THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CARIO, UNIV., VOL. (42) NO.2, 1998

COEFFICIENT OF DETERMINATION FOR RANDOM COEFFICIENT REGRESSION MODEL

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Key Words: Random coefficient regression model; Goodness of fit; Coefficient of determination; Panel data; Pooling cross section and time series data.

ABSTRACT

Random coefficient regression (RCR) model offers an approach by which to take into account nonsystematic changes in the parameters. Swamy (1970,1971) derived estimators of the RCR model parameters. Youssef (1997, 1998) used a Monte Carlo simulation to study some properties of Swamy's estimators in small sample size. In this paper, we derive the coefficient of determination, as a measure for goodness of fit, for the RCR model. Some properties of this measure will be discussed.

1. INTRODUCTION

Swamy random coefficient regression model offers an approach by which to take into account nonsystematic changes in the parameters. It is appropriate for the pooling cross sectional and time series model when the parameters assumed to vary randomly over the cross section units. A fixed parameter's model is a special case of the random coefficient model. Let the T observations on the i th cross sectional unit be written as

$$y_i = x_i\beta_i + e_i, \qquad (1)$$

Where $i=1, 2, \ldots, N$, $e_i \sim (0, \sigma_i^2 I)$, $E(e_i e_j') = 0$ if $i \neq j$, $\beta_i = \bar{\beta} + \upsilon_i$, $\upsilon_i \sim (0, V)$ and $E(\upsilon_i \upsilon_j') = 0$, $i \neq j$. Thus each individual in the sample has a unique coefficient vector β_i, but each individual's response is constant over time, and the β_i has a common mean $\bar{\beta}$. To estimate $\bar{\beta}$, we write all NT observations as

$$Y = \tilde{X}\,\bar{\beta} + \varepsilon, \tag{2}$$

where $Y' = (y_1', y_2', \ldots, y_N')$, $\varepsilon = X\upsilon + e$, and υ and e are similarly defined. The matrix $\tilde{X}$ is the matrix of stacked x_i's, $\tilde{X}' = (x_1', x_2', \ldots, x_N')$, and X is a block diagonal matrix, $X = \operatorname{diag}(x_1, x_2, \ldots, x_N)$. The covariance matrix of the composite error term ε is Ω which is block diagonal with i th block $\Omega_{ii} = x_i V x_i' + \sigma_i^2 I$. The generalized least squares estimator can be conveniently written as

$$\hat{\bar{\beta}} = \left(\tilde{X}'\Omega^{-1}\tilde{X}\right)^{-1} \tilde{X}'\Omega^{-1}Y$$
$$= \sum_{i=1}^{N} W_i b_i, \tag{3}$$

where

$$W_i = \left\{\sum_{j=1}^{N}\left[V + \sigma_j^2 (x_j'x_j)^{-1}\right]^{-1}\right\}^{-1} \left[V + \sigma_i^2 (x_i'x_i)^{-1}\right]^{-1}, \tag{4}$$

and

$$b_i = (x_i'x_i)^{-1} x_i'y_i. \tag{5}$$

The term in braces, { }, is $\left(\tilde{X}'\Omega^{-1}\tilde{X}\right)$, see Result (2) in the appendix, which provides a convenient basis for computing the covariance matrix.

In order to obtain feasible generalized least squares (GLS) estimators, we must obtain a consistent estimators of V and σ_i^2. Swamy (1970) show that σ_i^2 is consistently estimated by

$$\hat{\sigma}_i^2 = \frac{1}{T-K}(y_i - x_i b_i)'(y_i - x_i b_i), \tag{6}$$

and a consistent estimator of V is

$$\hat{V} = \frac{1}{N-1} S - \frac{1}{N}\sum_{i=1}^{N} \hat{\sigma}_i^2 (x_i'x_i)^{-1}, \tag{7}$$

Where

$$S = \sum_{i=1}^{N} (b_i - \bar{b})(b_i - \bar{b})', \tag{8}$$

and

$$\bar{b} = \frac{1}{N}\sum_{i=1}^{N} b_i \,. \tag{9}$$

Note that $\hat{V}$ is not guaranteed to be positive semidefinite. Swamy (1971) considered this problem and suggested an alternative for $\hat{V}$ as S/(N-1). This estimator is always positive semidefinite and consistent. Youssef (1997, 1998) studied the bias and the mean square error for the estimators $\bar{\beta}$ and V for small sample size using a Monte Carlo technique.

2. GOODNESS OF FIT FOR RCR MODEL

In the estimation of RCR model, as with a single equation model, one may wish to report summary statistics, reflecting some of the features or quality of the results obtained. In addition, one might wish to indicate the extent to which the fitted RCR model explains the variability in the data for the dependent variables. In this Section, a coefficient of determination, R^2, will be derived for the RCR model.

Buse(1973, 1979) defined a measure of goodness of fit for the GLS model $Y = X\beta + \varepsilon$, with $E(\varepsilon) = 0$ and var $(\varepsilon) = V$. It is assumed that the constant terms are included, so X partitioned as (s, Z), with $s' = (1,1,\ldots,1)$ and Z is explanatory variables. Buse's definition of the coefficient of determination for this model is given by

$$R_B^2 = \frac{\left(\hat{Y} - sb_0\right)' V^{-1}\left(\hat{Y} - sb_0\right)}{\left(Y - sb_0\right)' V^{-1}\left(Y - sb_0\right)}, \tag{10}$$

where $b_0 = \left(s'V^{-1}s\right)^{-1} s'V^{-1}Y = \left(s'V^{-1}s\right)^{-1} s'V^{-1}\hat{Y}$ is the estimator of the constant term under the restriction that all other coefficients are zero.

For simplicity, we assume that the variance of the error term, in model (2), will be known. If not, all relevant formulae will be replaced by a consistent estimators as in (6), and (7). A generalization of this measure for the RCR model

$$R_{RCR}^2 = \frac{\hat{Y}' \left(\Omega^{-1} - kW^{*'} W^{*}\right) \hat{Y}}{Y' \left(\Omega^{-1} - kW^{*'} W^{*}\right) Y}, \tag{11}$$

where

$$k = \sum_{i=1}^{N} \ell'\Omega_{ii}^{-1}\ell,$$

$$W^* = \begin{pmatrix} W_1^* & W_2^* & \cdots & W_N^* \end{pmatrix},$$

$$W_i^* = \frac{1}{k}\ell'\Omega_{ii}^{-1},$$

$$s' = \begin{pmatrix} \ell' & \ell' & \cdots & \ell' \end{pmatrix}_{1\times NT},$$

and

$$\ell' = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times T}.$$

The coefficient of determination for the RCR model on (11) can be viewed as the ratio of the estimated weighted variation to the total weighted variation in Y. For proof see Result (3) in the appendix.

3. SOME PROPERTIES OF THE COEFFICIENT OF DETERMINATION

The coefficient of determination of the RCR model has several desirable properties:

(I) R^2_{RCR} =1 if $\hat{Y} = Y$, which mean that a large R^2_{RCR} is evidence of a well fitting RCR model. R^2_{RCR} =0 implies that the inclusion of the regressor over and above the constant term offers no explanation. So

$$0 \le R^2_{RCR} \le 1, \tag{12}$$

(II) The coefficient of determination, R^2_{RCR}, is the square of the sample correlation coefficient, denoted by r^2_{RCR}, between $\left(I - sW^*\right)\Omega^{-\frac{1}{2}}Y$ and $\left(I - sW^*\right)\Omega^{-\frac{1}{2}}\hat{Y}$. That is

$$r^2_{RCR} = \frac{\left[\hat{Y}'\left(I - sW^*\right)'\Omega^{-1}\left(I - sW^*\right)Y\right]^2}{\hat{Y}'\left(I - sW^*\right)'\Omega^{-1}\left(I - sW^*\right)\hat{Y}.Y'\left(I - sW^*\right)'\Omega^{-1}\left(I - sW^*\right)Y}. \tag{13}$$

Because $\left(I - sW^{*}\right)'\Omega^{-1}\left(I - sW^{*}\right) = \Omega^{-1} - kW^{*'}W^{*}$, see equation A.9 in the appendix, we have $Y'\left(I - sW^{*}\right)'\Omega^{-1}\left(I - sW^{*}\right)Y = Y'\left(\Omega^{-1} - kW^{*'}W^{*}\right)Y$. The same holds for the other two quadratic forms. Further $\hat{Y}'\left(\Omega^{-1} - kW^{*'}W^{*}\right)Y = \hat{Y}'\left(\Omega^{-1} - kW^{*'}W^{*}\right)\hat{Y}$, because of $\left(I - sW^{*}\right)e = e$, and $x_i'\Omega_{ii}^{-1}e_i = 0$. So, we get

$$r_{RCR}^2 = R_{RCR}^2 . \qquad (14)$$

I) R_{RCR}^2 is invariant under changes of location dependent variable Y by $Y^* = Y + s\,\mu$, where μ is a scalar and it is not invariant if we consider a change of location of Y given by $Y^* = Y + \left(I_N \otimes \ell\right)\mu$, because of $\left(I_N \otimes \ell'\right)\left(\Omega^{-1} - kW^{*'}W^{*}\right)$ is not equal to zero, where $\mu' = \left(\mu_1, \mu_2, \ldots, \mu_N\right)$.

The estimated value of $Y^* = Y + s\,\mu$, where μ is a scalar, is $\hat{Y}^* = \hat{Y} + s\mu$ and because $s'\left(\Omega^{-1} - kW^{*'}W^{*}\right)s = 0$, we get

$$R_{RCR}^{*2} = \frac{\hat{Y}^{*'}\left(\Omega^{-1} - kW^{*'}W^{*}\right)\hat{Y}^{*}}{Y^{*'}\left(\Omega^{-1} - kW^{*'}W^{*}\right)Y^{*}}$$

$$= R_{RCR}^2 . \qquad (15)$$

٢) R_{RCR}^2 is invariant under changes of scale of the dependent variable Y given by $Y^* = \lambda\, Y$ and it is not invariant for $Y^* = \left(\Lambda \otimes I_T\right)Y$, with Λ a N x N diagonal matrix and Λ= diag ($\lambda_1, \lambda_2, \ldots, \lambda_N$). Premultiply equation (2) by $\left(\Lambda \otimes I_T\right)$, we get

$$Y^* = \tilde{X}\bar{\beta}^* + \varepsilon^*, \qquad (16)$$

where $Y^* = \left(\Lambda \otimes I_T\right)Y, \varepsilon^* = \left(\Lambda \otimes I_T\right)\varepsilon$, $\bar{\beta}^* = A\bar{\beta}$, $E\left(\varepsilon^*\right) = 0$, $\mathrm{var}(\varepsilon^*) = \Omega^*$ and $A' = \left(\lambda_1 I_K, \lambda_2 I_K, \ldots, \lambda_N I_K\right)$. The estimator $\hat{\bar{\beta}}^*$ of $\bar{\beta}^*$ is $A\,\hat{\bar{\beta}}$ and consequently $\hat{Y}^* = \tilde{X}A\hat{\bar{\beta}} = \left(\Lambda \otimes I\right)\hat{Y}$. It follows that

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$$R^{*2}_{RCR} = \frac{\hat{Y}^{*'}\left(\Omega^{*^{-1}} - k^{*}W^{**'}W^{**}\right)\hat{Y}^{*}}{Y^{*'}\left(\Omega^{*^{-1}} - kW^{**'}W^{**}\right)Y^{*}}, \qquad (17)$$

where

$$k^{*} = \sum_{i=1}^{N}\lambda_i^{-2}\ell'\Omega_{ii}^{-1}\ell$$

$$= k\lambda^{-2}, \qquad \text{if } \lambda_i = \lambda \text{ for all i}, \qquad (18)$$

$$W_i^{**} = \frac{1}{k^{*}}\ell'\Omega^{*^{-2}}_{ii}$$

$$= W_i^{*}, \qquad \text{if } \lambda_i = \lambda \text{ for all } i, \qquad (19)$$

and

$$(\Lambda \otimes I)\Omega^{*^{-1}}(\Lambda \otimes I) = \Omega^{-1}. \qquad (20)$$

Substitute (18), (19), and (20) in (17), we found that $R^{*2}_{RCR} \neq R^2_{RCR}$ and the equality hold if the scalar parameters are equal for all cross sectional units.

(V)When $W^{*} = \frac{1}{NT}s'$ and $\Omega = I\sigma^2$, the coefficient of determination of the GLS can be obtained, as special case from (11), as follows:

$$R^2_{RCR} = \frac{\hat{Y}'\left(I - \frac{1}{NT}ss'\right)\hat{Y}}{Y'\left(I - \frac{1}{NT}ss'\right)Y}, \qquad (21)$$

APPENDIX

In this appendix, we introduce some useful results that we used in the analysis.

RESULT (1):

$$(1)\ x_i'\left[x_i V x_i' + \sigma_i^2 I\right]^{-1} x_i = \left[V + \sigma_i^2 (x_i'x_i)^{-1}\right]^{-1}. \qquad (A.1)$$

$$(2)\ x_i'\left[x_i V x_i' + \sigma_i^2 I\right]^{-1} y_i = \left[V + \sigma_i^2 (x_i'x_i)^{-1}\right]^{-1} b_i. \qquad (A.2)$$

Proof A Taylor series expansion can be applied to Ω_{ii}^{-1} as

$$\Omega_{ii}^{-1} = \left[\sigma_i^2 I + x_i V x_i'\right]^{-1}$$
$$= \frac{1}{\sigma_i^2}\left[I - \Delta + \Delta\Delta - \Delta\Delta\Delta + \ldots\right], \qquad \text{(A.3)}$$

where

$$\Delta = \frac{1}{\sigma_i^2} x_i V x_i .$$

(1) The left hand side of (A.1) can be written, using (A.3), as follows:

$$x_i'\left[x_i V x_i' + \sigma_i^2 I\right]^{-1} x_i = \frac{1}{\sigma_i^2}\left[x_i' x_i - \frac{1}{\sigma_i^2} x_i' x_i V x_i' x_i + \left(\frac{1}{\sigma_i^2}\right)^2 x_i' x_i V x_i' x_i V x_i' x_i - \cdots\right]$$
$$= \frac{1}{\sigma_i^2}(x_i' x_i)\left[I + \frac{1}{\sigma_i^2} V (x_i' x_i)\right]^{-1}$$
$$= \left[V + \sigma_i^2 (x_i' x_i)^{-1}\right]^{-1}.$$

(2) Since $x_i'\left[x_i V x_i' + \sigma_i^2 I\right]^{-1} e_i = 0$, then

$$x_i'\left[x_i V x_i' + \sigma_i^2 I\right]^{-1} y_i = \left[V + \sigma_i^2 (x_i' x_i)^{-1}\right]^{-1} \hat{y}_i .$$

Using the result (A.1), we get

$$x_i'\left[x_i V x_i' + \sigma_i^2 I\right]^{-1} y_i = \left[V + \sigma_i^2 (x_i' x_i)^{-1}\right]^{-1} b_i .$$

RESULT (2):

$$\tilde{X}'\Omega^{-1}\tilde{X}' = \sum_{I=1}^{N} \left[V + \sigma_i^2 (x_i' x_i)^{-1}\right]^{-1}. \qquad \text{(A.4)}$$

Proof

$$\tilde{X}'\Omega^{-1}\tilde{X}' = \begin{pmatrix} x_1' & x_2' & \cdots & x_N' \end{pmatrix} \begin{pmatrix} \Omega_{11}^{-1} & 0 & \cdots & 0 \\ 0 & \Omega_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega_{NN}^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$= \sum_{i=1}^{N} x_i' \Omega_{ii}^{-1} x_i ,$$

using the Result (1), we get

$$\tilde{X}'\Omega^{-1}\tilde{X}' = \sum_{I=1}^{N} \left[V + \sigma_i^2 (x_i'x_i)^{-1}\right]^{-1}.$$

RESULT (3): *A coefficient of determination of the RCR model is*

$$R^2_{RCR} = \frac{\hat{Y}' \left(\Omega^{-1} - kW^{*'} W^{*}\right) \hat{Y}}{Y' \left(\Omega^{-1} - kW^{*'} W^{*}\right) Y}. \tag{A.5}$$

Proof Assume that every equation in (1) contains a constant term, so we can rewrite equation (2) as

$$Y = Z\overline{\beta}_Z + s\,\overline{\beta}_0 + \varepsilon .$$

To find R^2_{RCR} for this model, we have to estimate the constant term under the restriction that all other coefficients are zero. That is

$$\begin{aligned}
\hat{\overline{\beta}}_0 &= \left(s\Omega^{-1}s\right)^{-1} s\Omega^{-1}Y. \\
&= \left(\sum_{i=1}^{N} \ell'\Omega_{ii}^{-1}\ell\right)^{-1} \sum_{i=1}^{N} \ell'\Omega_{ii}^{-1}y_i \\
&= \sum_{i=1}^{N} W_i^{*} y_i \\
&= W^{*}Y \\
&= W^{*}\hat{Y}.
\end{aligned} \tag{A.6}$$

The equality holds because $W^{*} e = 0$. Following the procedure of Buse's definition, we find that the deviation value of Y from the weighted mean is

$$\begin{aligned}
Y - s\hat{\overline{\beta}}_0 &= Y - sW^{*}Y \\
&= (I - sW^{*})\hat{Y}.
\end{aligned} \tag{A.7}$$

Similarly,

$$\begin{aligned}
\hat{Y} - s\hat{\overline{\beta}}_0 &= \hat{Y} - sW^{*}\hat{Y} \\
&= (I - sW^{*})\hat{Y}.
\end{aligned} \tag{A.8}$$

Also

$$(I - sW^*)'\Omega^{-1}(I - sW^*) = (I - k\,W^{*'}\,W^*\Omega)\,\Omega^{-1}(I - k\,\Omega W^{*'}\,W^*)$$
$$= (\Omega^{-1} - k\,W^{*'}\,W^*)\,(I - k\,\Omega W^{*'}\,W^*)$$
$$= (\Omega^{-1} - k\,W^{*'}\,W^*). \qquad (A.9)$$

Applying (A.7), (A.8), and (A.9) in equation (10), the result in (A.5) will be satisfied.

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