghazal and Farag [7]).

The statistical analysis of estimate $\hat{f}^{(T)}_{XX}(\lambda)$ of $f_{XX}(\lambda)$ on crossed intervals in discrete time processes is considered in Ghazal and Farag [6], and our paper is an extension of the latter in continuous time processes. The estimation of spectral density on crossed intervals mainly based on study the properties of the expanded finite Fourier transform in discrete or in continuous time processes (Ghazal, Hennawy and Farag [5]; Ghazal and Farag [7]). The asymptotic normality of the estimate of $f_{XX}(\lambda)$ is considered in Lii and Masry [8]; Brillinger ([1], [2]) while the asymptotic Wishart distribution of estimates of spectral density is established in the latter.

In this paper we study the statistical properties of spectral density estimate $\hat{f}^{(T)}_{XX}(\lambda)$ $(-\infty < \lambda < \infty)$ on crossed intervals using data wndow. In Section 2 we introduce the notation and some recent results which will be used later. Details can be found in Ghazal and Farag [7]. In Section 3 the asymptotic expressions of the first, second-order moments and cumulant of $\hat{f}^{(T)}_{XX}(\lambda)$ are given, and some results in this area are presented. Section 4 contains some useful results on the asymptotic distribution of $\hat{f}^{(T)}_{XX}(\lambda)$, where two limiting distributions are seen to appear in the case of estimate spectral density. Under one limiting process it tends to be χ^2 distribution and under it tends to be a Wishart distribution.

2. Preliminaries

Let $X(t)$ $(-\infty < t < \infty)$ be a strictly stationary r vector-valued continuous time series with mean zero. Let $C_{XX}(u)$ $(-\infty < u < \infty)$ be an autocovariance function and spectral density function $f_{XX}(\lambda)$ $(-\infty < \lambda < \infty)$ be an rxr matrix of second-order spectral densities which are given by

$$C_{XX}(u) = E\{X(t)\, X(t+u)\} = \int_{-\infty}^{\infty} f_{XX}(\lambda)\exp(iu\lambda)\,d\lambda\ , \qquad (2.1)$$

$$(-\infty < t,\ u < \infty)\ ,\ (-\infty < \lambda < \infty)$$

$$f_{XX}(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} c_{XX}(u) \exp(-iu\lambda)\, du \qquad (2.2)$$

$$(-\infty < \lambda < \infty)$$

respectively. Suppose

$$\int_{-\infty}^{\infty} |c_{XX}(u)|\, du < \infty, \qquad (2.3)$$

where $|c_{XX}(u)|$ denotes the matrix of absolute values.

Now, given a sample of observed values of $X(t)$ for $(t = 0, 1, \dots, T)$, let $h(t)$ be a data window, where $h(t)$ equal zero outside $[0\ , T-1]$. Put $T = L\ N + M - 1$, where L is a number of crossed intervals which contains $N + M - 1$ observations, $0 \le M < N$. We construct on ℓ - intervals observation:

$$X(\ell N), X(\ell N + 1), \dots, X[(\ell + 1)N - 1], \qquad \ell = 0, 1, \dots, L-1$$

the expanded finite Fourier transform which can be represented as

$$d_X^{\ell N}(\lambda) = \frac{1}{\sqrt{2\pi \int_{(\ell-1)N}^{\ell N + M - 1} h[t - (\ell - 1)N]\, dt}}$$

$$\times \int_{(\ell-1)N}^{\ell N + M - 1} h[t - (\ell - 1)N]\, X(t) \exp(-it\lambda)\, dt \qquad (2.4)$$

$$\ell = 1, \dots, L$$

We shal define the estimate spectral density on ℓ - intervals observation by

$$\hat{f}_{XX}^{(T)}(\lambda) = \frac{1}{L-1} \int_{1}^{L} I_{XX}^{\ell N}(\lambda)\, d\ell\,, \quad \ell = 1, \dots, L \qquad (2.5)$$

where $I_{XX}^{\ell N}(\lambda) = d_X^{\ell N}(\lambda)\, \bar{d}_X^{\ell N}(\lambda)$, (2.6)

$(-\infty < \lambda < \infty),\ \ell = 1, \dots, L$

is the expanded periodogram on ℓ - intervals observation. The bar denotes complex conjugate.

Set

$$c_{a_1,\dots,a_k}(t_1,\dots,t_{k-1})$$

$$= \operatorname{cum}\left\{X_{a_1}(t_1+\tau),\dots,X_{a_{k-1}}(t_{k-1}+\tau),X_{a_k}(\tau)\right\} \quad (2.7)$$

$$(a_1,\dots,a_k = 1,\dots,r;\ -\infty < t_1,\dots,t_{k-1},\tau < \infty;\ k = 1,2,\dots)$$

using the assumed stationarity. We then set down:

Assumption I. X(t) is a strictly stationary series all of its moments exist. For each j = 1, 2, ... , k-1 and k-tuple $a_1,\dots,a_k$ we have

$$\int_{-T}^{T}\cdots\int_{-T}^{T}\left|u_j\, c_{a_1,\dots,a_k}(u_1,\dots,u_{k-1})\right| du_1\dots du_{k-1} < \infty\ ,\quad k = 2, 3, \dots$$

(2.8)

If X(t) satisfies Assumption I we may define its k-th order cumulant spectrum by

$$f_{a_1,\dots,a_k}(\lambda_1,\dots,\lambda_{k-1}) = \frac{1}{(2\pi)^{k-1}}$$

$$\times \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} c_{a_1,\dots,a_k}(u_1,\dots,u_{k-1}) \exp\left(-i\sum_{j=1}^{k-1}\lambda_j u_j\right) du_1\dots du_{k-1}$$

$$(a_1,\dots,a_k = 1,\dots,r;\ -\infty < \lambda < \infty;\ k = 2,3,\dots), \quad (2.9)$$

and we note that $f_{a_1,\dots,a_k}(\lambda_1,\dots,\lambda_{k-1})$ is bounded, uniformly continuous, and $f^2(\lambda)=f(\lambda)$ .Ghazal and Farag [6] studied the statistical analysis of spectral density estimate on crossed intervals in discrete time series. In the next section we will construct the moments statistics of spectral density estimate and study the limiting of these moments in continuous time series using data window.

3. Asymptotic Expressions of the Estimation of Spectral Density

We shall consider estimate $\hat{f}^{(T)}_{XX}(\lambda)\,(-\infty<\lambda<\infty)$ given by formula (2.5). The general expressions for the first, second-order moments and cumulants of the estimate spectral density $\hat{f}^{(T)}_{XX}(\lambda)$ is given by the following theorem:

Theorem (3.1). Let X(t) satisfy Assumption I and have mean zero. If $h_\alpha(t)$, $\alpha=1,2,\dots,r$ $(-\infty<t<\infty)$ is bounded and has bounded variation, then

$$E\left\{\hat{f}^{(T)}_{ab}(\lambda)\right\}=\int_{-\infty}^{\infty}A^{L}(\alpha)f_{ab}(\alpha)d\alpha\ , \tag{3.1}$$

where

$$A^{L}(\lambda-\alpha)=\frac{1}{L-1}\times\int_{1}^{L}(2\pi)^{-1}\left[H_a^N(0)\,H_b^N(0)\right]^{-\frac{1}{2}}H_a^N(\lambda-\alpha)\overline{H}_b^N(\lambda-\alpha)d\ell\ , \tag{3.2}$$

and

$$H^N_{a_1,\dots,a_k}(\sum_{j=1}^{k}\lambda_j)=\int_{0}^{N+M-1}h_{a_1}(z)\dots h_{a_k}(z)\exp(-i\sum_{j=1}^{k}\lambda_j z)dz\ ,$$

$$z=t-(\ell-1)N\,,\ j=1,\dots,k \tag{3.3}$$

$$\mathrm{Cov}\left\{\hat{f}^{(T)}_{a_1 b_1}(\lambda_1), \hat{f}^{(T)}_{a_2 b_2}(\lambda_2)\right\} = \frac{1}{L-1}$$

$$\times \left\{(N+M-1)^{-1}\left(H^N_{a_1,a_2}(0) H^N_{b_1,b_2}(0)\right)^{-\frac{1}{2}} H^N_{a_1,a_2}(\lambda_1+\lambda_2)\right.$$

$$\times \bar{H}^N_{b_1,b_2}(\lambda_1+\lambda_2) f_{a_1 a_2}(\lambda_1) f_{b_1 b_2}(-\lambda_1) + (N+M-1)^{-1}$$

$$\times \left(H^N_{a_1,b_2}(0) H^N_{b_1,a_2}(0)\right)^{-\frac{1}{2}} H^N_{a_1,b_2}(\lambda_1-\lambda_2)$$

$$\times \bar{H}^N_{b_1,a_2}(\lambda_1-\lambda_2) f_{a_1 b_2}(\lambda_1) f_{b_1 a_2}(-\lambda_1)$$

$$+(N+M-1)^{-\frac{3}{2}} R^N(\lambda_1,\lambda_2) + O\left\{(N+M-1)^{-1}\right\}, \qquad (3.4)$$

The error term is uniform in $\lambda_1, \ldots, \lambda_k$, and there is a finite K such that

$$\left|R^N(\lambda_1,\lambda_2)\right| \le K \left\{\left(\left|H^N_{a_1,a_2}(0)\right|\right)^{-\frac{1}{2}} \left|H^N_{a_1,a_2}(\lambda_1+\lambda_2)\right|\right.$$

$$\left. + \left(\left|H^N_{a_1,b_2}(0)\right|\right)^{-\frac{1}{2}} \left|H^N_{a_1,b_2}(\lambda_1-\lambda_2)\right|\right\} \qquad (3.5)$$

and

$$\mathrm{Cum}\left\{\hat{f}^{(T)}_{a_1 b_1}(\lambda_1), \ldots, \hat{f}^{(T)}_{a_k b_k}(\lambda_k)\right\} = O\left\{\left[(L-1)(N+M-1)^{-K+1}\right]\right\},$$

$$, K = 2, 3, \ldots \qquad (3.6)$$

Proof: Formula (3.1) comes directly from formulas (2.5), (2.2) and then substituting about $t-(\ell-1)N = Z$ in the resulting equation. To deduce the covariance of $\hat{f}_{XX}^{(T)}(\lambda)$, we must state the following lemma:

Lemma (3.1). Let Assumption I be satisfied and have mean zero. If $h_\alpha(t)$, $\alpha = 1, 2, \ldots, r$ $(-\infty < t < \infty)$ is bounded and has bounded variation, then

$$\mathrm{Cum}\left\{d_{a_1}^N(\lambda_1), \ldots, d_{a_k}^N(\lambda_k)\right\} = \left(\frac{(2\pi)^{k-2}}{(N+M-1)^{k-1}}\right)^{\frac{1}{2}} \left(H_{a_1,\ldots,a_k}^N(0)\right)^{-\frac{1}{2}}$$

$$\times H_{a_1,\ldots,a_k}^N\left(\sum_{j=1}^k \lambda_j\right) f_{a_1\ldots a_k}(\lambda_1, \ldots, \lambda_{k-1})$$

$$\times \exp\left\{-i\sum_{j=1}^k \lambda_j(\ell-1)N\right\} + O\left\{(N+M-1)^{-\frac{k}{2}}\right\}, \qquad (3.7)$$

where $H_{a_1,\ldots,a_k}^N\left(\sum_{j=1}^k \lambda_j\right)$ is given by (3.3).

The proof of this lemma is basically a repetition of the proof of Theorem (2.1) in Ghazal and Farag [7]. Let

$= (L-1)^{-2}\ \mathrm{Cov}\left\{\hat{f}_{a_1b_1}^{(T)}(\lambda_1), \hat{f}_{a_2b_2}^{(T)}(\lambda_2)\right\}$; then by (Theorem 2.3.2, p.g. 21) in Brillinger [3], relation (3.7) and the notation $H_{ab}^N(\lambda), H_{abcd}^N(\lambda) = O(N+M-1)$, we have

$$\operatorname{Cov}\left\{\hat{f}^{(T)}_{a_1b_1}(\lambda_1), \hat{f}^{(T)}_{a_2b_2}(\lambda_2)\right\} = \frac{1}{(L-1)^2}$$

$$\times \int_1^L \int_1^L \left\{ (N+M-1)^{-1} \left(H^N_{a_1,a_2}(0) H^N_{b_1,b_2}(0) \right)^{-\frac{1}{2}} \right.$$

$$\times H^N_{a_1,a_2}(\lambda_1+\lambda_2)\, \bar{H}^N_{b_1,b_2}(\lambda_1+\lambda_2)\, f_{a_1a_2}(\lambda_1) f_{b_1b_2}(-\lambda_1)$$

$$+ (N+M-1)^{-1} \left(H^N_{a_1,b_2}(0) H^N_{b_1,a_2}(0) \right)^{-\frac{1}{2}} H^N_{a_1,b_2}(\lambda_1-\lambda_2)$$

$$\times \bar{H}^N_{b_1,a_2}(\lambda_1-\lambda_2)\, f_{a_1b_2}(\lambda_1) f_{b_1a_2}(-\lambda_1)$$

$$+ \frac{O\left\{(N+M-1)^{-1}\right\}}{\sqrt{N+M-1}} \left\{ \left(H^N_{a_1,a_2}(0) \right)^{-\frac{1}{2}} H^N_{a_1,a_2}(\lambda_1+\lambda_2) \right.$$

$$\times f_{a_1a_2}(\lambda_1) \exp\left\{-i(\lambda_1+\lambda_2)(\ell-1)N\right\}$$

$$+ \left(H^N_{a_1,b_2}(0) \right)^{-\frac{1}{2}} H^N_{a_1,b_2}(\lambda_1-\lambda_2)\, f_{a_1b_2}(\lambda_1)$$

$$\left. \left. \times \exp\left\{-i(\lambda_1-\lambda_2)(\ell-1)N\right\} \right\} + O\left\{(N+M-1)^{-1}\right\} \right\} d\ell_1 d\ell_2,$$

Now (3.4) follows when $\ell_1 = \ell_2$. Finally, from (2.6) we have

$$\operatorname{Cum}\left\{\hat{f}^{(T)}_{a_1 b_1}(\lambda_1), \dots, \hat{f}^{(T)}_{a_k b_k}(\lambda_k)\right\} = \frac{1}{(L-1)^k}$$

$$\times \int_1^L \dots \int_1^L h_k(\lambda_1, \dots, \lambda_k) d\ell_1 \dots d\ell_k \tag{3.8}$$

where
$h_k(\lambda_1, \dots, \lambda_k)$

$$= \operatorname{Cum}\left\{I^N_{a_1 b_1}(\lambda_1), \dots, I^N_{a_k b_k}(\lambda_k)\right\} = \sum_{\vartheta} \operatorname{Cum}\left\{C_{\vartheta_1}\right\} \dots \operatorname{Cum}\left\{C_{\vartheta_p}\right\} \tag{3.9}$$

where $\left[C_{\vartheta_1}, \dots, C_{\vartheta_p}\right]$ is an indecomposable partition of the elements of the table

$$\begin{matrix} d^N_{a_1}(\lambda_1), d^N_{b_1}(-\lambda_1) & & d^N_{c_1}(\mu_1), d^N_{d_1}(\vartheta_1) \\ \vdots & \rightarrow & \vdots \\ d^N_{a_k}(\lambda_k), d^N_{b_k}(-\lambda_k) & & d^N_{c_k}(\mu_k), d^N_{d_k}(\vartheta_k) \end{matrix}$$

and the summation in (3.9) extends over all such indecomposable partition [See Brillinger [2], Theorem 2.3.2]. Note that in the using the transformed table above, $\operatorname{Cum}\left\{d^N_{c_j}(\mu_j), d^N_{d_j}(\vartheta_j)\right\}$ denotes the cumulant of all random variables $\left\{d^N_{c_j}(\mu_j), d^N_{d_j}(\vartheta_j)\right\}$ excluding the cases with $\mu_j = -\vartheta_j = \lambda_m$ for some j, m. Then by (3.7), (3.9) becomes

$$h_k(\lambda_1, \dots, \lambda_k) = \sum_{\vartheta} \left[(N+M-1)^{-\frac{k}{2}} \left\{\prod_{i=1}^{k} H^N_{c_i, d_i}(0)\right\}^{-\frac{1}{2}}\right.$$

$$\times \left\{ \prod_{i=1}^{k} H^{N}_{c_i,d_i}(\mu_i + \vartheta_i) f_{c_i d_i}(\vartheta_i) \right\}$$

$$\times \exp\left\{ -i \sum_{j=1}^{k} \left(\mu_j + \vartheta_j\right)(\ell - 1)N \right\} + \sum_{i=1}^{k} O\left\{ (N + M - 1)^{-1} \right\}$$

$$\times \left\{ (N + M - 1)^{-\frac{1}{2}} \left(H^{N}_{c_i,d_i}(0) \right)^{-\frac{1}{2}} \left(H^{N}_{c_i,d_i}(\mu_i + \vartheta_i) \right) \right.$$

$$\left. \times \exp\left\{ -i(\mu_i + \vartheta_i)(\ell - 1)N \right\} f_{c_i d_i}(\mu_i) \right\} \Big],$$

With the notation $H^{N}_{ab}(\lambda) = O(N + M - 1)$, then relation (3.6) is obtained, which completes the proof of the theorem.

From the previous theorem, we can derive the following corollary which indicates that $\hat{f}^{(T)}_{ab}(\lambda)$ is an asymptotically unbiased estimate of $f_{ab}(\lambda)$ if $\lambda \equiv 0 \pmod{2\pi}$.

Corollary (3.1). Under the conditions of the theorem and if $h_a(z) = h_b(z) = 1$, $t - (\ell - 1)N = Z$, $\ell = 1, \ldots, L$ and a, b. $= 1, \ldots, r$, then

$$\lim_{N \to \infty} E\left\{ \hat{f}^{(T)}_{ab}(\lambda) \right\} = f_{ab}(\lambda), \quad -\infty < \lambda < \infty \qquad (3.10)$$

if $\lambda \not\equiv 0 \pmod{2\pi}$.

Proof. Put $\lambda = \alpha - \gamma$ into expression (3.1) and (3.2), we get

$$E\left\{\hat{f}_{ab}^{(T)}(\lambda)\right\} = \frac{1}{L-1}$$

$$\times \int_{1}^{L}\left\{\int_{-\infty}^{\infty}(2\pi)^{-1}\left(H_a^N(0)H_b^N(0)\right)^{-\frac{1}{2}} H_a^N(\gamma)\overline{H}_b^N(\gamma) f_{ab}(\lambda-\gamma)\,d\gamma\right\} d\ell,$$

As $f_{ab}(\lambda-\gamma)$ is a uniformly continuous function of γ, then Ghazal and Farag [7] (Theorem 4.1) indicates that the inner integral tends to $f_{ab}(\lambda)$ as $N \to \infty$, $\lambda \neq 0 \pmod{2\pi}$. This gives the indicated result.

In Corollary (3.2) below we make use the function

$$\eta(\lambda) = 1 \quad \text{if } \lambda \equiv 0 \pmod{2\pi}$$
$$= 0 \quad \text{otherwise} \qquad (3.11)$$

This is a periodic extension of the Kronecker delta function

$$\delta(\lambda) = 1 \quad \text{if } \lambda = 0$$
$$= 0 \quad \text{otherwise} \qquad (3.12)$$

The statistical dependence of $\hat{f}_{a_1b_1}^{(T)}(\lambda_1)$ and $\hat{f}_{a_2b_2}^{(T)}(\lambda_2)$ is seen to fall off as the functions $H_{a,b}^N(\lambda)$ fall off. In the limit Theorem (3.1) becomes:

Corollary (3.2). Under the condition of Theorem (3.1); suppose $\lambda_1 \pm \lambda_2 \equiv 0 \pmod{2\pi}$; $h_a(t) = h_b(t) = 1$; $a, b = 1, \ldots, r$ and that L does not depend on N. Then

$$\lim_{N\to\infty} \mathrm{Cov}\left\{\hat{f}^{(T)}_{a_1b_1}(\lambda_1), \hat{f}^{(T)}_{a_2b_2}(\lambda_2)\right\} =$$

$$\frac{\eta(\lambda_1+\lambda_2) f_{a_1a_2}(\lambda_1) f_{b_1b_2}(-\lambda_1) + \eta(\lambda_1-\lambda_2) f_{a_1b_2}(\lambda_1) f_{b_1a_2}(-\lambda_1)}{L-1} \tag{3.13}$$

for $\lambda_1, \lambda_2 \neq 0 \pmod{2\pi}$, and if $\lambda_1 \pm \lambda_2 \neq 0 \pmod{2\pi}$, then

$$\lim_{N\to\infty} \mathrm{Cov}\left\{\hat{f}^{(T)}_{a_1b_1}(\lambda_1), \hat{f}^{(T)}_{a_2b_2}(\lambda_2)\right\} = 0 \tag{3.14}$$

The proof comes directly by using the following notations:

(i) $H^{N}_{ab}(\lambda) = (N+M-1)$, when $h_a(t) = h_b(t) = 1$ and $\lambda = 0 \pmod{2\pi}$.

(ii) $\lim_{N\to\infty} H^{N}_{ab}(0) = \infty$.

In the case of $\lambda_1 = \pm\lambda_2$, Corollary (3.2) indicates the following corollary:

Corollary (3.3). Under the conditions of Theorem (3.1) and Corollary (3.2) and if $\lambda_1 = \pm\lambda_2$ then

$$\lim_{N\to\infty} \mathrm{Var}\, \hat{f}^{(T)}_{ab}(\lambda) = \frac{f^2(\lambda)}{L-1}, \tag{3.15}$$

Turning to the k-th order cumulant of $\hat{f}^{(T)}_{XX}(\lambda)$, we have the following corollary:

Corollary (3.4). Under the conditions of Theorem (3.1), then

$$\mathrm{Cum}\left\{\hat{f}^{(T)}_{a_1b_1}(\lambda_1), \ldots, \hat{f}^{(T)}_{a_kb_k}(\lambda_k)\right\} \to 0 \text{ as } N \to \infty, \qquad (3.16)$$

The proof comes directly from equation (3.6).

4. Asymptotic χ^2 and Wishart Distribution of Spectral Estimates

This section is concerned with finding the asymptotic distribution of $\hat{f}^{(T)}_{XX}(\lambda)$, on crossed intervals which based on the study of asymptotic distributions of the expanded finite Fourier transform $d_X^{\ell N}(\lambda)$ and periodograms.

Theorem (4.1). Let $X(t)$ $(-\infty < t < \infty)$ be an r vector-valued series satisfying Assumption I and $h_a(u)$ $(-\infty < u < \infty)$ bounded, has bounded variations and equal zero outside $[0, T-1]$. Let $d_X^{\ell N}(\lambda)$ be defined as (2.4) for $-\infty < \lambda < \infty$, $\ell = 1, \ldots, L$ and $a = 1, \ldots, r$. Then $d_X^{\ell N}(\lambda) = d_a^{\ell N}(\lambda)$ are asymptotically independent $N_r^c\left(0, f_{ab}(\lambda)\right)$ variates if $\lambda \not\equiv 0 \pmod{\pi}$ and asymptotically $N_r\left(0, f_{ab}(\lambda)\right)$ variates if $\lambda = \pm\pi, \pm 3\pi, \ldots$, as $N \to \infty$.

Proof. We begin by noting that

$$E\, d_a^{\ell N}(\lambda) = \frac{1}{\sqrt{2\pi \int_{(\ell-1)N}^{\ell N+M-1} h_a\left[t-(\ell-1)N\right]dt}}$$

$$\int_{(\ell-1)N}^{\ell N+M-1} h_a\left[t-(\ell-1)N\right]\exp(-it\lambda)\,dt\, EX_a(t) = 0,$$

Then $E d_a^{\ell N}(\lambda) = 0$ if $\lambda \neq 0 \pmod{\pi}$ and $\lambda = \pm\pi, \pm 3\pi, \ldots$, using Lemma 3.5 in Ghazal and Farag [7] and the notation

$\int_{(\ell-1)N}^{\ell N+M-1} h_a\left[t-(\ell-1)N\right]dt = \infty$. We therefor see that the first cumulant of

$d_X^{\ell N}(\lambda)$ behaves in the manner required by the theorem. Next we note that

$$\operatorname{Cov}\left\{d_a^{\ell N}(\lambda_1), d_b^{\ell N}(\lambda_2)\right\} = \int_{-\infty}^{\infty} f_{ab}(\vartheta)\Phi_{a,b}^{N}(\vartheta-\lambda_1,\vartheta-\lambda_2)d\vartheta \qquad (4.1)$$

where

$$\Phi_{a,b}^{N}(\vartheta-\lambda_1,\vartheta-\lambda_2) = (2\pi)^{-1}$$

$$\times\left(\int_{(\ell-1)N}^{\ell N+M-1}\int_{(\ell-1)N}^{\ell N+M-1} h_a\left[t_1-(\ell-1)N\right]h_b\left[t_2-(\ell-1)N\right]dt_1\,dt_2\right)^{-\frac{1}{2}}$$

$$\times\,\varphi_a^N(\vartheta-\lambda_1)\overline{\varphi}_b^N(\vartheta-\lambda_2)$$

with $\varphi^N(x) = \int_{(\ell-1)N}^{\ell N+M-1} h\left[t-(\ell-1)N\right]\exp(-itx)dt$

Then equation (4.1) tends to 0 if $\lambda_1 \pm \lambda_2 \equiv 0 \pmod{\pi}$. If $\pm\lambda_1 = \mp\lambda_2 \pmod{\pi}$, it tends to $f_{ab}(\lambda)$ (using Corollary (3.1), in Ghazal and Farag [7]). This indicates that the second-order cumulant behaviour required by the theorem holds.

Finally, we note from Lemma (3.1), that
$\text{Cum}\left\{d_{a_1}^N(\lambda_1), \ldots, d_{a_k}^N(\lambda_k)\right\}$ tends to 0 as $N \to \infty$ if $k > 2$ because
$H_{a_1,\ldots,a_k}^N(\lambda) = O(N+M-1)$.

Putting the above results together, we see that the cumulants of the variates at issue, and the conjugates of these variates, tends to the cumulants of a normal distribution. The conclusion of the theorem now follows from Brillinger [2] (Theorem P4.5, p.g. 403) since the distribution is determined by its moments.

In Theorem (4.1) we saw that the expanded finite Fourier transform of the same frequency, $\lambda \neq 0 \pmod{\pi}$, which is constructed on crossed intervals were asymptotically independent $N_r^c\left(0, f_{ab}(\lambda)\right)$ variates. This result suggests that the periodogram will have two limiting distribution, the first is χ^2 - distribution, and the other is Wishart distribution which we can express about these distributions in Theorems (4.2), and (4.3).

Theorem (4.2). Let $X(t)$ $(-\infty < t < \infty)$ be an r vector-valued series satisfying Assumption I and $h(u)$ $(-\infty < u < \infty)$ bounded, has bounded variations and equal zero outside $[0, T-1]$. Let

$$I_{XX}^{\ell N}(\lambda) = d_X^{\ell N}(\lambda)\bar{d}_X^{\ell N}(\lambda) \tag{4.2}$$

$\ell = 1, \ldots, L$; $(-\infty < \lambda < \infty)$. Then as $N \to \infty$, $I_{XX}^{\ell N}(\lambda)$, $\ell = 1, \ldots, L$ are asymptotically independent $\dfrac{f_{XX}(\lambda)\chi_2^2}{2}$ variate if $\lambda \neq 0 \pmod{\pi}$ and asymptotically idependent $\dfrac{f_{XX}(\lambda)\chi_1^2}{2}$ variate if $\lambda = \pm\pi, \pm 3\pi, \ldots$.

Proof. Theorem (4.1) indicates that Re $d_X^{\ell N}(\lambda)$, Im $d_X^{\ell N}(\lambda)$ are asymptotically independent $N_r\left(0, \frac{1}{2} f_{XX}(\lambda)\right)$ variates. It follows from Theorem P 5.1, p.g. 413, in Brillinger [2] that

$$I_{XX}^{\ell N}(\lambda) = \left\{\left[\mathrm{Re}\ d_X^{\ell N}(\lambda)\right]^2 + \left[\mathrm{Im}\ d_X^{\ell N}(\lambda)\right]^2\right\}$$

is asymptotically $f_{XX}(\lambda)\chi_2^2 / 2$. The asymptotic independence for different values follows in the same manner from the asymptotic independence of the $d_X^{\ell N}(\lambda)$.

This result will suggest a useful means of constructing spectral estimates later.

Theorem (4.3). Let $X(t)$ $(-\infty < t < \infty)$ be an r vector-valued series satisfying Assumption I and $h(u)$ $(-\infty < u < \infty)$ bounded, has bounded variations and equal zero outside $[0, T-1]$. Let $I_{XX}^{\ell N}(\lambda)$ be defined as (4.2). Then $I_{XX}^{\ell N}(\lambda)$ $\ell = 1, \dots, L$ are asymptotically independent $W_r^c(1, f_{XX}(\lambda))$ variates if $\lambda \not\equiv 0 \pmod \pi$ and asymptotically independent $W_r(1, f_{XX}(\lambda))$ variates if $\lambda = \pm\pi, \pm 3\pi, \dots$, as $N \to \infty$.

Proof. Theorem (4.1) indicates that $d_X^{\ell N}(\lambda_1), \dots, d_X^{\ell N}(\lambda_J)$ are asymptotically independent $N_r^c(0, f_{XX}(\lambda))$ variates while the Re $d_X^{\ell N}(\lambda_j)$ and Im $d_X^{\ell N}(\lambda_j)$ are asymptotically independent $N_r(0, f_{XX}(\lambda))$ variates. Now, Theorem P 5.1, p.g. 413, in Brillinger [2] indicates that

$$I_{XX}^{\ell N}(\lambda) = d_X^{\ell N}(\lambda_j)\bar{d}_X^{\ell N}(\lambda_j), \quad j = 1, \dots, J$$

are asymptotically independent $W_r^c(1, f_{XX}(\lambda))$ variates. The asymptotic independence for different values of j follows in the same manner from the asymptotic independence of the $d_X^{\ell N}(\lambda_j)$, $j = 1, \ldots, J$, which completes the proof.

The asymptotic distribution of $\hat{f}_{XX}^{(T)}(\lambda)$ under certain regularity conditions is indicated in the following :

Theorem (4.4). Let X(t) $(-\infty < t < \infty)$ be an r vector-valued series satisfying Assumption I and $h(u)$ $(-\infty < u < \infty)$ bounded, has bounded variations and equal zero outside $[0, T-1]$. Let $I_{XX}^{\ell N}(\lambda)$ be defined as (4.2). Let

$$\hat{f}_{XX}^{(T)}(\lambda) = \frac{1}{L-1} \int_1^L I_{XX}^{\ell N}(\lambda)\, d\ell\ , \quad \ell = 1, \ldots, L \tag{4.3}$$

where $T = LN + M - 1$. Then $\hat{f}_{XX}^{(T)}(\lambda)$ is asymptotically $\dfrac{f_{XX}(\lambda)\chi^2_{2(L-1)}}{2(L-1)}$ if $\lambda \neq 0 \pmod{\pi}$ and asymptotically independent $\dfrac{f_{XX}(\lambda)\chi^2_{L-1}}{2(L-1)}$ if $\lambda = \pm\pi, \pm 3\pi, \ldots$, as $N \to \infty$.

The proof comes directly from Theorem (4.2) and Theorem P 5.1 , p.g. 413, in Brillinger [2].

Theorem (4.5). Let X(t) $(-\infty < t < \infty)$ be an r vector-valued series satisfying Assumption I and $h(u)$ $(-\infty < u < \infty)$ bounded, has bounded variations and equal zero outside $[0, T-1]$. Let $\hat{f}_{XX}^{(T)}(\lambda)$ be defined as (4.3) where $T = LN + M - 1$. Then $\hat{f}_{XX}^{(T)}(\lambda)$ is asymptotically $(L-1)^{-1} W_r^c(L-1, f_{XX}(\lambda))$ if $\lambda \neq 0 \pmod{\pi}$ and asymptotically $(L-1)^{-1} W_r(L-1, f_{XX}(\lambda))$ $\lambda = \pm\pi, \pm 3\pi, \ldots$, as $N \to \infty$.

The proof comes directly from Theorem (4.3) and Theorem P 5.1 , p.g. 413, in Brillinger [2] .

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