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Empirical Bayes Approach For Accelerated Life Tests Considering the Weibull Distribution

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Abstract:

In this paper, we present a general technique for construction of the empirical Bayes (EB) methods considering accelerated life tests with m of the stress levels. Estimates of prior density based on orthogonal expansions are proposed for the conditional density. We obtain the EB estimators of the parameters of the inverse power model, and the parameters of the Weibull distribution.

This is a first part of a two-part series on the empirical Bayes approach of estimation using accelerated data. The theory and application are described briefly here, and step-by-step procedures for a method for EB estimation of the parameters of the Weibull distribution and the reliability function in the coming Part.

Key Words: Empirical Bayes - accelerated life testing - orthogonal polynomial - Laplace's method - reliability function.

1- Introduction

Accelerated life testing (ALT) of products and materials is used to get information quickly on their life distributions. Such testing involves subjecting the test units to conditions that are more sever than normal. This results in shorter lives than would be observed under normal conditions. Accelerated test conditions are typically produced by testing units at steady high levels of temperature, voltage, pressure, vibration, cycling rate, load, etc., or some combination of them. The use of certain accelerating or stress variables is a well-established engineering practice for many products and materials.

The results obtained at the more sever or accelerated conditions are extrapolated to the normal conditions to obtain an estimate of the life distribution under normal conditions. Such testing provides saving in time and cost compared with testing at normal conditions. Indeed, for many products and materials, life at normal conditions is so lengthy that testing at those conditions is completely out of the question.

A model that connects the distribution of accelerated failure times to the distribution of the failure time under usual conditions is then used, and the parameters in the model are estimated from accelerated life time data.

Let v be a one or higher dimensional stress applied to a device. In most of all applications, the relation between stress v and corresponding life time distribution was assumed to be given by the dependence of a statistical parameter λ on the applied stress v. Several authors have considered the problem of analyzing ALT by the classical and Bayesian approach (see for example, Nelson (1990), Mann, Schafer and Singpuralla (1974)). Now Bayesian analysis is carried out in the following way. If life time data $D = (t_{ij}, j = 1, 2, ..., n_i, i = 1, 2, ..., m)$ are observed for stress combination $v_1, v_2, ..., v_m$ the posterior distribution $\pi(\theta / D)$ of θ is obtained by Bayes theorem $\pi(\theta / D) \propto g(\theta) L(\theta; D)$, where $g(\theta)$ is the prior distribution of θ and $L(\theta;D)$ is likelihood function. Under the squared error loss, the Bayes estimator is the posterior mean $E(\theta/t)$, here θ is the set of all possible values of the parameter.

In the empirical Bayes (EB) procedure due principally of (Robbins (1956,1964) the prior distribution function is not known and usually it is estimated from a sample $D = (t_{ij}, j = 1, 2, ..., n_i, i = 1, 2, ..., m)$. This estimated used to obtain posterior distribution of θ by using Bayes rule. If one or more sample t_{i,n_i+1}, for every $i = 1, 2, ..., m$, is observed, then this sample and the posterior distribution are used to obtain EB estimator of the parameter θ_{EB}, an estimated of θ, corresponding to t_{i,n_i+1} or to estimate the value some function of the parameters.

Robbins (1956, 1964) was able to provide a set of consistent estimators, which, when the loss function was bounded in the parameter, are asymptotically optimal (a.o.). Therefore, we consider a sequence of iid pairs $\{(\theta_i, t_i), i \geq 1\}$ where θ_1 is generated according to an unknown (prior) distribution G and given θ_1, t_1 has distribution $f(t_1 / \theta_1)$. The θ's are non- observable and the t's are observable sequentially. At stage n+1, for any positive integer n, the problem is to estimate θ_{n+1}, under squared error loss, based on the past data $t_1, t_2, ..., t_n$ and the current data t_{n+1}. The (Bayes) risk incurred by an EB estimator $\hat{\delta}$ in estimating θ_{n+1} is given by

$$R_n(\hat{\delta}, G) = E\{\hat{\delta}(t_1, t_2, ... t_n; t_{n+1}) - \theta_{n+1}\}^2,$$

where as the (Bayes) risk $R(G)$ of δ_G, the Bayes estimator, is given by

$$R(G) = E\{\delta_G(t_{n+1}) - \theta_{n+1}\}^2,$$

the later being constant with respect to n. It can easily be checked that

$$R_n(\delta , G) \geq R(G)$$

for every finite n . An EB estimation δ is said to be asymptotically optimal {see Robbins (1956, 1964) } if

$$D_n(\hat{\delta} , G) = R_n(\delta , G) - R(G) \rightarrow 0, \text{as } n \rightarrow \infty ,$$

for all priors G ; which means that the performance of δ is asymptotically (as the amount of past data grows) as good as that of the Bayes estimator δ_G , no matter what the prior distribution G is. δ is said to be uniformly asymptotically optimal if the above convergence is uniform in G .

In this paper, we assume that the lifetime of any item under consideration is assumed to follow a Weibull distribution with a scale parameter λ and a shape parameter β, both of which are unknown; i.e. ,

$$F(t/\lambda,\beta) = 1 - \exp(-\lambda t^{\beta}) \quad , t > 0 \qquad (1)$$

we assume that the stress v affects only the scale parameter of the Weibull distribution λ through the inverse power law model, we assume that the shape parameter β is invariant with the stress, i.e. , it dose not change with v. The inverse Power law model is used when the life of a system is inversely proportional to an applied stress. The underlying failure rate distribution can be log-normal, or exponential, but the weibull distribution seems to be used more often than the others.

Two common applications of the inverse power law are for voltage stress, and for fatigue due to alternating stress. Fatigue may be low-cycle, such as that resulting from temperature cycling, or high-cycle, resulting from mechanical vibration. The general form of the inverse power law is

$$\lambda = c / s^{\alpha} \qquad (2)$$

where c , α are the physical parameters and consider an EB approach to be estimated, and s = 1/v is the applied stress.

In section 2, orthogonal expansion methods are used to show the relation between associated functions with more than one parameter and orthogonal polynomials. This technique, has been introduced by Huang ,Fu and Pao (1994) in the case of univariate and bivariate distribution with one parameter.

2 - General procedure

Consider the families of continuous distribution whose probability density function $f(x/\theta)$ where $\theta = (\theta_1, \theta_2, ..., \theta_L)$. Suppose the prior distribution given by, $G(\theta)$ is absolutely continuous with respectively density $g(\theta)$, then the marginal density is given by

$$f(x) = \int_{\theta} f(x/\theta) g(\theta) d\theta \tag{3}$$

We assume that f and $g \in L^2(0,\infty)$ and we approximate f and g by partial sums of their expansion with respect to the orthogonal polynomial on $(0,\infty)$, which are denoted by $L_k(\eta)$, such that

$$\int_0^\infty L_j(\eta) L_i(\eta) e^{-\eta} d\eta = \delta_{ij}$$

where δ_{ij} is the kroneckel delta, and $L_k(\eta)$ is define by

$$L_k(\eta) = \frac{e^{\eta}}{k!} \frac{d^k}{d\eta^k} \eta^k e^{-\eta} \quad \text{for } k = 0,1,2,... \tag{4}$$

Define also the polynomial

$$P_{k_1,k_2,...}(\theta_1,\theta_2,...) = \prod_j L_{k_j}(\theta_j),\ k = 0, 1, 2,.. \tag{5}$$

They are orthogonal and complete on $(0,\infty)$ with respect to the weight function $e^{-\Sigma\theta}$. Thus, for example orthogonality is defined as, for every k, we have

$$\iint_{\Omega} P_{k,i}(\theta_i,\theta_j) P_{k,j}(\theta_i,\theta_j) e^{-(\theta_i+\theta_j)} d\theta_i d\theta_j = \delta_{ij}$$

We assume that the density function of the random vector $\underline{\theta}$, $g(\underline{\theta})$ to be in the span of $P(\underline{\theta})$ in $L^2(e^{-\Sigma\theta_i},(0,\infty))$.

We now form the function

$$g(\theta) = \sum_i b_i p_i(\theta) e^{-\Sigma\theta/2} \tag{6}$$

thus (3) may be expressed as approximately,

$$f(x) = \int_{\Omega} f(x/\theta) \sum_i b_i P_i(\theta) e^{-\Sigma\theta_i/2} d\theta \tag{7}$$

New, we define the function

$$\phi_i(x) = \int_\Omega f(x/\theta)P_i(\theta)e^{-\Sigma\theta}d\theta \qquad (8)$$

so that $$f(x) = \Sigma b_i \phi_i(x) \qquad (9)$$

Suppose there exist associate sequence $\lambda(x)$ such that

$$E(\lambda(x)) = p(\underline{\theta})e^{-\Sigma\theta/2} \qquad (10)$$

where x is a random variable from the marginal density f(x) defined by (3), we note that $\phi_i(x)$ and $\lambda_j(x)$ are orthogonal, for every i and j, it follows that the associate coefficient b of f(x) in the expansion of $\phi_i(x)$ can be obtained as

$$b = \int f(\underline{x})\lambda(\underline{x})dx = E_x(\lambda(x)). \qquad (11)$$

The expectation is taken with respect to the marginal density function f(x). For single population, Prasal and Singh (1990) and Singh and Prasal (1989) have constructed a class of empirical Bayes estimators of the parameters through estimating the prior distribution. Without loss of generality in this paper, we seek to select the sequence of function $\hat{b}_i$, based on the orthogonal polynomial, such that the integrated mean square error of $\hat{g}_N(\theta)$ goes to zero as $N \rightarrow \infty$. We now use (11) to estimate b and use this in turn to estimate the prior density g.

3- Estimation of The Prior Distribution

Consider continuous probability densities which belongs to form $f(x/\theta)$ for the univariate random variable, with three parameters. Let x_1, $x_2, \ldots, x_N$ be the iid samples from the density function defined by (3).

From (6) and (11), an estimator $\hat{g}_N$ of g based on orthogonal expansions is proposed by the following

$$\hat{g}_N(\theta,\alpha,\beta) = \sum_{l=0}^{q_3(N)} \sum_{j=0}^{q_2(N)} \sum_{i=0}^{q_1(N)} \hat{b}_{i,j,l} \; p_{i,j,l}(\theta,\alpha,\beta)e^{-(\theta+\alpha+\beta)/2} \qquad (12)$$

where the statistics $\hat{b}_{i,j,l}$ is computed from a random sample of size N from the density (3) such that,

$$\hat{b}_{i,j,l} = (1/N)\sum_{k=1}^{N} \lambda_{i,j,l}(x_k) \tag{13}$$

and $q_1(N)$, $q_2(N)$ and $q_3(N)$ are positive integer valued functions of N such that $q_1(N)$, $q_2(N)$, $q_3(N) \to \infty$, with $\frac{q_1(N)q_2(N)q_3(N)}{N} \to 0$ as $N \to \infty$.

Huang ,Fu and Pao (1994) show that, the set $S = [\lambda_{i,j,l}, i,j,l = 0, 1, \dots]$, is linearly independent.

Theorem (1) :

Let $g(\theta, \alpha, \beta)$ can be expanded by (6) and if $\lambda_{i,j,l}(\underline{x})$, $i,j,l = 0, 1, \dots$ is uniformly bounded, then $\hat{g}(\theta,\alpha,\beta)$ is a strongly consist estimator of $g(\theta, \alpha, \beta)$ at continuity point, where $q_1(N)$, $q_2(N)$, $q_3(N) \to \infty$, with $\frac{q_1(N)q_2(N)q_3(N)}{N} \to 0$ as $N \to \infty$.

proof : (see [Huang ,Fu and Pao (1994)]).

By the extending Parseval's formula, we have the following lemma.

Lemma (1) :

Let $h(\theta, \alpha, \beta) \in L^2(e^{-(\theta+\alpha+\beta)}, (0, \infty))$ and suppose that

$$h(\theta,\alpha,\beta) = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} b_{i,j,l}\, p_{i,j,l}(\theta,\alpha,\beta) \tag{14}$$

where $p_{i,j,l}(\theta, \alpha, \beta)$ defined by (5). Then

$$\iiint h^2(\theta,\alpha,\beta)e^{-(\theta+\alpha+\beta)}d\theta d\alpha d\beta = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} b_{i,j,l} < \infty$$

and $\sum_{l=q_3(N)+1}^{\infty}\ \sum_{j=q_2(N)+1}^{\infty}\ \sum_{i=q_1(N)+1}^{\infty} b^2_{i,j,l} \to 0$ as $N \to \infty$.

Theorem (2) :

Let $x_1, x_2, \dots, x_n$ be iid random variables with density function given (3), let the orthogonal polynomial in $L^2(e^{-(\theta+\alpha+\beta)}, (0, \infty))$. Then, the estimator (12) is integrated mean square consistent for g, provided that

$\gamma_N \to 0$ as $N \to \infty$
where

$$\gamma_N = \sum_{l=0}^{q_3(N)} \sum_{j=0}^{q_2(N)} \sum_{i=0}^{q_1(N)} \text{var}(\lambda_{i,j,l}(x))$$

proof.

We first observe that $\hat{b}_{i,j,l}$ is unbiased consistent estimator of $b_{i,j,l}$ the coefficient of g. This follows form,

$$E(\hat{b}_{i,j,l}) = (1/N) \sum_{k=1}^{N} E(\lambda_{i,j,l}(x_k)) = b_{i,j,l} \tag{15}$$

The mean integrated square error (MISE) is given by.

$$E\iiint_0 [\hat{g}_N(\theta,\alpha,\beta) - g(\theta,\alpha,\beta)]^2 d\theta d\alpha d\beta = E \sum_{l=0}^{q_3(N)} \sum_{j=0}^{q_2(N)} \sum_{i=0}^{q_1(N)} [\hat{b}_{i,j,l} - b_{i,j,l}]^2$$

$$+ \sum_{l=q_3(N)+1}^{\infty} \sum_{j=q_2(N)+1}^{\infty} \sum_{i=q_1(N)+1}^{\infty} b^2_{i,j,l} = \sum_{l=0}^{q_3(N)} \sum_{j=0}^{q_2(N)} \sum_{i=0}^{q_1(N)} E[\hat{b}_{i,j,l} - b_{i,j,l}]^2$$

$$+ \sum_{l=q(N)+1}^{\infty} \sum_{j=q(N)+1}^{\infty} \sum_{i=q(N)+1}^{\infty} b^2_{i,j,l} \tag{16}$$

Using (14) and (15), then the first term in (16) may be expressed as,

$$E[\hat{b}_{i,j,l} - b_{i,j,l}]^2 = (1/N)\,\text{var}(\lambda_{i,j,l}(x)) \tag{17}$$

Since the set $S = (\lambda_{i,j,l}\ , i,j,l = 0,1,\ldots)$ is linearly independent, then

$$\sum_{l=0}^{q_3(N)} \sum_{j=0}^{q_2(N)} \sum_{i=0}^{q_1(N)} E[\hat{b}_{i,j,l} - b_{i,j,l}]^2 = (1/N) \sum_{l=0}^{q_3(N)} \sum_{j=0}^{q_2(N)} \sum_{i=0}^{q_1(N)} \text{var}(\lambda_{i,j,l}(x)) \to 0 \text{ as}$$

$N \to \infty$. since $\lambda_{i,j,l}$ is uniformly bounded in $(0, \infty)$. By lemma (1) we note that the second term of (16) goes to zero as $N \to \infty$. Hence the conclusion. □

Lemma (2)

Let the hypothesis of theorem (2) hold, then

$$E[\hat{g}(\theta,\alpha,\beta) - g(\theta,\alpha,\beta)]^2 \to 0 \text{ as } N \to \infty.$$

We should find a sequence of function $\{ \lambda_{i,j,l}(x)\ , i,j,l = 0,1,\ldots\}$ satisfies (11) and (14), this means that, we need to drive function $\lambda_{i,j,l}(x)$ such that, for every $i,j,l = 0, 1, \ldots, N$

$$\int f(x/\theta,\alpha,\beta)\lambda_{i,j,l}(x)dx = p_{i,j,l}(\theta,\alpha,\beta)e^{-(\theta+\alpha+\beta)\Omega} \qquad (17)$$

and then we can use the estimator of the prior distribution (12). In practical we often have a nice special cases when the exact solution $\lambda_{i,j,l}$ (x) of equation (17) exists. Since the right hand side of equation (17) is entirely known. it can be solved by the Fourier or Mellin transformation or Laplace method [see Penskaga (1994) and Tierney, Kass and Kadane (1989)]

Suppose we can observe n_i+1 sample from a marginal density f(t) under stress v_i , i= 1, 2,... , m, then we are interested in estimating the prior density g(θ, α, β) based on the first n_i ,i= 1, 2, ... ,m sample and to make use an addition sample t_{i,n_i+1} to estimate the parameters θ, α and β. Using the mean square loss error, then, the EB estimator of θ, α and β based on $t_{i,j}$, j = 1, 2,... , n_i +1, i = 1, 2 ... , m, from f(t), is given by

$$\hat{\psi}_N = \frac{\iiint \psi f(t/\theta,\alpha,\beta)\hat{g}(\theta,\alpha,\beta)d\theta d\alpha d\beta}{\iiint f(t/\theta,\alpha,\beta)\hat{g}(\theta,\alpha,\beta)d\theta d\alpha d\beta}$$

and ψ =θ ,α ,and β

In the rest of this paper, we study the above procedure of the empirical Bayes in the of the Weibull life time distribution.

4 - Empirical Bayes estimator for the Weibull distribution

Consider T as a random variable, which denoting the life of an unit with a Weibull density given by (1). We assume a stress variable v affecting the scale parameter λ, but with common shape parameter β for all stress level, with m levels of a stress variable, assume the inverse power rule model given (2), such that

$$\lambda_k = \theta.v_k^{\alpha}, \quad \text{for } k = 1,2,\ldots, m$$

where θ and α are unknown parameters. Thus, with n_k units at the beginning of each test with v, we have the observations given by $t_{1,k}, t_{2,k}, \ldots, t_{n_k,k}$, for k = 1, 2, ... , m, such that $\sum_{k=1}^{m} n_k = N$. Our objective now is to obtain estimators for the parameters θ and α , and to predict λ_u , the value of the scale parameter at use conditions stress v_u.

Considering the data of **m stress levels $v_1, v_2, \ldots, v_m$ taken at** random, then, the likelihood function for θ, α and β is given by

$$L(\theta,\alpha,\beta) = \prod_{k=1}^{m}\prod_{\ell=1}^{n_k} f(t_{k,\ell},\theta,\alpha,\beta)$$

$$= (\theta\beta)^{N} \prod_{k=1}^{m}\prod_{\ell=1}^{n_k} (t_{k,\ell})^{\beta-1} v_k^{\alpha} \exp\Big(-\theta \sum_{k=1}^{m}\sum_{\ell=1}^{n_k} v_k^{\alpha} (t_{k,\ell})^{\beta}\Big) \qquad (18)$$

Usually, in engineering applications different prior could be used in Bayesian analysis of the model. Since we have difficulties to get exact posterior or predictive densities of interest, our objective here is estimating the parameters by the EB approach. Firstly, we construct an estimator for the unknown prior distribution based on the orthogonal polynomial, then we derive the estimator of the parameters with respect to the estimated prior, otherwise we derive the sequence $\lambda_{i,j,l}(t_{k,\ell})$ for i,j,l = 1, 2, ... such that under stress v_k, we have

$$\int_0^{\infty} \lambda_{i,j,l}(t_{k,\ell})(t_{k,\ell})^{\beta-1} e^{-\theta v_k^{\alpha} t_{k,\ell}^{\beta}}\, dt = p_{i,j,l}(\theta,\alpha,\beta) e^{-(\theta+\alpha+\beta)/2} \qquad (19)$$

Using the Laplace's method for approximation of integrals [see Tierney, Kass and Kadane (1989)] we find an approximate sequence

$$\tilde{\lambda}_{i,j,l}(\dot{t}_k) = \sqrt{\frac{\beta-1}{2\pi\beta}}\, \frac{1}{\sqrt{\theta (v_k)^{\alpha} \dot{t}_k^{\beta}}}\, p_{i,j,l}(\theta,\alpha,\beta) e^{\theta v_k^{\alpha} \dot{t}_k^{\beta}} \qquad (20)$$

where $\dot{t}_k = \max(t_{k,\ell}, \ell = 1, 2, \ldots, n_k)$ for all k = 1, 2, ... , m. Accordingly, if the right hand side of (20) holds, we can obtain the estimates of the coefficient $b_{i,j,l}$ which is given by

$$\hat{b}_{i,j,l} = \sum_{k=1}^{m} \frac{n_k}{N} \tilde{\lambda}_{i,j,l}(\dot{t}_k) \qquad (21)$$

From (12) the estimator of the joint prior distribution $\hat{g}_N(\theta,\alpha,\beta)$ is given by

$$\hat{g}_N(\theta,\alpha,\beta) = \sum_{l=1}^{q_3(N)} \sum_{j=1}^{q_2(N)} \sum_{i=1}^{q_1(N)} \sum_{k=1}^{m} \frac{n_k}{N} \tilde{\lambda}_{i,j,l}(\dot{t}_k) p_{i,j,l}(\theta,\alpha,\beta) e^{-(\theta+\alpha+\beta)/2} \qquad (22)$$

Hereafter, we let t_k, for k = 1, 2, .., m, θ, α and β denote the $(n_k+1)^{th}$ observations t_{k+1}, θ_{k+1}, α_{k+1} and β_{k+1} respectively. Then the EB estimation of θ, α and β under squared error loss by utilizing all n_k+1 observations are given

$$\hat{\theta}_{EB} = \frac{\iiint \theta J(\theta,\alpha,\beta) d\theta d\alpha d\beta}{\iiint J(\theta,\alpha,\beta) d\theta d\alpha d\beta}$$

$$\hat{\alpha}_{EB} = \frac{\iiint \alpha J(\theta,\alpha,\beta) d\theta d\alpha d\beta}{\iiint J(\theta,\alpha,\beta) d\theta d\alpha d\beta}$$

and $$\hat{\beta}_{EB} = \frac{\iiint \beta J(\theta,\alpha,\beta) d\theta d\alpha d\beta}{\iiint J(\theta,\alpha,\beta) d\theta d\alpha d\beta} \qquad (23)$$

where

$$J(\theta,\alpha,\beta) = (\theta\beta)^N \prod_{k=1}^{m} v_k^{\alpha} (t_{k,n_k+1})^{\beta-1} \exp[-\theta \sum_{k=1}^{m} v_k^{\alpha} (t_{k,n_k+1})^{\beta}] . \hat{g}_N(\theta,\alpha,\beta).$$

The EB estimator of the scale parameter λ_u under the usual stress v_u can be obtained by using the following equation

$$\hat{\lambda}_u = \hat{\theta}_{EB} (v_u)^{\hat{\alpha}_{EB}} \qquad (24)$$

Canavos(1973) finds, for the Poisson distribution, that the EB point estimate of reliability function obtained by substituting the EB estimate of the failure rate into reliability function has a uniformly smaller EB risk than the estimator based on the expectation of the reliability. In our case, the EB estimator of the reliability functional mission time t_0, is given by

$$\hat{R}_u(t_0) = e^{-\hat{\lambda}_u (t_0)^{\hat{\beta}_{EB}}}$$

where $\hat{\lambda}_u$ and $\hat{\beta}_{EB}$ are given by (23) and (24)

5- Conclusions

In this paper we obtained formulas for the approximate prior density and empirical Bayes estimators under the squared loss. Since the accelerated life tests in reliability analysis are important in all applications, the obtained results could be of great practical interest. We obtain the empirical estimators of the parameter of the inverse power model, and the shape and scale parameters of the Weibull distribution .We also could obtain the estimators of the reliability function .

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