THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CAIRO UNIV., VOL., 42, NO.1, 1998

Sensitivity Analysis to Bayesian Multiple Hypothesis Testing Using Goal Programming

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Abstract:

The decision in the Bayesian hypothesis testing depends on the posterior probability distribution of the associated parameter and the loss function. In turn, the posterior probability distribution depends on the prior probability distribution of the parameter. Goal programming is used in this paper to study the robustness of the decision to the changes in the posterior probability and\or the loss function. The approach depends on the fact the hypothesis-testing problem can be reformulated in the language of a single stage decision theory. Three types of sensitivity analysis are considered in the paper; sensitivity analysis for the posterior (prior) probabilities keeping the loss function fixed, sensitivity analysis on the loss function keeping the probabilities fixed and sensitivity analysis on both the probabilities and the losses. The bounds of changes in the probability and\or the losses are determined by using the linear goal programming in the first and second case. For the joint sensitivity on both the probabilities and the losses, the problem is converted to a multiobjective program then to a nonlinear goal program. By using the suggested approach, the global optimal solution is attained in the first and second cases because of using linear goal programs. The nonlinearity in the third case can be managed by the priority ranking in the achievement function.

***Key words*: Multiple hypothesis testing, Bayesian approach, Prior probabilities, Posterior probabilities, Loss function, Goal programming.**

1- Introduction

The statistical hypothesis testing problem can be defined as: Having an unknown parameter θ which is known to be from a will defined set Θ, we want to know whether $\theta \in \Theta_0$ or $\theta \in \Theta_1$ where:

$\Theta_0 \cup \Theta_1 = \Theta$ and $\Theta_0 \cap \Theta_1 = \varnothing$. By using a set of observations x (sample) whose density $p(x|\theta)$ depends on θ, one can perform a test to decide whether $\theta \in \Theta_0$ or $\theta \in \Theta_1$. This problem is written in the following form:

$H_0 : \theta \in \Theta_0$ as the null hypothesis and $H_1 : \theta \in \Theta_1$ as the alternative hypothesis.

In the classical statistics the test is decided by a rejection region R defined as [10,13] :

R = {x; observing x would lead to the rejection of H_0}.

R is determined by using two types of probabilities; the probability of type I error; α; and the probability of type II error; β;. α and β are defined as:

$\alpha = p(R| \theta \in \Theta_0)$ and $\beta = 1 - p(R| \theta \in \Theta_1)$.

The Bayesian approach is more straightforward. The decision is taken based on the prior and posterior probabilities of θ. The prior probabilities of θ are defined as [3,13]:

$\pi_0 = p(\theta \in \Theta_0)$ and $\pi_1 = p(\theta \in \Theta_0)$ and the posterior probabilities are given by:

$p_o = p(\theta \in \Theta_0 |x)$ and $p_1 = p(\theta \in \Theta_1 |x)$. By using the prior and posterior probabilities of θ, the Bayes factor B is defined as:

$B = p_0\pi_1/p_1\pi_0$.

The Bayes factor can be interpreted as the odds in favor of H_0 against H_1 given by the data and is used to choose between H_0 and H_1.

The comparison between the classical approach and the Bayesian approach can be found in [3,13]. On the other hand the hypothesis testing problem can be reformulated in the language of decision theory [3,13] and in this case the losses of taking the false decision is to be taken into consideration together with the prior and posterior probabilities. The robustness of the decision to the changes of the prior and posterior probabilities and/or the loss function is the ultimate goal of this paper. Section 2 of this paper presents the sensitivity analysis in the general decision theory problem. The sensitivity analysis in the Bayesian multiple hypothesis testing is considered in section 3 and numerical examples for illustration are given in section 4.

Sensitivity analysis is performed through this paper by using goal programming. The general form of goal programming is as follows:

Lexicographically minimize Z:

$$Z = \{ g_t(n_i,v_i) \mid t=1,2,\dots,T\} \quad (1)$$

Subject to:

$$f_i(y) + n_i - v_i = b_i \qquad i=1,2,\dots, I \quad (2)$$

$$y \geq 0 , n_i \geq 0 , v_i \geq 0 \qquad i=1,2,\dots, I \quad (3)$$

$$n_i . v_i = 0 \qquad i=1,2,\dots, I \quad (4)$$

Where y is the vector of the decision variables, $f_i(.)$ is a real valued function ; n_i and v_i are the negative and positive deviational variables, respectively, for the i^{th} goal , $g_t(.)$ is a real valued function for the goal with priority level t ; and T is the number of priority levels . An intensive review on goal programming formulations, solutions and applications can be found in [14,16,17]. One of the important properties of goal programming [4,11] is that :

$n_i + v_i = | f_i(y) - b_i |$

This means that if we need to minimize $| f_i(y) - b_i |$, then the equivalent goal program can be attained by putting $g_t(n_i,v_i)= n_i + v_i$ in (1) and solving (1)-(4). This property will be used through this paper to reformulate the resulting models to goal programs.

2- Sensitivity analysis in decision theory

Single stage decision theory is defined as choosing one alternative from among a finite number of alternatives. Many criteria were suggested to determine the best alternative [2,12]. If more than one state of nature is known to exist, and the probability of each state is known, then the best alternative is the one that maximizes the expected value of the payoffs. We can also define the next best alternative as the alternative with the closest expected payoff to the best-expected payoff.

Let $A = \{ A_1, A_2, ..., A_m)$ be the set of alternatives, $S = \{S_1, S_2, ..., S_n)$ be the states of nature, and $P(S=S_j) = p_j$ be the probability that S_j occurs. Let the payoff matrix be defined by $H = \{h_{ij}\}$ where h_{ij} is the payoff when the alternative A_i is selected and the state of nature turns out to be S_j. The best alternative depends on both p_j and h_{ij}. The sensitivity analysis of the single stage decision making involves determining bounds of changes in the parameters (p_j and/or h_{ij}) at which the optimal action remains the same. The problem as defined here was studied in [5,6] and nonlinear programming was used.To study the sensitivity analysis on probabilities, let the optimal alternative be A_k i.e :

$$\sum_{j=1}^{n} h_{kj}\, p_j \geq \sum_{j=1}^{n} h_{ij} p_j \quad \text{for all } i \neq k \qquad (5)$$

and let A_r be the next best action .

If $X = (x_1\ x_2 \ldots x_n)$ represents an arbitrary probability vector; then the changes from P to X can be measured by the following or other forms :

$$D(X,P) = \left(\sum_{j=1}^{n} (x_j - p_j)^2\right)^{1/2} \qquad (6)$$

$$D(X,P) = \sum_{j=1}^{n} |x_j - p_j| \qquad (7)$$

We will use expression (7) in this paper because it leads to a linear goal program. The problem now is converted to finding the values of x_j that:

$$\text{Minimize } D(X,P) = \sum_{j=1}^{n} |x_j - p_j| \qquad (8)$$

Subject to:

$$\sum_{j=1}^{n} h_{rj}\, x_j = \sum_{j=1}^{n} h_{kj}\, x_j \qquad (9)$$

$$\sum_{j=1}^{n} h_{rj}\, x_j \geq \sum_{j=1}^{n} h_{ij} x_j \quad \text{for all } i \neq r \text{ and } i \neq k \qquad (10)$$

$$\sum_{j=1}^{n} x_j = 1 \qquad (11)$$

$$x_j \geq 0 \qquad j=1,2,\ldots,n \qquad (12)$$

Also, to study the sensitivity analysis on the payoffs, let the changes in h_{ij} be tij . This means that the new values of the payoffs become $h_{ij} + t_{ij}$. Holding the values of p_j fixed, what are the minimum values of t_{ij} that makes A_r as best as A_k and better than other actions? . To answer this question the following program will be solved

$$\text{Minimze} \sum_{i=1}^{m} \sum_{j=1}^{n} |t_{ij}| \tag{13}$$

Subject to :

$$\sum_{j=1}^{n} (h_{rj}+t_{rj})\, p_j = \sum_{j=1}^{n} (h_{kj}+t_{kj}) p_j \tag{14}$$

$$\sum_{j=1}^{n} (h_{rj}+t_{rj})\, p_j \geq \sum_{j=1}^{n} (h_{ij}+t_{ij}) p_j \quad \text{for all } i \neq r \text{ and } k \tag{15}$$

To find the minimum changes in both the probabilities and the payoffs until A_r becomes the best action, the previous two programs will be combined in the following multi-objective program :

$$\text{Minimize} \sum_{j=1}^{n} |x_j - p_j| \,,\, \sum_{i=1}^{m} \sum_{j=1}^{n} |t_{ij}| \tag{16}$$

Subject to:

$$\sum_{j=1}^{n} (h_{rj}+t_{rj})\, x_j = \sum_{j=1}^{n} (h_{kj}+t_{kj}) x_j \tag{17}$$

$$\sum_{j=1}^{n} (h_{rj}+t_{rj})\, x_j \geq \sum_{j=1}^{n} (h_{ij}+t_{ij}) x_j \quad \text{for all } i \neq r \text{ and } i \neq k \tag{18}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{19}$$

$$x_j \geq 0 \qquad j=1,2,\dots,n \tag{20}$$

Each of the first two mathematical programs (8)-(12) and (13)-(15) are special cases o f the third program (16)-(20). Putting $t_{ij} = 0$ in (16)-(20) gives the program (8)-(12) while putting $x_j = p_j$ gives program (13)-(15). In the following the programs (8)-(12) and (13)-(15) will be transformed to a linear goal program while the third program (16)-(20) will be transformed to a nonlinear goal program. The linear goal program can be easily solved using any of the sequential or multiphase simplex methods, while the nonlinear goal program can be solved by using the sequential approach of nonlinear programming [4,7,8,9].

First case : determining the changes in the probabilities :

To convert the program (8)-(12) to a goal program, the objective function will be transformed to a goal constraint at priority level 2, the other constraints will be considered at priority level 1 (absolute goals); and by adding the negative and positive deviational variables the following goal program is obtained:

Find x_j (j=1,2,...,n) so as to lexicographically minimize Z_l :

$$Z_l = \{ v_{rk} + \sum_{i=1}^{m} u_{ri} + u + v, \sum_{j=1}^{n} (w^1_j + w^2_j) \} \quad (21)$$

Subject to:

$$x_j + w^1_j - w^2_j = p_j \quad j=1,2,\ldots,n \quad (22)$$

$$\sum_{j=1}^{n} h_{rj} x_j - \sum_{j=1}^{n} h_{kj} x_j + u_{rk} - v_{rk} = 0 \quad (23)$$

$$\sum_{j=1}^{n} h_{rj} x_j - \sum_{j=1}^{n} h_{ij} x_j + u_{ri} - v_{ri} = 0 \ \text{ for all } i \neq r \text{ and } i \neq k \quad (24)$$

$$\sum_{j=1}^{n} x_j + u + v = 1 \quad (25)$$

$$x_j \geq 0 \quad j=1,2,\ldots,n \quad (26)$$

In the program (21)-(26) minimizing $u_{rk} + v_{rk}$ is equivalent to satisfying (5) i.e. to make the alternative A_r as best as A_k, minimizing $\sum u_{ri}$ is equivalent to satisfying (6) i.e. to make A_r better than other alternatives except A_k, minimizing u+v is equivalent to satisfying (7), and minimizing $\sum_{j=1}^{n}(w^1_j + w^2_j)$ is equivalent to minimizing $\sum_{j=1}^{n} |x_j - p_j|$ as a property of goal programming.

The goal program (21)-(26) is linear and can be solved by the sequential or multiphase approaches [8] .

Second case : determining the changes in the payoffs :

Using the same approach of the first case, the program (13)-(15) can be transformed to the following goal program:

Find t_{ij} that lexicographically minimize Z_2:

$$Z_2 = \{v_{rk} + \sum_{i=1}^{m} u_{ri} , \sum_{i=1}^{m} \sum_{j=1}^{n} (w_{ij}^1 + w_{ij}^2)\} \quad (27)$$

Subject to:

$$\sum_{j=1}^{n} (h_{rj} + t_{rj})p_j - \sum_{j=1}^{n} (h_{kj} + t_{kj})p_j + u_{rk} - v_{rk} = 0 \quad (28)$$

$$\sum_{j=1}^{n} (h_{rj} + t_{rj})\, p_j - \sum_{j=1}^{n} (h_{ij} + t_{ij})p_j + u_{ri} - v_{ri} = 0 \text{ for all } i \neq r \text{ and } i \neq k \quad (29)$$

$$t_{ij} + w_{ij}^1 - w_{ij}^2 = 0 \quad i=1,2,\ldots,m \quad j=1,2,\ldots,n \quad (30)$$

$$u_{rk}, v_{rk}, u_{ri}, v_{ri}, w_{ij}^1, w_{ij}^2 \geq 0 \quad i=1,2,\ldots,m \quad j=1,2,\ldots,n \quad (31)$$

In the program (27)-(31) minimizing $u_{rk} + v_{rk}$ is equivalent to satisfying (10) i.e. to make the alternative A_r as best as A_k, minimizing $\sum u_{ri}$ is equivalent to satisfying (11) i.e. to make A_r better than other alternatives except A_k, and minimizing $\sum_{i=1}^{m} \sum_{j=1}^{n} (w_{ij}^1 + w_{ij}^2)$ is equivalent to minimizing $\sum_{i=1}^{m} \sum_{j=1}^{n} |t_{ij}|$ as a property of goal programming.

The goal program (27)-(31) is linear and can be solved by the sequential or multiphase approaches [8].

Third case : determining the simultaneous changes in the probabilities and the payoffs :

To transform the program (16) – (20) to a goal program, the same approach of the previous cases will be used. The resulting program in this case is the following nonlinear goal program.

Find x_j and t_{ij} so as to lexicographically minimize Z_3 :

$$Z_3 = \{ v_{rk} + \sum_{i=1}^{m} u_{ri} + u + v , \sum_{j=1}^{n} (w^1_j + w^2_j) , \sum_{i=1}^{m} \sum_{j=1}^{n} (w_{ij}{}^1 + w_{ij}{}^2)) \qquad (32)$$

Subject to :

$$\sum_{j=1}^{n} (h_{rj}+t_{rj})\, x_j - \sum_{j=1}^{n} (h_{kj}+t_{kj})x_j + u_{rk} - v_{rk} = 0 \qquad (33)$$

$$\sum_{j=1}^{n} (h_{rj}+t_{rj})\, x_j - \sum_{j=1}^{n} (h_{ij}+t_{ij})x_j + u_{ri} - v_{ri} = 0 \quad \text{for all } i \neq r \text{ and } i \neq k \qquad (34)$$

$$\sum_{j=1}^{n} x_j + u + v = 1 \qquad (35)$$

$$x_j + w^1_j - w^2_j = p_j \qquad j=1,2,\dots,n \qquad (36)$$

$$t_{ij} + w_{ij}{}^1 - w_{ij}{}^2 = 0 \quad i=1,2,\dots,m \quad j=1,2,\dots,n \qquad (37)$$

$x_j \geq 0$, all deviational variables ≥ 0, and t_{ij} are unrestricted (38)

The nonlinear goal program (32) - (38) can be solved using the sequential penalty algorithm [4,8].

3-Sensitivity analysis of the decisions

in the Bayesian multiple hypothesis testing

The statistical hypothesis testing can be thought of as a decision-making problem, in which one has to choose between two (or more) hypotheses. Let us assume that we have the following hypotheses:

H_i: $\theta \in A_i$ $\quad$ i=1,2,...,K (39)

Where A_i (i=1,2,...,K) are mutually exclusive and exhaustive sets. Furthermore, from a Bayesian point of view, the decision-maker can assign a prior distribution of θ and using both the prior information and the sample datum the posterior distribution of θ, $P(H_i) = P(\theta \in A_i)$,can be computed. The other input to the decision problem in the case of the hypothesis testing is the loss function. Let L(i,j) be the loss occurring if H_i is accepted while H_j is true. This situation can be represented by a single stage decision problem. The states of nature in this case are S_j={ H_j is true} (j=1,2,...,K) . The posterior probabilities are $P(S_j)$=P(H_j is true). The alternatives are $A_i \equiv$ accept H_i (i=1,2...,K). The associated losses are $\{h_{ij}\}$ (i=1,2,...,K,j=1,2,...,K) where:

$h_{ii}=0$, $h_{ij}=L(i,j)$ $\quad i \neq j$ (40)

According to the Expected Value Criteria, the optimal decision is the one with minimum expected loss. It can be proved that H_i is to be accepted if :

$$\sum_{j=1}^{K} h_{ij} P(H_j) < \sum_{j=1}^{K} h_{rj} P(H_j) \qquad r=1,2,...,K , r \neq i \tag{41}$$

In the case of having only two hypotheses, H_1 and H_2, H_1 is to be accepted if :

$P(H_1)/P(H_2) < h_{21}/h_{12}$ (42)

The left hand side of (42) is called the posterior odds ratio in favor of H_1 in the Bayesian approach while it is called the likelhood ratio in the classical approach (assuming equal prior probabilities for H_1 and H_2). The loss ratio in the right hand side is not used in the classical approach, instead the classical statistician can determine a rejection region in terms of the likelhood ratio and to some value of α (level of significance) [11].

Now, starting from a specific posterior distribution, if the best alternative is H_i and the next to best is H_r, then how sensitive is our decision to the changes in the posterior distribution function and therefore to the changes in the prior distribution?

The suggested model in section 2 (first case) can be used to answer the previous question as follows:

Lexicographically minimize Z_4 :

$$Z_4 = \{u_{ri} + \sum_{k=1}^{K} v_{rk} + u + v, \sum_{j=1}^{K} (w^1_j + w^2_j)\} \tag{42}$$

Subject to:

$$x_j + w^1_j - w^2_j = P(H_j) \quad j=1,2,\ldots,k \tag{43}$$

$$\sum_{j=1}^{K} h_{rj}\, x_j - \sum_{j=1}^{K} h_{ij} x_j + u_{ri} - v_{ri} = 0 \tag{44}$$

$$\sum_{j=1}^{K} h_{rj}\, x_j - \sum_{j=1}^{K} h_{kj} x_j + u_{rk} - v_{rk} = 0; \quad k=1,2,3,\ldots,K \; ; k \neq r \; ; k \neq i \tag{45}$$

$$\sum_{j=1}^{K} x_j + u + v = 1 \tag{46}$$

$$x_j \geq 0 \qquad j=1,2,\ldots,K \tag{47}$$

The goal program (42) – (47) finds the minimum changes in the posterior probabilities (equation 43) that makes the r^{th} decision as best as the i^{th} decision (equation 44) and better than other decisions (equation 45).

To study the robustness of the decision to the changes in the loss function, the goal program (27) – (31) can be used as follows:

Find t_{ij} that lexicographically minimize Z_2:

$$Z_2 = \{v_{rk} + \sum_{i=1}^{K} u_{ri} \, , \sum_{i=1}^{K} \sum_{j=1}^{K} (w_{ij}^1 + w_{ij}^2)\} \tag{48}$$

Subject to:

$$\sum_{j=1}^{K} [(h_{rj}+t_{rj})p_j - (h_{kj}+t_{kj})p_j] + u_{rk} - v_{rk} = 0 \tag{49}$$

$$\sum_{j=1}^{K} [(h_{rj}+t_{rj})p_j - (h_{ij}+t_{ij})p_j] + u_{ri} - v_{ri} = 0 \quad \text{for all } i \neq r \text{ and } i \neq k \tag{50}$$

$$t_{ij} + w_{ij}^1 - w_{ij}^2 = 0 \quad i=1,2,\ldots,K \quad j=1,2,\ldots,K \tag{51}$$

$$u_{rk}, v_{rk}, u_{ri}, v_{ri}, w_{ij}^1, w_{ij}^2 \geq 0 \quad i=1,2,\ldots,K \quad j=1,2,\ldots,K \tag{52}$$

The goal program (48) – (52) finds the minimum changes in the losses (equation 51) that makes the r^{th} decision as best as the i^{th} decision (equation 49) and better than other decisions (equation 51).

If both the posterior probabilities and the loss function are to be changed, the bounds of changes in each that keep the current decision as best as the next to best one, can be determined by solving the following goal program:

Find x_j and t_{ij} so as to lexicographically minimize Z_3 :

$$Z_3 = \{ v_{rk} + \sum_{i=1}^{K} u_{ri} + u + v \, , \sum_{j=1}^{K} (w_j^1 + w_j^2) \, , \sum_{i=1}^{K} \sum_{j=1}^{K} (w_{ij}^1 + w_{ij}^2)\} \tag{53}$$

Subject to :

$$\sum_{j=1}^{K} (h_{rj}+t_{rj})\, x_j - \sum_{j=1}^{K} (h_{kj}+t_{kj})x_j + u_{rk} - v_{rk}=0 \quad (54)$$

$$\sum_{j=1}^{K} (h_{rj}+t_{rj})\, x_j - \sum_{j=1}^{K} (h_{ij}+t_{ij})x_j + u_{ri} - v_{ri}=0 \text{ for all } i \neq r \text{ and } i \neq k \quad (55)$$

$$\sum_{j=1}^{K} x_j + u + v = 1 \quad (56)$$

$$x_j + w^1_j - w^2_j = p_j \qquad j=1,2,\ldots,K \quad (57)$$

$$t_{ij} + w_{ij}^1 - w_{ij}^2 = 0 \quad i=1,2,\ldots,K \quad j=1,2,\ldots,K \quad (58)$$

$x_j \geq 0$, all deviational variables ≥ 0, and t_{ij} are unrestricted (59)

The goal program (53) – (59) finds the minimum changes in both the posterior probabilities (equation 57) and the losses (equation 58) that make the r^{th} decision as best as the i^{th} decision (equation 54) and better than other decisions (equation 55).

4- numerical examples

Bayesian hypotheses testing for the mean of a normal distribution:

Suppose that a sample of size n=10 is drawn from a normal distribution with unknown mean μ and known variance $\sigma^2=400$ and the sample mean is 200. If the prior distribution of μ is normal with mean $M_1=220$ and variance $\sigma_1^2=25$, then the posterior distribution of μ is normal with mean M_2 and variance σ_2^2 [11]. M_2 is the weighted average of the prior mean and the sample mean, the weights being the reciprocals of the respective variances. The reciprocal of σ_2^2 is equal to the sum of the reciprocal of the prior variance and the reciprocal of the variance of M. i.e.:$M_2 = (W_1 M_1 + W_2 M)/(W_1 + W_2)$ and $1/\sigma_2^2 = W_1 + W_2$ where $W_1 = 1/\sigma_1^2$ and $W_2 = n/\sigma^2$ [11]. In this example $M_2 = 212.3$ and $\sigma_2^2 = 15.38$.

Starting with these information, if one is going to choose between the following three hypotheses:

H_1: $210 \leq \mu \leq 220$ H_2: $\mu < 210$ and H_3: $\mu > 220$ then the posterior probabilities of each of H_1, H_2, and H_3 being true can be computed from the posterior distributions of μ as follows:

Hypotheses H_j	H_1	H_2	H_3
Posterior probability P (H_j)	0.695	0.28	0.025

Suppose that the loss function is defined by $L(i,j)=100(\mu_i - \mu_j)^2$, where:

$\mu_i = E(\mu \mid \mu \in H_i)$. By using the definition of the expected value of truncated normal distribution, $\mu_1=213.6$, $\mu_2=207.6$, and $\mu_3=221.2$. Accordingly the losses can be presented in the following Table:

Alternatives	H_1 is true $P(H_1)=0.695$	H_2 is true $P(H_2)=0.28$	H_3 is true $P(H_3)=0.025$
Accept H_1	0	3600	5776
Accept H_2	3600	0	18496
Accept H_3	5776	18496	0

Using the expected loss principle, the best decision is to accept H_1 and the next to best decision is to accept H_2. To determine the minimum changes in the posterior distribution that makes H_2 the best alternative, the following goal program is to be solved:

Find x_1, x_2 and x_3 so as to:

Lexicographically minimize Z_4 :

$$Z_4=\{u_{21}+v_{21}+v_{23}+u+v,\ w'_1+w^2_1 + w'_2+w^2_2 + w'_3+w^2_3\} \quad (48)$$

Subject to:

$x_1 + w^1_1 - w^2_1 = 0.695$ (49)

$x_2 + w^1_2 - w^2_2 = 0.28$ (50)

$x_3 + w^1_3 - w^2_3 = 0.025$ (51)

$3600\ x_1 - 3600\ x_2 + 12720\ x_3 + u_{21} - v_{21} = 0$ (52)

$-2176\ x_1 - 18496\ x_2 + 18496\ x_3 + u_{23} - v_{23} = 0$ (53)

$x_1 + x_2 + x_3 + u + v = 1$ (54)

$x_j \geq 0 \qquad j=1,2,3$ (55)

By using the Micro-Manger Software, the optimal solution of the LGP (48)-(55) is: x_1*=0.5 , x_2*= 0.5 and x3*= 0 and in this case the expected losses of accepting H_1 is equal to the expected losses of accepting H_2 (1800) . This result can be used to determine the associated changes in the parameters of prior and posterior distributions. For example if the prior distribution becomes normal but with parameters M_1=240 and σ_1^2=50 . Also another sample with the same size (n=10) was drawn from the same population and gave mean M=180. What is the effect of these changes on the best decision? To answer this question all we need is to calculate the posterior probabilities of each hypothesis being true and check if the changes in these posterior probabilities are within the ranges that was determined by the goal program (48)-(55) or not. The new posterior distribution is normal with mean M_2= 206.67 and variance σ_2^2 = 22.22. accordingly the posterior probabilities of the hypothesis being true are:

$P(H_1)$= 0.237 , $p(H_2)$=0.76 , $p(H_3)$=0.003

These posterior probabilities are out of the allowed range to accept H_1. H_1 is no longer the best decision according to the new information. $P(H_1)$=0.237 is less than its lower bound (0.5) and $p(H_2)=0.76$ is higher than its largest bound (0.5). This result can be checked if we calculate the expected loss for each decision.

By using the same approach together with the suggested goal programs in section 3, the robustness of the decision to the changes in the loss function can be investigated for this example.

Conclusion

Goal programming is used in this paper to study the changes in the components of the Bayesian hypothesis testing problems. The paper includes three types of sensitivity analysis. Sensitivity analysis for the prior and hence the posterior probabilities of the tested parameter keeping the loss function fixed, sensitivity analysis on the loss function keeping the prior and\ posterior probabilities fixed and joint sensitivity analysis on both the posterior probabilities and the loss function. The suggested approach can be used for the general single stage decision theory. The resulting programs are linear goal programs in the first and second cases and nonlinear goal program in the joint sensitivity analysis case.

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