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Admissibility when Estimating the Population Counterpart of a U-Statistic of Order 2 Employing a Pseudo Bayesian Argument

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In this paper, a pseudo Bayesian argument due to Basu (1971) is employed to construct an estimator for a class of parametric functions namely the population counterpart of a one sample U-statistic of order 2. This estimator is, then, shown to be admissible using a technique given by Meeden and Ghosh (1983). Some admissibility results given by Ghosh and Meeden (1983) and Mazloum (1988, 1990) are shown to be special cases of the admissibility result obtained in this paper.

1. Introduction

In keeping the standard formulation of finite population sampling problems, we suppose that a population consists of units labeled 1,2,...,N and that attached to each unit i is the value of a single characteristic y_i. The vector $y = (y_1,y_2,\ldots,y_N)$ is the unknown state of nature and is assumed to belong to $\Theta = R^N$, the N^{th} dimensional Euclidean space. A subset $s = \{i_1,i_2,\ldots,i_n\}$ of $\{1,2,\ldots,N\}$ is called a sample of size n. A probability distribution, p, defined on the set S of all possible samples from this distribution is called a design. Suppose that for estimating some real valued function, say $\gamma(y)$, with squared error loss, one uses an estimator, say e(s,y) [e(s,y) depends on y only through y(s) where $y(s) = (y_{i_1}, y_{i_2},\ldots,y_{i_n})$] along with a design p then (p,e) is a typical decision strategy with risk function:

$$r(p,e;y) = \sum_{s\in S} [e(s,y) - \gamma(y)]^2 \, p(s) \qquad (1.1)$$

An estimator e is said to be admissible when using a design p if there does not exist any other estimator e' such that

$$r(p,e';y) \leq r(p,e;y) \qquad \text{for all } y\in\Theta$$

with strict inequality for at least one y.

In this paper, we will concern ourselves with the case where the function $\gamma(y)$ of interest is the population counterpart of a one sample U-statistic of order 2, namely:

$$U(y) = \frac{1}{\binom{N}{2}} \sum_{\beta\in B} \xi(y_{\beta_1}, y_{\beta_2}) \tag{1.2}$$

where $\xi(.,.)$ is any symmetric function and $B = \{\beta | \beta = (\beta_1,\beta_2)$ is one of the $\binom{N}{2}$ unordered subsets of 2 integers chosen without replacement from the set $(1,2,...,N)\}$. U(y) is a class of parametric functions of the population whose sample counterpart, called "U-statistic", is defined for a given sample of size $n \geq 2$ as follows:

$$U(y(s)) = \frac{1}{\binom{n}{2}} \sum_{\beta^2\in B^2} \xi(y_{\beta_1^2}, y_{\beta_2^2}) \tag{1.3}$$

where $B^2 = \{\beta^2 | \beta^2 = (\beta_1^2,\beta_2^2)$ is one of the $\binom{n}{2}$ unordered subsets of 2 integers chosen without replacement from the set $(1,2,...,n)\}$.

In section 2, following a line of argument given by Basu (1971), an estimator of U(y) is constructed. In section 3, this estimator is shown to be admissible using the simple but powerful tools given by Meeden and Ghosh (1983) for proving admissibility of estimators in finite population sampling problems. Finally, in section 4, we show that some of the admissibility results given by Ghosh and Meeden (1983) and Mazloum (1988 & 1990) are just special cases of the admissibility result obtained in this paper.

2. The Proposed Estimator

Basu (1971) has presented an argument that led to an estimator for the population total. We, now, follow his line of argument to construct an estimator for U(y).

From (1.2), U(y) can be rewritten as:

$$U(y) = \frac{1}{\binom{N}{2}} \sum_{t=0}^{2} \sum_{\beta^t \in B^t} \xi(y_{\beta_1^t}, v_{\beta_2^t}) \tag{2.1}$$

where $B^t = \{\beta^t | \beta^t = (\beta_1^t, \beta_2^t)$ is one of the $\binom{n}{t}\binom{N-n}{2-t}$ unordered subsets of 2 integers chosen without replacement from the set $(1,2,\ldots,N)$ where t of them chosen from the set $(i_1, i_2, \ldots, i_n)$ and (2-t) chosen from the set $(1,\ldots,N) \cap (i_1,\ldots,i_n)^c\}$.

Suppose that before observing the sample, the statistician is willing to make a prior guess for the value y_i for each i. If m_i denotes his prior guess for y_i, then $m = (m_1, m_2, \ldots, m_N)$ is the vector of his prior guesses. After the sample s is observed, the ratios y_i/m_i, $i \epsilon s$ become known. If these ratios are approximately equal, then the unobserved ratios will probably take on similar values as well. This suggests that for any $i^* \notin s$ and $j^* \notin s$ where $i^* \neq j^*$:

$$Pr\left(\frac{y_{i^*}}{m_{i^*}} = \frac{y_i}{m_i} \,\Big|\, s\right) = \frac{1}{n} \qquad \text{for } i \epsilon s$$

and

$$Pr\left(\frac{y_{i^*}}{m_{i^*}} = \frac{y_i}{m_i}, \frac{y_{j^*}}{m_{j^*}} = \frac{y_j}{m_j} \,\Big|\, s\right)$$

$$= Pr\left(\frac{y_{i^*}}{m_{i^*}} = \frac{y_i}{m_i} \,\Big|\, s\right) . Pr\left(\frac{y_{j^*}}{m_{j^*}} = \frac{y_j}{m_j} \,\Big|\, \frac{y_{i^*}}{m_{i^*}} = \frac{y_i}{m_i}, s\right)$$

$$= \begin{cases} \left(\frac{1}{n}\right)\left(\frac{2}{n+1}\right) & \text{if } i = j;\ i,j \epsilon s \\ \left(\frac{1}{n}\right)\left(\frac{1}{n+1}\right) & \text{if } i \neq j;\ i,j \epsilon s \end{cases}$$

Now, the proposed estimator $\hat{U}$ for U(y) could be obtained as follows:

$$\hat{U} = E[U(y)|s]$$

$$= \frac{1}{\binom{N}{2}} \Big[\sum_{\beta^2 \epsilon B^2} \xi(y_{\beta_1^2}, y_{\beta_2^2}) + \sum_{\beta^1 \epsilon B^1} E\{\xi(y_{\beta_1^1}, y_{\beta_2^1})|s\}$$

$$+ \sum_{\beta^o \epsilon B^o} E\{\xi(y_{\beta_1^o}, y_{\beta_2^o})|s\}\Big] \tag{2.2}$$

Without loss of generality, assume that the first coordinate of ξ in the second term of (2.2) is the observed value. Hence,

$$\hat{U} = \frac{1}{\binom{N}{2}}\Big[\sum_{\beta^2 \epsilon B^2} \xi(y_{\beta_1^2}, y_{\beta_2^2}) + \frac{1}{n} \sum_{\beta^1 \epsilon B^1} \sum_{i \epsilon s} \xi(y_{\beta_1^1}, \frac{y_i}{m_i} m_{\beta_2^1})$$

$$+ \frac{1}{n(n+1)} \sum_{\beta^o \epsilon B^o} \{2 \sum_{i \epsilon s} \xi(\frac{y_i}{m_i} m_{\beta_1^o}, \frac{y_i}{m_i} m_{\beta_2^o})$$

$$+ \sum_{i \epsilon s} \sum_{\substack{i'=s \\ i \neq i'}} \xi(\frac{y_i}{m_i} m_{\beta_1^o}, \frac{y_{i'}}{m_{i'}} m_{\beta_2^o})\}\Big] \tag{2.3}$$

3. Admissibility of the Estimator $\hat{U}$

Before we present the main result of this section, we give the following definition:

Definition

Let $\lambda^1, \ldots, \lambda^K$ be an ordered set of prior distributions that are mutually singular (i.e. with mutually exclusive supports). The Bayes class with respect to $\lambda^1, \ldots, \lambda^K$ say $D(\lambda^1, \ldots, \lambda^K)$ is defined inductively as follows: for $K = 1$, $D(\lambda^1)$ is the collection of all Bayes rules with respect to λ^1 , for $K > 1$, $D(\lambda^1, \ldots, \lambda^K)$ is the set of all Bayes rules within $D(\lambda^1, \ldots, \lambda^{K-1})$ with respect to λ^K. A decision rule in $D(\lambda^1, \ldots, \lambda^K)$ is called a stepwise Bayes rule with respect to $(\lambda^1, \ldots, \lambda^K)$.

It has been demonstrated in Meeden and Ghosh (1981)[see also Brown (1981) and Hsuan (1979)] that a unique stepwise Bayes rule is admissible. We now use this idea to prove the following theorem:

Theorem

For estimating U(y) [given by (1.1)] using squared error loss, the estimator $\hat{U}$ given by (2.3) is admissible under any design such that $n \geq 2$.

Proof

We, now, follow a line of argument of Meeden and Ghosh (1983) to prove the admissibility of the estimator $\hat{U}$. The first step in this line of argument is to observe that admissibility results for the parameter space R^N follow as corollaries to admissibility results for the finite, so called scale-load, parameter sets of Hartley and Rao (1968,1969). That is, if $\alpha_1,\ldots,\alpha_r$, where $1 \leq r \leq N$, are distinct real numbers, then:

$$\bar{\theta}(\alpha_1,\ldots,\alpha_r) = \{y: \frac{y_i}{m_i} = \alpha_j \text{ for some } j = 1,\ldots,r \text{ and for all } i = 1,\ldots,N\}$$

is the appropriate parameter space for the scale-load situation.

Hence, to prove the theorem, it sufficies to take the parameter space to be $\bar{\theta}(\alpha_1,\ldots,\alpha_r)$ and show that, under any design with $n \geq 2$ and squared error loss, $\hat{U}$ is unique stepwise Bayes against some set of mutually singular priors.

First, we need the following notations: let $\bar{\bar{O}}(\alpha_1,\ldots,\alpha_r) = \{y: \frac{y_i}{m_i} = \alpha_j$ for some $j = 1,\ldots,r$; for all $i = 1,\ldots,N$ and each α_j appears at least once for $j = 1,\ldots,r\}$.

If $y \in \bar{\bar{\Theta}}(\alpha_1,\ldots,\alpha_r)$ we say that y is of order r for $\alpha_1,\ldots,\alpha_r$. Similarly, if y(s) is a sample point with $r \le n$ we say that y(s) is of order r for $\alpha_1,\ldots,\alpha_r$ if each y_i/m_i equals one of the r values $\alpha_1,\ldots,\alpha_r$ and if for each value α_j, there exists at least one i_ℓ for which $y_{i_\ell}/m_{i_\ell} = \alpha_j$. If $y \in \bar{\bar{\Theta}}(\alpha_1,\ldots,\alpha_r)$, let $w_y(j)$ be the number of (y_i/m_i)'s which are equal to α_j. Note that for each j, $w_y(j) \ge 1$ and $\sum_{j=1}^{r} w_y(j) = N$. If y(s) is a sample point of order r for $\alpha_1,\ldots,\alpha_r$, let $w_y(j,s)$ be the number of observed (y_i/m_i)'s, $i \in s$ which are equal to α_j. It is clear that $w_y(j,s) \ge 1$ and $\sum_{j=1}^{r} w_y(j,s) = n$.

Recall that under squared error loss, the Bayes estimate of U(y) at an observed sample y(s) against some prior is just the posterior mean, i.e.,

$$E[U(y)|y(s)] = \frac{1}{\binom{N}{2}}\Big[\sum_{\beta^2 \in B^2} \xi(y_{\beta_1^2}, y_{\beta_2^2}) + \sum_{\beta^1 \in B^1} E\{\xi(y_{\beta_1^1}, y_{\beta_2^1})|y(s)\} + \sum_{\beta^o \in B^o} E\{\xi(y_{\beta_1^o}, y_{\beta_2^o})|y(s)\}\Big] \quad (3.1)$$

We, now, exhibit a family of mutually singular prior distributions against which $\hat{U}$ is unique stepwise Bayes.

The first prior λ^1 assigns mass 1/r to each point in the set $\bigcup_{j=1}^{r} \bar{\bar{\Theta}}(\alpha_j)$. For any point in this set, say $\bar{\bar{\Theta}}(\alpha_j)$, all the observed ratios in a data point s are α_j. Hence, the Bayes estimate is:

$$E[U(y)|y(s)] = \frac{1}{\binom{N}{2}}\Big[\sum_{\beta^2 \in B^2} \xi(\alpha_j m_{\beta_1^2}, \alpha_j m_{\beta_2^2}) + \sum_{\beta^1 \in B^1} \xi(\alpha_j m_{\beta_1^1}, \alpha_j m_{\beta_2^1}) + \sum_{\beta^o \in B^o} \xi(\alpha_j m_{\beta_1^o}, \alpha_j m_{\beta_2^o})\Big]$$

which is just $\hat{U}$.

Next, the prior λ^2 is defined on the set $\bar{\bar{\Theta}}(\alpha_1,\alpha_2)$ as follows:

$$\lambda^2(y) \propto \int_0^1 q^{w_y(1)-1} (1-q)^{w_y(2)-1} dq = \frac{\Gamma[w_y(1)]\Gamma[w_y(2)]}{\Gamma[N]}$$

The sample points that are consistent under λ^2 are those of order less than or equal to 2. However, the sample points of order 1 have been taken care of. Now, for a sample point y(s) of order 2 for α_1 and α_2, the marginal probability is given by:

$$\lambda^2(y(s)) \propto \frac{\Gamma[w_y(1,s)]\Gamma[w_y(2,s)]}{\Gamma[n]}$$

For $i^* \notin s$ and $j^* \notin s$ with $i^* \neq j^*$, it is easy to show that

$$\Pr(\frac{y_{i*}}{m_{i*}} = \alpha_j | y(s)) = \frac{w_y(j,s)}{n} \tag{3.2}$$

and

$$\Pr(\frac{y_{i*}}{m_{i*}} = \alpha_j, \frac{y_{j*}}{m_{j*}} = \alpha_{j'} | y(s))$$

$$= \begin{cases} \frac{w_y(j,s)[w_y(j,s) + 1]}{n(n+1)} & \text{if } j = j' \\ \frac{w_y(j,s)w_y(j',s)}{n(n+1)} & \text{if } j \neq j' \end{cases} \tag{3.3}$$

where j = 1,2 and j' = 1,2.

Hence, using (3.2) and (3.3) we get:

$$E[\xi(y_{\beta_1^1}, y_{\beta_2^1}) | y(s)] = \sum_{j=1}^{2} \xi(y_{\beta_1^1}, \alpha_j m_{\beta_2^1}) \frac{w_y(j,s)}{n}$$

$$= \frac{1}{n} \sum_{i \in s} \xi(y_{\beta_1^1}, \frac{y_i}{m_i} m_{\beta_2^1}) \tag{3.4}$$

Also,

$$E[\xi(y_{\beta_1^o},y_{\beta_2^o})|y(s)] = \sum_{j=1}^{2} \xi(\alpha_j m_{\beta_1^o},\alpha_j m_{\beta_2^o})\frac{w_y(j,s)[w_y(j,s)+1]}{n(n+1)}$$

$$+ \sum_{j=1}^{2}\sum_{\substack{j'=1 \\ j\neq j'}}^{2} \xi(\alpha_j m_{\beta_1^o},\alpha_{j'} m_{\beta_2^o})\frac{w_y(j,s)w_y(j',s)}{n(n+1)}$$

$$= \frac{1}{n(n+1)} \Big[\sum_{j=1}^{2} \xi(\alpha_j m_{\beta_1^o},\alpha_j m_{\beta_2^o}) w_y(j,s)$$

$$+ \sum_{j=1}^{2}\sum_{j'=1}^{2} \xi(\alpha_j m_{\beta_1^o},\alpha_{j'} m_{\beta_2^o}) w_y(j,s) w_y(j',s)\Big]$$

$$= \frac{1}{n(n+1)} \Big[\sum_{i\epsilon s} \xi(\frac{y_i}{m_i} m_{\beta_1^o}, \frac{y_i}{m_i} m_{\beta_2^o})$$

$$+ \sum_{i\epsilon s}\sum_{i'\epsilon s} \xi(\frac{y_i}{m_i} m_{\beta_1^o}, \frac{y_{i'}}{m_{i'}} m_{\beta_2^o}) \Big] \qquad (3.5)$$

Now, (3.1), (3.4) and (3.5) imply:

$$E[U(y)|y(s)] = \frac{1}{\binom{N}{2}} \Big[\sum_{\beta^2\epsilon B^2} \xi(y_{\beta_1^2},y_{\beta_2^2})$$

$$+ \frac{1}{n} \sum_{\beta^1\epsilon B^1}\sum_{i\epsilon s} \xi(y_{\beta_1^1}, \frac{y_i}{m_i} m_{\beta_2^1})$$

$$+ \frac{1}{n(n+1)} \{2 \sum_{i\epsilon s} \xi(\frac{y_i}{m_i} m_{\beta_1^o}, \frac{y_i}{m_i} m_{\beta_2^o})$$

$$+ \sum_{i\epsilon s}\sum_{\substack{i'\epsilon s \\ i\neq i'}} \xi(\frac{y_i}{m_i} m_{\beta_1^o}, \frac{y_{i'}}{m_{i'}} m_{\beta_2^o})\}\Big]$$

which is just $\hat{U}$.

Note that this will also be the case when defining any prior of the type λ^2 on any set $\bar{\bar{\Theta}}(\alpha_i,\alpha_j)$, $1 \leq i < j \leq r$. In fact, it would have been better if we defined λ^2 on the set $\bigcup_{\{i<j\}} \bar{\bar{\Theta}}(\alpha_i,\alpha_j)$ and the proof would have been exactly the same as above. However, to avoid the complexity of notations, we defined λ^2 just on $\bar{\bar{\Theta}}(\alpha_1,\alpha_2)$.

Assume that λ^2 was defined on the set $\bigcup_{\{i<j\}} \bar{\bar{\Theta}}(\alpha_i, \alpha_j)$ and define the next prior λ^3 on the set $\bigcup_{\{i<j<k\}} \bar{\bar{\Theta}}(\alpha_i, \alpha_j, \alpha_k)$ as follows:

$$\lambda^3(y) \propto \int_0^1 \int_0^1 q_1^{w_y(i)-1} q_2^{w_y(j)-1} (1-q_1-q_2)^{w_y(k)-1} dq_1\, dq_2$$

$$= \frac{\Gamma[w_y(i)]\Gamma[w_y(j)]\Gamma[w_y(k)]}{\Gamma(N)}$$

The data points that are consistent under this prior but not under any of the previous priors are those of order 3 for some α_i, α_j and α_k. For any such point, the marginal probability is easily seen to be:

$$\lambda^3(y(s)) \propto \frac{\Gamma[w_y(i,s)]\Gamma[w_y(j,s)]\Gamma[w_y(k,s)]}{\Gamma(n)}$$

In this case, following the same steps as above, it is easy to show that the Bayes estimate against λ^3 is just $\hat{U}$.

Continuing in this way until all possible data points are covered, we see that $\hat{U}$ is unique stepwise Bayes against that set of priors and hence is admissible and the proof is complete.

4. Special Cases

Case (1)

If $\xi(y_i, y_j) = \frac{1}{2}(y_i - y_j)^2$ then

$$U(y) = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2 \qquad (4.1)$$

where $\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i$, and

$$\hat{U} = \frac{1}{N(N-1)}\Big[\sum_{\beta^2 \varepsilon B^2} (y_{\beta_1^2} - y_{\beta_2^2})^2 + \frac{1}{n} \sum_{\beta^1 \varepsilon B^1} \sum_{i \varepsilon s} (y_{\beta_1^1} - \frac{y_i}{m_i} m_{\beta_2^1})^2 \Big]$$

$$+ \frac{1}{n(n+1)} \sum_{\beta^o \varepsilon B^o} \Big[2 \sum_{i \varepsilon s} (\frac{y_i}{m_i} m_{\beta_1^o} - \frac{y_i}{m_i} m_{\beta_2^o})^2 + \sum_{i \varepsilon s} \sum_{\substack{i' \varepsilon s \\ i \neq i'}} (\frac{y_i}{m_i} m_{\beta_1^o} - \frac{y_{i'}}{m_{i'}} m_{\beta_2^o})^2 \Big]$$

Adding and subtracting $\bar{y}$ (the sample mean) in each term, simple algebra leads to :

$$\hat{U} = \frac{1}{N-1} \sum_{i \varepsilon s} (y_i - \bar{y})^2 + \frac{1}{nN} \sum_{i^* \notin s} \sum_{i \varepsilon s} (\frac{y_i}{m_i} m_{i^*} - \bar{y})^2$$

$$- \frac{1}{N(N-1)n(n+1)} \sum_{i^* \notin s} \sum_{\substack{j^* \notin s \\ i^* \neq j^*}} \cdot \Big[2 \sum_{i \varepsilon s} (\frac{y_i}{m_i} m_{i^*} - \bar{y})(\frac{y_i}{m_i} m_{j^*} - \bar{y})$$

$$+ \sum_{i \varepsilon s} \sum_{\substack{i' \varepsilon s \\ i \neq i'}} (\frac{y_i}{m_i} m_{i^*} - \bar{y})(\frac{y_{i'}}{m_{i'}} m_{j^*} - \bar{y})\Big] \qquad (4.2)$$

Consequently, for estimating the population variance given by (4.1) using squared error loss, the estimator $\hat{U}$ given by (4.2) is admissible under any design. This estimator was first constructed in Mazloum (1990).

Case (2)

If all the m_i's are equal then

$$\hat{U} = \frac{1}{\binom{N}{n}}\Big[\sum_{\beta^2 \varepsilon B^2} \xi(y_{\beta_1^2}, y_{\beta_2^2}) + \frac{1}{n} \sum_{\beta^1 \varepsilon B^1} \sum_{i \varepsilon s} \xi(y_{\beta_1^1}, y_i)$$

$$+ \frac{1}{n(n+1)} \sum_{\beta^o \varepsilon B^o} \{2 \sum_{i \varepsilon s} \xi(y_i, y_i) + \sum_{i \varepsilon s} \sum_{\substack{i' \varepsilon s \\ i \neq i'}} \xi(y_i, y_{i'})\}\Big] \qquad (4.3)$$

If, in addition, we assume that

$$\xi(y_i, y_i) = 0 \qquad \forall\ i \tag{4.4}$$

then (4.4) becomes:

$$\hat{U} = \frac{1}{\binom{N}{2}}\Big[\sum_{\beta^2 \in B^2} \xi(y_{\beta_2^2}, y_{\beta_2^2}) + \frac{2(N-n)}{n} \sum_{\substack{j\in s \\ j<i}} \sum_{i \in s} \xi(y_j, y_i) + \frac{2\binom{N-n}{2}}{n(n+1)} \sum_{\substack{i\in s \\ i<i'}} \sum_{i' \in s} \xi(y_i, y_{i'})\Big]$$

$$= \frac{\binom{n}{2}}{\binom{N}{2}} \left[1 + \frac{2(N-n)}{n} + \frac{2\binom{N-n}{2}}{n(n+1)}\right] U(y(s)) \tag{4.5}$$

where U(y(s)) is the sample U-statistic given by (1.3).

Hence, for estimating the population counterpart of a U-statistic of order 2 using squared error loss, the estimator $\hat{U}$ given by (4.5) where $\xi(\cdot,\cdot) = 0$ if the two coordinates are equal is admissible. This estimator is a special case of the admissible estimator of the population counterpart of a U-statistic of order K constructed in Mazloum (1988) under the assumption that $\xi(\cdot, \dots,) = 0$ if two or more of its coordinates are equal.

<u>Case (3)</u>

If all the m_i's are equal and $\xi(y_i, y_j) = \frac{1}{2}(y_i - y_j)^2$ then

$$U(y) = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2 .$$

and

$$\hat{U} = \frac{(N+1)}{(N-1)(n+1)} \sum_{i \in s} (y_i - \bar{y})^2 \tag{4.6}$$

Hence, for estimating the population variance, the estimator $\hat{U}$ given by (4.6) is admissible. This estimator was first constructed in Ghosh and Meeden (1983).

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