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Tests For the New Renewal Better than Used Classes of Life distributions

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ABSTRACT

The new renewal better than used property is one of the newly introduced aging properties in the Literature . This property and its dual play an important role in formulating repair or replacement policies. In this work, we test exponentiality versus the new renewal better than used property or its dual . For this purpose the total time on test transform technique is used and an asymptotic distribution of this test is established.

1- Introduction :

Classes of life distributions in reliability theory have been classified according to the aging properties in different ways. One of the classifications is based on the monotonicity of the failure rates or functions their of . Such classifications are discussed by Bickel, and Doksum (1969), Bryson and siddiqui (1969), Rolski (1975) Barlow and Proschan (1981), Klefsjo (1983), Loh (1984) and Deshpand etal (1986). Asecond type of classification is based on the mean residual life of a Component with life distribution F. This is discussed in detail by Abouammoh (1988). Abouammoh and Khalique (1987) have discussed the testing of these classes against exponentiallity for Complete samples using total time on test transform technique .

Another classification based on renewal failure rate was introduced by Abouammoh and Ahmed (1992).

Suppose a unit with distribution F is in operation. Upon Failure, the unit is replaced by a sequence of mutually independent units, independent of the first unit, and identically distributed with the same life distribution F. The Life of the unit in operation is given by the Stationary renewal distribution

$$W_F(t) = (1/\mu) \int_0^t \bar{F}(\mu)\, du, \quad o \le t < \infty \qquad (1)$$

Where $\mu = \int_0^{\infty} \bar{F}(u)\, du$, is the mean of the distribution F, and $\bar{F}(.) = 1 - F(.)$. By Com paring the failure rates $r(t)$ and $r_{W_F}(t)$ for $t \geq 0$, Abouammoh and Ahmed defined the following classes

Definition 1. Alife distribution F on $[0, \infty)$ with failure rate $r_F(t)$ is said to be new better than renwal failure rate (NBURFR) if $r_F(0) \leq r_{W_F}(t) \quad \forall\, t \geq 0$.

Difinition 2. A life distribution F on $[0, \infty)$ with failure rate $r_F(t)$ is said to be new better than average renewal failure rate (NBARFR) if

$$r_F(0) \leq t^{-1} \int_0^t r_{W_F}(u)\, du \qquad \forall\, t \geq 0.$$

They showed the relation ship of those classes with other classes of life distributions and studied their closure properties under mixtures and Convolutions .

In (1994) Abouammoh and Ahmed introduced the new renewal better than used classes of life distribution (NRBU).

Using (1), one get the survival of renewal distribution as

$$\bar{W}_F(t) = \frac{1}{\mu} \int_t^{\infty} \bar{F}(u)\, du$$

and the failure rate of W is given by

$$r_w(t) = -[\,1/\bar{W}_F(t)\,] \frac{d}{dt} \bar{W}_F(t) = \bar{F}(t) \Big/ \int_t^{\infty} \bar{F}(u)\, du = \frac{1}{\mu(t)}$$

Where $\mu(t)$ is the mean remaining life of the parent distributian F at time t. The mean remaining life of W is given by

$$\mu_{w}(t) = [\,1 / \bar{W}(t)\,] \int_t^{\infty} \bar{W}(x)\ dx$$

$$= [\,1 / \int_t^{\infty} \bar{F}(u)\,du\,] \int_t^{\infty} \int_x^{\infty} \bar{F}(u)\,du\ dx$$

$$= \int_t^{\infty} \mu(x)\,\bar{F}(x)\,dx \;/\; \mu(t)\,\bar{F}(t).$$

the mean remaining life of the parent distribution F is defined as ,

$$\mu(t) = E[\,T-t \setminus T > t\,] = [\,1 / \bar{F}(t)\,] \int_t^{\infty} \bar{F}(u)\,du .$$

For Comparison between $\mu(t)$ and $\mu_{w}(t)$, let T be anonnegative Continuous random variable with distribution F such that $F(0-) = 0$, Survival $\bar{F}$ and density f. Let T_w be a renewal random variable of T with distribution W given by relation (1), Survival given by $\bar{W}$ and density function $w(t) = \frac{d}{dt} W(t)$. Let T_x be the random variable of the Conditional distribution $\bar{F}_x(t) = \bar{F}(t+x) / \bar{F}(x)$.

Definition 3. Arandom variable T, its distribution F, its survival $\bar{F}$ or its density f is said to have new renewal better than used parent property, denoted by NRBU if $T_t \leq^{st} T_w$ ie T_t is strictly less than or equal to T_w.

The definition implies that T is NRBU if

$$\bar{F}_x(t) \leq \bar{W}(t),\ t \geq 0 \qquad (2)$$

The dual class new renewal worse than used parent property denoted by NRWU is defined by reversing the inquality in (2) T is, of course, NRBU and NRWU if and only if it is an exponentail random variable.

Definition 3 is introduced by Abouammoh and Ahmed (1994) They have studied the reliability properties of the NRBU class and suggested the testing of exponentiality against this class via total time on test transform . El Arishy and khalique (1995). have formulated the test statistic for testing exponenliality versus the NRBU or its dual using total time on test transform technique and simulated the percentile points for different sample sizes which indicated that a suitable transformation may be found to obtain normality of the test statistic .

In this paper, we establish the asymptotic distribution of the test statistic given by El Arishy and Khalique . Theoretically. In section 2 , we present some preliminaries, and section 3. In cludes presentation of this test and the aasymptotic distribution of the test statistic. With comments and conclusions in section 4 .

2. Preliminary results

In this section we introduce some results which will be useful to establish the main problem .

Lemma. 1. let $X_1, X_2, \ldots X_n$ be arandom sample from the exponential distribution with parameter $\lambda > 0$, let $X_{(1)} \leq \ldots \leq X_{(n)}$ be the corresponding order statistics, and let

$B_k = (n - k + 1)(X_{(k)} - X_{(k-1)})$, then

$$f(B_1, B_2, \ldots, B_n) = \prod_{k=1}^{n} \lambda e^{-\lambda B_k} \tag{3}$$

Proof : - (i) let $n = 1$, then $B_1 = X_{(1)}$ and

$$f(B_1) = f(X_1) = \lambda e^{-\lambda B_1} \quad \therefore \quad (3) \text{ holds}$$

(ii) let $n = 2$; then

$B_1 = 2 X_{(1)}$ and $B_2 = (X_{(2)} - X_{(1)})$

The joint density of $X_{(1)}$ and $X_{(2)}$ is given by

$$f(X_{(1)}, X_{(2)}) = 2 \lambda^2 e^{-\lambda(x_1 + x_2)}$$

By using variable transformation then

$$f(B_1, B_2) = \lambda^2 \; e^{-\lambda(B_1 + B_2)} = \prod_{k=1}^{2} \lambda \; e^{-\lambda B_k} \;, \quad \infty > B_k > 0$$

then (3) holds

and so on by repeated applications of this argument, we may show that (3) holds for all n.

Lemma 2: Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with parameter $\lambda > 0$. Let $w_i = \sum_{j=1}^{i} X_j$ and let $Y_i = X_i/W_n$, $i = 1, 2, \ldots, n-1$. Then

$$f(y_1, y_2, \ldots, y_{n-1}) = (n-1)! \;, \quad n > 1 \tag{4}$$

with $1 > y_1 + y_2 + \ldots + y_{n-1} > 0$

Proof: (i) let n = 2, let $Y_1 = \frac{X_1}{W_2}$, $w_2 = X_1 + X_2$

The joint density of X_1 and X_2 is given by

$$f(X_1, X_2) = \lambda^2 \, e^{-\lambda(X_1 + X_2)} \qquad \infty > X_1, X_2 > 0$$

by using variable transformation then

$$f(y_1, w_2) = \lambda^2 \, w_2 \, e^{-\lambda w_2} \qquad \infty > w_2 > 0$$

$$1 > y_1 > 0$$

integrating over w_2, we get

$$f(y_1) = \Gamma(2) = 1! \qquad \therefore (4) \text{ hold}$$

(ii) let n = 3, let $Y_1 = X_1/w_3$, $Y_2 = X_2/w_3$, $w_3 = X_1 + X_2 + X_3$

$$f(X_1, X_2, X_3) = \lambda^3 \, e^{-\lambda(X_1 + X_2 + X_3)} \qquad \infty > X_1, X_2, X_3 > 0$$

by using variable transformation then

$$f(y_1, y_2, w_3) = \lambda^3 \, w_3^2 \, e^{-\lambda w_3} \qquad \infty > w_3 > 0$$

$$1 > y_1 + y_2 > 0$$

integrating over w_3, we get

$$f(y_1,y_2) = \Gamma 3 = 2! \qquad \therefore (4) \text{hold}$$

and so on by repeated applications of this argument, we may show that (4) holds for all n.

Theorem 1: Let $X_1,X_2,...,X_n$ be arandom sample from the exponential distribution with parametr λ $\lambda > 0$, let, $Y_i = X_i/w_n$ with $w_i = \sum_{j=1}^{i} X_j$ and let

$$S = \sum_{i=1}^{n-1} (n-i)Y_j$$

Then the M.G.f of S is given by

$$E(e^{tS}) = [(e^t-1)/t]^{n-1} \quad , n > 1 \qquad (5)$$

Proof: i) let n = 2 , $S = Y_1$

The moment generating function of S is,

$$E(e^{tS}) = \int_0^1 e^{ty_1} f(y_1)dy_1 \qquad \text{by using lemma 2. at } n = 2$$

$$\therefore E(e^{tS}) = (e^t-1)/t \quad \text{then (5) holds}$$

ii) let n = 3, $S = 2Y_1 + Y_2$

$$E(e^{tS}) = \int_0^1 \int_0^{1-y_2} e^{2ty_1+ty_2} f(y_1,y_2)dy_1dy_2 \qquad \text{by using lemma 2 at } n = 3$$

$$\therefore E(e^{tS}) = \int_0^1 \int_0^{1-y_2} 2e^{2ty_1+ty_2} dy_1dy_2$$

$$= [(e^t-1)/t]^2 \qquad \text{then (5) holds}$$

iii) let n = 4 , $S = 3Y_1 + 2Y_2 + Y_3$

$$E(e^{tS}) = \int_0^1 \int_0^{1-Y_3} \int_0^{1-Y_2-Y_3} e^{3ty_1+2ty_2+ty_3} f(y_1,y_2,y_3)dy_1dy_2dy_3.$$

$$= 6 \int_0^1 \int_0^{1-Y_3} \int_0^{1-Y_2-Y_3} e^{3ty_1+2ty_2+ty_3} dy_1dy_2dy_3.$$

$[(e^t-1)/t]^3$ (5) holds

and so on, by repeated applications of this argument, we may show that (5) holds for all n.

Lemma 3: let S be as defined in Theorem 1, then

$E(S) = (n-1)/2$, and $Var(S) = (n-1)/12$.

Proof:

From theorem 1, we have $E(e^{tS}) = [(e^t-1)/t]^{n-1}$

let $M_t = \frac{e^t - 1}{t} = \left[1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots\right]$

by differentiating M_t^{n-1} with respect to t

$$\frac{\partial M_t^{n-1}}{\partial t} = (n-1)M_t^{n-2}\left[\frac{1}{2} + \frac{t}{3} + \frac{t^2}{8} + \ldots\right] \quad (6)$$

then $E(S) = \frac{\partial M_t^{n-1}}{\partial t}\Big|_{t=0} = \frac{n-1}{2}$

$$\frac{\partial^2 M_t^{n-1}}{\partial t^2} = (n-1)(n-2)M_t^{n-3}\left[\frac{1}{2} + \frac{t}{3} + \frac{t^2}{8} + \ldots\right]^2 + (n-1)M_t^{n-2}\left[\frac{1}{3} + \frac{t}{4} + \ldots\right]$$

then $E(S^2) = \frac{\partial^2 M_t^{n-1}}{\partial t^2}\Big|_{t=0} = \frac{(n-1)(n-2)}{4} + \frac{n-1}{3} = \frac{n-1}{12}[3n-2]$ (7)

From (6) and (7), we get

$$Var(S) = E(S^2) - [E(S)]^2 = \frac{n-1}{12}$$

3. Testing exponentiality against NRBU class

In this section, we use the total time on test transforn (TTT-transform) technique for testing exponentiality versus NRBU property, and establish the asymptotic distribution of the test statsitic of this test.

3.1 The hypothesis testing.

we test, $H_o : F(t) = 1\text{-}\exp(-\lambda t)$, $\lambda > 0, t \geq 0$ versus

$H_1 : F(t)$ is in Ω

Where Ω denotes the NRBU or the dual NRWU classes of life distributions.

3.2 The TTT-transform

The TTT-transform of a life distribution F with finite mean

$$\mu = \int_0^\infty \bar{F}(x)dx, \qquad x < \infty \quad \text{is given by}$$

$$\phi_F(t) = \mu^{-1} \int_0^{F^{-1}(t)} \bar{F}(x)dx, \qquad , 0 < t < 1$$

where $F^{-1}(t) = \inf\{x : F(x) \geq t\}$.
For an exponential distribution, $\phi(t) = t$, $0 \leq t 1$.

Suppose that n components with identical and independent life distribution F are subjected for testing. Let their observed life times be $t_1, t_2, \ldots, t_n$ with ordered sample as

$$o = t_{(o)} \leq t_{(1)} \leq \ldots \leq t_{(n)} .$$

Define

$$D_k = (n-k+1)(t_{(k)} - t_{(k-1)}) , \quad k = 1,2,\ldots,n.$$

Then $\xi_j = \sum_{k=1}^{j} D_k$ estimates the TTT-transform at $t_{(j)}$

where $\xi_o = 0$.

The quantity $S_j = \xi \xi_j / \xi_n \xi$ estimates the scaled TTT-transform and the departure of S_j from $\phi \phi(t)$ of exponential distribution measures the departure of its distribution from the exponential distribution.

The TTT-transform technique was introduced by Barlow and Campo (1975) and later on used by many authors including Klefsjo (1983) to test various aging criteria against exponential. In fact a life distribution is uniquely determined by its TTT-Transform.

3.3 The Test Statistic

El Arishy and Khalique (1995) have presented the test statsitic for testing exponentiality versus NRBU. This test statistic was given as

$$U = \frac{n(n+1)}{2} - \sum_{i=1}^{n}\sum_{j=1}^{n} F(F^{-1}(\tfrac{i}{n}) + F^{-1}(\tfrac{j}{n})) + \frac{n-1}{2}\sum_{j=1}^{n} S_j$$

After simplification U can be reduced to the following form

$$U = (n-1)\,Z \quad \text{where} \quad Z = \tfrac{1}{2}\left[\sum_{j=1}^{n-1} S_j - \frac{(n-1)}{2}\right]$$

In this work, we use the test statistic in the form

$$U^* = Z/\sqrt{n-1}$$

Theorem 2: Let U^* as defined above be the test statistic for testing

$H_o : F(t) = 1-e^{-t}$ vs $H_1 : F \subset \Omega$.

Then under H_o, the moment generating function (M.G.F.) of U^* is given by

$$M.G.f(U^*) = E(e^{hU^*}) = \left[(e^{h/4(\sqrt{n-1})} - e^{-h/4(\sqrt{n-1})}) / h/2(\sqrt{n-1})\right]^{n-1}$$

for all n > 1, and -a < h < a for some a > 0.

Proof:

$$E(e^{hU^*}) = E(e^{hZ/\sqrt{n-1}}) = e^{-(n-1)h^*/2}\, E(e^{(h^* \sum_{j=1}^{n-1} S_j)}) \qquad (8)$$

where $h^* = h/2\ (\sqrt{n-1})$.

$$\sum_{j=1}^{n-1} S_j = \sum_{j=1}^{n-1}\sum_{k=1}^{j} D_k \Big/ \sum_{k=1}^{n} D_k$$, under H_o and by using Lemma 1, we find that

$D_1, D_2, \ldots, D_n$ are i.i.d. exponential with parameter λ By using Theorem 1. one can get

$$E(e^{(h^* \sum_{j=1}^{n-1} S_j)}) = [\,e^{h^*}-1/h^*]^{n-1}$$

substitue in (8) we get

$$E(e^{hU^*}) = e^{-(n-1)h^*/2}\,[(e^{h^*}-1)/h^*]^{n-1}$$

$$= [(e^{h^*/2} - e^{-h^*/2})/h^*]^{n-1}$$

and the result is straight forward

Lemma 4: Let U^* be as defined above, then

$$E(U^*) = \text{zero}\ , \ \text{and}\ Var(U^*) = \tfrac{1}{48}\ .$$

Proof: $$U^* = \frac{1}{2\sqrt{n-1}}\left[\sum_{j=1}^{n-1} S_j - \frac{n-1}{2}\right]$$

using lemma (3) the result is straight forward.

The following theorem includes the asymptotic distribution of U^* using the moment generating function technique.

Theorem 3: Let U^* be the test statistic for testing $H_o : F(t) = 1-e^{-t}$ vs $H_1 : F \subset \Omega$ then under H_o, as $n \to \infty$ and if $E(U^*) < \infty$, then U^* is asymptoticaly distributed normally with mean zero and variance $\frac{1}{48}$.

Proof: The moment generating function of U^* is given in theorem 2. by

$$M.G.F(U^*) = [[\exp(h/4(\sqrt{n-1}))-\exp(-h/4(\sqrt{n-1}))]/h/2(\sqrt{n-1}))]^{n-1} \quad (9)$$

By expanding the exponential function in (9), one get,

$$M.G.F(U^*) = [\{[1+(h/4(\sqrt{n-1})+h^2/32(\sqrt{n-1})^2+h^3/6(4(\sqrt{n-1}))^3+...] -$$

$$[1-h/4(\sqrt{n-1}) + h^2/32(\sqrt{n-1})^2 - h^3/6(4(\sqrt{n-1})^3+...]\}/(h/2(\sqrt{n-1})]^{n-1}$$

After simplification, we get

$$M.G.F(U^*) = [1+\frac{1}{16(n-1)}\frac{h2}{3!}+\frac{1}{64(n-1)^2}\frac{h4}{5!}+...]^{n-1}$$

$$= [1+\frac{1}{n-1}V]^{n-1} \quad (10)$$

where $V = \frac{1}{48}\frac{h2}{2}+\frac{1}{16(n-1)}\frac{h4}{5!}+\frac{1}{512(n-1)^2}\frac{h6}{7!}+...$ is bounded.

By taking the limit of equation (10) as $n \to \infty$

we get $\lim_{n\to\infty} M.G.F(U^*) = e^{\frac{1}{48}\frac{h^2}{2}}$ which is the moment generating function of a normal distribution with mean zero and variance $\frac{1}{48}$

4- Comments and Conclusions

New renewal better than used is an important class of life distributions due to its possible applications in repair and replacement policies. A procedure to test exponentiality versus this property is given, and the asymptotic distribution of the test statistic U^* is established because the exact distribution is difficult to find.

Theorem 3. Shows theoretically that U^* can be asymptotically distributed normal with mean zero and variance $\frac{1}{48}$ when n is very large.

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