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Bayesian Estimation of the Reliability Function of a Two-parameter Cauchy Distribution

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Abstract

In this paper, we obtained an apprixImate Bayes procedure for the estimation of the reliability function of a two-parameter Cauchy distribution under a predictive distribution approach of Sinha and Guttman (1988) using Jeffrys' non-informative prior. Based on a Monto Carlo study and Mathematica programs, such approximate Bayes estimator is compared with those of Howlader and Weiss (1988b) and Maximum likelihood.

Key words : Cauchy distribution ; Reliability function ; Non-informative prior ; A predictive distribution approach.

1 Introduction

The special form of the Pearson Type VII distribution, with probability density function

$$f(x\,;\theta,\lambda) = (\pi\lambda)^{-1}\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-1}, \lambda > 0, \qquad (1)$$

is called the Cauchy distribution. The reliability function is

$$R(t) = 1 - F(t) = \frac{1}{2} - \pi^{-1} \arctan \left[\frac{t-\theta}{\lambda} \right] \tag{2}$$

for various (fixed) value of t.

The parameters θ and λ are location and scale parameters, respectively. The distribution is symmetrical about $x = \theta$. The median is θ; the upper and lower quartiles are $\theta \pm \lambda$. The distribution does not posses finite moments of order greater than or equal 1, and so does not posses a finite expected value or standard deviation. However, θ and λ are location and scale parameters, respectivily, and may be regarded as being analogous to mean and standard deviation.The Cauchy distribution is often used in extreme cases to model heavy-tail disributions, such as those which arise in outlier analyses.

For the problem of estimating θ when λ is known, Copas (1975) and Gabrielsen (1982) showed that the joint likelihood function for θ and λ is unimodal. Hence the two-parameter situation is easier to handel than just the location-parameter alone. For Bayesian inference Franck (1981) considered the problem of testing of normal versus the Cauchy, and Spiegelhalter (1985) used some of Franck's results to obtain exact Bayes estimators for θ and λ under a non-informative prior, for odd values of n larger than 3 and for even values of n. Howlader and Weiss (1988a) pointed out that the exact formulas given by Spiegelhalter (1985) are difficult to compute and require great computational precision as these estimtes are very unstable and often blow up in values. Through an emperical Monte Carlo study that they carried out, they also observed that the exact method often grossly overestimated λ, especially for small values of n.

Howlader and Weiss (1988a) derived some approximate Bayesian estimators by using a method of approximating ratios of integrals [due to Lindley (1980)]. They then showed that these approximate Bayesian estimators perform very well in comparsion to the maximum likelihood estimator (MLE). Also Howlader and Weiss (1988b) derived an approximate Bayes procedure for the estimation of the reliability function of a two-parameter Cauchy distribution using Jeffreys' non-informative prior with a squared-error loss function. and with a log-odds ratio squared-error loss function. Based on a Monte Carlo simulation study, two such Bayes estimators of the reliability function are compared with the maximum likelihood estimator at $\theta = 5$, $\lambda = 1$ at n=7,15,30 where R(t), t=1....9.

They showed that the three procedures are fairy competive, and the maximum likelihood does generly well. Sinha and Guttman (1988) suggested a predictive distribution approach for the estimation of the reliability function of a two-parameter Weibull distribution using Jeffreys' non-informative prior.

2 Reliability Function: A Predictive Approach

Based on n independent observations $\underline{x}= (x_1, x_2, ..., x_n)$ from the Cauchy density (1), the likelihood function for θ and λ is

$$\ell = \pi^{-n}\lambda^{n}\prod_{i=1}^{n}\left[\lambda^2 + (x_i - \theta)^2\right]^{-1}. \tag{3}$$

Then, the log likelihood function

$$\ell = -nln\pi + n\ln\lambda - \sum_{i=1}^{n} ln\left[\lambda^2 + (x_i - \theta)^2\right]. \tag{4}$$

Then, the likelihood equations for θ and λ to be

$$\frac{\partial ln\ell}{\partial\theta} = \sum_{i=1}^{n}\frac{2\,(x_i - \theta)}{\lambda^2 + (x_i - \theta)^2} = 0, \tag{5}$$

$$\frac{\partial ln\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\frac{2\lambda}{\lambda^2 + (x_i - \theta)^2} = 0. \tag{6}$$

Suppose little is known a-priori about θ and λ so that Jeffreys' (1983) vague prior, say $g(\theta, \lambda)$ is appropriate for this situation, that is

$$g(\theta, \lambda) = \frac{1}{\lambda}. \tag{7}$$

From (3) and (7) we have the joint posterior of θ and λ is given $\underline{x}$, by

$$h(\theta, \lambda\,|\underline{x}) = \frac{l_1(\theta, \lambda)}{\int_0^{\infty}\int_{-\infty}^{\infty} l_1(\theta, \lambda)\; d\theta d\lambda} \tag{8}$$

where

$$l_1(\theta, \lambda) = \lambda^{n-1}\prod_{i=1}^{n}\left[\lambda^2 + (x_i - \theta)^2\right]^{-1}. \tag{9}$$

By following Sinha and Guttman's (1988) approach, an approximate Bayes estimator of R(t) is

$$
\begin{aligned}
\tilde{R}(t) &= E\left[R(t)\mid \underline{x}\right] = \frac{\int_0^\infty \int_{-\infty}^\infty R(t) I_1(\theta,\lambda)\, d\theta d\lambda}{\int_0^\infty \int_{-\infty}^\infty I_1(\theta,\lambda)\, d\theta d\lambda} \\
&= \frac{\int_0^\infty \int_{-\infty}^\infty P(X>t)\, I_1(\theta,\lambda)\, d\theta d\lambda}{\int_0^\infty \int_{-\infty}^\infty I_1(\theta,\lambda) d\theta d\lambda} \\
&= \frac{\int_0^\infty \int_{-\infty}^\infty \left\{\int_t^\infty f(x\,;\,\theta,\lambda)\, dx\right\} I_1(\theta,\lambda)\, d\theta d\lambda}{\int_0^\infty \int_{-\infty}^\infty I_1(\theta,\lambda)\, d\theta d\lambda} \\
&\quad \frac{\int_t^\infty \left\{\int_{-\infty}^\infty \int_{-\infty}^\infty u_1(\theta,\lambda,x)\, I_1(\theta,\lambda)\, d\theta d\lambda\right\} dx}{\pi \int_0^\infty \int_{-\infty}^\infty I_1(\theta,\lambda)\, d\theta d\lambda}
\end{aligned} \tag{10}
$$

where

$$
u_1(\theta,\lambda,x) = \frac{1}{\lambda}\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-1}. \tag{11}
$$

Then (10) become

$$
\tilde{R}(t) = \frac{1}{\pi}\int_t^\infty \left\{E\left[u_1(\theta,\lambda,x)\mid \underline{x}\right]\right\} dx. \tag{12}
$$

The ratio of integrals in $E[u_1(\theta,\lambda,x)\mid \underline{x}]$ in (12) does not seem to take any close form. Howlader and Weiss (1988b) derived approximate experssion for $E[w(\underline{\theta})\mid \underline{x}]$ where $w(\underline{\theta})$ is arbitrary function in $\underline{\theta}$, $\underline{\theta}=\theta_1,\dots,\theta_m$. Now according to Howlader and Weiss (1988b) (12) becomes

$$
\begin{aligned}
\tilde{R}(t) &= \frac{1}{\pi}\left[\hat{z}_1 - \frac{1}{2}\left\{\frac{1}{A}\hat{z}_4 + \frac{1}{B}\left(\hat{z}_5 - \frac{2}{\lambda}\hat{z}_3\right)\right\}\right. \\
&\quad \left. + \frac{C}{2A^2}\hat{z}_2 + \frac{D}{2B^2}\hat{z}_3\right].
\end{aligned} \tag{13}
$$

All functions of the right-hand side of (13) are to be evaluate at the maximum likelihood estimates of (θ, λ) and where

$$A = 2\sum_{i=1}^{n} w_i - 4\widehat{\lambda}^2 \sum_{i=1}^{n} w_i^2,$$

$$B = \frac{-n}{\widehat{\lambda}^2} - A\ ,$$

$$C = 4\sum_{i=1}^{n} \left(x_i - \widehat{\theta}\right)\left(i - 4\widehat{\lambda}^2 w_i\right) w_i^2,$$

$$D = \frac{2n}{\widehat{\lambda}^3} + 4\widehat{\lambda}\sum_{i=1}^{n}\left(3 - 4\widehat{\lambda^2} w_i\right) w_i^2\ .$$

$$w_i = \left[\widehat{\lambda}^2 + \left(x_i - \widehat{\theta}\right)^2\right]^{-1}\ ,$$

to evaluate $\widehat{z}_i\ , i = 1, \ldots 5$ in (13) we first used Mathematica to evaluate the following integrals

$$\begin{aligned} z_1 &= \int_t^{\infty} u_1(\theta, \lambda, y) dy = \frac{1}{\lambda}\int_t^{\infty}\left[1 + \left(\frac{x-\theta}{\lambda}\right)^2\right]^{-1} dx \\ &= \arctan\left[\frac{\lambda}{t-\theta}\right]\ , \end{aligned} \tag{14}$$

$$\begin{aligned} z_2 &= \int_t^{\infty} \frac{\partial u_1(\theta, \lambda, x)}{\partial \theta} dx = \frac{1}{\lambda^3}\int_t^{\infty} 2(x-\theta)\left[1 + \left(\frac{x-\theta}{\lambda}\right)^2\right]^{-2} dx \\ &= \frac{\lambda}{\theta^2 + \lambda^2 - 2\theta t + t^2}\ , \end{aligned} \tag{15}$$

$$\begin{aligned} z_3 &= \int_t^{\infty} \frac{\partial u_1(\theta, \lambda, x)}{\partial \lambda} dx \\ &= \int_t^{\infty}\left\{\frac{2\,(x-\theta)^2\left[1 + \left(\frac{x-\theta}{\lambda}\right)^2\right]^{-2}}{\lambda^4} - \frac{\left[1 + \left(\frac{x-\theta}{\lambda}\right)^2\right]^{-1}}{\lambda^2}\right\} dx \\ &= \frac{(t-\theta)}{\theta^2 + \lambda^2 - 2\theta t + t^2}\ , \end{aligned} \tag{16}$$

$$
\begin{aligned}
z_4 &= \int_t^{\infty} \frac{\partial^2 u_1(\theta,\lambda,x)}{\partial\theta^2} dx \\
&= \int_t^{\infty} \left\{ \frac{8(x-\theta)^2\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-3}}{\lambda^5} - \frac{2\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-2}}{\lambda^3} \right\} dx \\
&= \frac{2\lambda(t-\theta)}{\left(\theta^2+\lambda^2-2\theta t+t^2\right)^2}, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
z_5 &= \int_t^{\infty} \frac{\partial^2 u_1(\theta,\lambda,x)}{\partial\lambda^2} dx \\
&= \int_t^{\infty} \left\{ \frac{8(x-\theta)^4\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-3}}{\lambda^7} - \frac{10(x-\theta)^2\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-2}}{\lambda^5} \right. \\
&\quad \left. + \frac{2\left[1+\left(\frac{x-\theta}{\lambda}\right)^2\right]^{-1}}{\lambda^3} \right\} dx. \\
&= \frac{-2\lambda(t-\theta)}{\left(\theta^2+\lambda^2-2\theta t+t^2\right)^2}, \qquad (18)
\end{aligned}
$$

all z_i, i=1,...5 from (14) to (18) are evaluated at the maximum likelihood estimates of (θ,λ) to obtain $\hat{z}_i$, i=1,...5 in (13) .

3 Monte Carlo Study

In order to compare our estimator (13) of the reliability function with those of Howlader and Weiss (1988b), say $\overline{R}(t)$, and MLE, say $\widehat{R}(t)$, 1000 (=N) samples of sizes n=10,20,30,40,50 were generated from p.d.f. in (1) with $\theta = 5, \lambda = 1$. The study performed using Mathematica programs (version 2.2). The mean of N estimates and the corresponding (emperical) mean square error (MSE)

$$
\text{MSE} = \frac{\text{Sum of squares of the N deviations estimates from the true value}}{N}
$$

were computed where R(t), t=10. We reported the results in table (1). The entries within the parentheses indicate the corresponding mean square errors. To compare the efficiencies of the three estimators of the reliability function, the relative error (RER)

$$\text{RER} = \frac{\sqrt{MSE}}{\text{True value of the reliability function}}$$

was computed and the results in table (2).

Table (1)

True Reliability 0.0628			
	Estimates		
n	$\tilde{R}(t)$	$\bar{R}(t)$	$\hat{R}(t)$
10	0.0961 (0.0075288)	0.0804 (0.00479266)	0.0691 (0.003197)
20	0.0819 (0.000747951)	0.0687 (0.000301853)	0.0592 (0.000209765)
30	0.0774 (0.000297768)	0.0649 (0.0000618655)	0.0559 (0.0000894938)
40	0.0758 (0.000183947)	0.0635 (0.0000108589)	0.0546 (0.0000746716)
50	0.0749 (0.00016021)	0.0628 ($8.06386\ 10^{-6}$)	0.0540 0.0000829731)

Table (2)

n	$\tilde{R}(t)$	$\bar{R}(t)$	$\hat{R}(t)$
10	1.3809	1.1018	0.9170
20	0.4353	0.2765	0.2305
30	0.2746	0.1252	0.1506
40	0.2159	0.0524	0.1375
50	0.2014	0.0452	0.1450

The Monte Carlo study indicates that

1- From table (1) the MSE of $\tilde{R}(t)$, $\bar{R}(t)$, $\hat{R}(t)$ decrease as n increases. Also the bias of $\tilde{R}(t)$, $\bar{R}(t)$ decrease as n increases.

2- From table (3) and n=10, n=20, MLE is more efficient than the two approixmate Bayes estimators.

3- From table (3) and n$\geq$ 30, Bayes estimator under squared error loss function is more efficient than those of maximum likelihood and our estimator.

Table (3)

n	$\frac{RER(\tilde{R}(t))}{RER(\bar{R}(t))}$	$\frac{RER(\hat{R}(t))}{RER(\bar{R}(t))}$	$\frac{RER(\tilde{R}(t))}{RER(\hat{R}(t))}$	$\frac{RER(\bar{R}(t))}{RER(\hat{R}(t))}$
10	—	—	1.5059	1.2015
20	—	—	1.8885	1.996
30	2.1933	1.2029	—	—
40	4.1202	2.6240	—	—
50	4.4558	3.2080	—	—

4 Conclusions

For small samples MLE more efficient than the two approximate Bayes estimators of the reliability distribution function under Cauchy distribution. But for n$\geq$30 Bayes estimator of Howlader and Weiss (1988b) is more efficient than those of MLE and our estimator which under a predictive approach of Sinha and Guttman (1988).

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