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The W Test of Uniformity (Complete Sample)

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Summary

This research work is concerned with deriving a test statistic for testing the null hypothesis that a complete random sample was drawn from a uniform distribution with unknown parameters. The derivation is based on some ideas used first by Shapiro & Wilk (1965). In addition, this derived test has been compared in terms of power with some other well known tests for uniformity.

Key words and phrases: Uniformity; Test statistic; Order statistics; Moments; Distribution function; Power comparison.

1 Introduction

In 1965, Shapiro & Wilk gave their test for normality. The principles underlying the construction of their test statistic can be applied to any distribution with location and scale parameters such as exponential and uniform distributions.

Let $X_1, X_2, \ldots, X_n$ represent a random sample of size n from a continuous distribution, where the sample mean is $\bar{X}$, and let this sample be arranged in order of increasing magnitudes to get the order statistics $Y_1 \leq Y_2 \leq \ldots \leq Y_n$. It is required here to investigate the null hypothesis that this sample was drawn from a uniform distribution, $U(\lambda_1, \lambda_2)$, where λ_1 and λ_2 can be either known or unknown parameters.

2 Construction of the Test

It follows from section 1 that the probability density function (p.d.f.) under consideration is

$$f_X(x;\lambda_1,\lambda_2)=\frac{1}{\lambda_2-\lambda_1}\ ,\ \lambda_1\le x\le\lambda_2\ ;\ -\infty<\lambda_1<\lambda_2<\infty\ .$$

It is of interest to reparameterise this equation by setting

$$\lambda_1=\mu-\sqrt{3}\sigma\ ,\ \lambda_2=\mu+\sqrt{3}\sigma\ ;\ -\infty<\mu<\infty\ ,\ \sigma>0\ .$$

Then the p.d.f. above becomes

$$f_X(x;\mu,\sigma)=\frac{1}{2\sqrt{3}\sigma}\ ,\ \mu-\sqrt{3}\sigma\le x\le\mu+\sqrt{3}\sigma\ .$$

It is easy of course to realize that μ and σ are respectively location and scale parameters. In addition, μ and σ^2 are the mean and variance of X.

It is well known that Y_1 and Y_n are jointly sufficient and complete for μ and σ. It is possible to construct unbiased estimators of μ and σ, or λ_1 and λ_2, that are functions of the complete sufficient statistics. These estimators are uniformly minimum-variance unbiased estimators (UMVUE). One can check that

$$\hat{\mu}=\frac{1}{2}(Y_1+Y_n)$$

and

$$\hat{\sigma}=\frac{n+1}{2\sqrt{3}\,(n-1)}\,(Y_n-Y_1)$$

are indeed UMVUEs of μ and σ respectively.

Let S^2 be the usual estimator of $(n-1)\sigma^2$ where

$$S^2=\sum_{i=1}^{n}\,[X_i-\bar{X}\,]^2\ .$$

The comparison between S^2 and the estimator $\hat{\sigma}$ can be applied to test whether the ordered sample $Y_1,\ldots\ldots,Y_n$ was taken from a uniform distribution. This follows the ideas of Shapiro & Wilk (1965,1972).

The proposed statistic for testing for uniformity is

$$W=k_n\frac{S^2}{(Y_n-Y_1)^2}\ ,$$

where k_n depends only on n. This test statistic is location and scale invariant. In addition, its distribution does not depend on the parameters of the uniform distribution (Ferguson, 1967, pp.242-247). Therefore, when the time comes to calculate the moments of W, it is possible to assume that the uniform distribution is U(0,1).

According to the fact that W as given above is distributed independently of its denominator and using the mathematical expectation of this denominator (Basu, 1955 and Lloyd, 1952), it can be shown after some elementary algebra that

$$E(W)=k_n\frac{(n+1)\,(n+2)}{12n}\ .$$

Thus the mean of W is equal to 1 if the numerical coefficient k_n is $[12n/(n+1)(n+2)]$. This choice is made for the following reason: if the mean of W does not depend on n, then the tabulation of the percentage points of the test should be made more easily and also the interpolation in the table should be easier. Therefore, the final formula of W is

$$W=\frac{12nS^2}{(n+1)(n+2)(Y_n-Y_1)^2} \; .$$

3 The Percentage Points and Bounds of W

The approximate percentage points of W can be obtained by fitting Pearson Curves (Johnson et.al., 1963) using the standardised third and fourth central moments of W. To get these two moments, the derivation of $E(W^r)$ has to be explained first. It has already been mentioned that Y_1 and Y_n are jointly sufficient and complete statistics, and W is distributed independently of the parameters of the uniform distribution. Then, W is distributed independently of Y_1 and Y_n (Basu, 1955, page 378). $(Y_n-Y_1)^2$ is a function of Y_1 and Y_n. Therefore, W is independent of $(Y_n-Y_1)^2$. The general formula of $E(W^r)$, where r=1,2,3,4,..., can be concluded as follows :
take W to the power r,

$$W^r=[\frac{12n}{(n+1)(n+2)}]^r \; \frac{S^{2r}}{(Y_n-Y_1)^{2r}} \; ,$$

multiply both sides by $(Y_n-Y_1)^{2r}$ to get

$$(Y_n-Y_1)^{2r} \; W^r=[\frac{12n}{(n+1)(n+2)}]^r \; S^{2r} \; ,$$

then, the mathematical expectation of both sides is

$$E[(Y_n-Y_1)^{2r} \; W^r]=[\frac{12n}{(n+1)(n+2)}]^r \; E[S^{2r}] \; ,$$

hence, because of the fact that W and $(Y_n-Y_1)^2$ are independent,

$$E[(Y_n-Y_1)^{2r}] \quad E[W^r]=[\frac{12n}{(n+1)(n+2)}]^r \quad E[S^{2r}]$$

therefore,

$$E(W^r)=[\frac{12n}{(n+1)(n+2)}]^r \quad [\frac{E(S^{2r})}{E[(Y_n-Y_1)^{2r}]}]$$

Considering the fact that the sample range (Y_n-Y_1) has a beta distribution (David, 1981, pp.11-13), it can be proved that E(W)=1. The r-th central moment of W about its mean is $\mu_r=E[(W-1)^r]$. Then, the standardised third and fourth central moments β_1 and β_2 are

$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{40(2n^3+n^2+83n-296)^2}{49(n-2)(2n^2+n-13)^3},$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{3(28n^5-36n^4-417n^3-316n^2+11995n-25450)}{7(n-2)(2n^2+n-13)^2}.$$

β_1 and β_2 tend to have the values 0 and 3 respectively for large sample sizes. Indeed it is reasonable to suppose that the limiting distribution of W is normal with mean 1 and variance given only in terms of n. Table 1 shows how quickly the convergence occurs. The resulting non-standardised percentage points of W are given in Table 2 for several sample sizes.

In regard to the lower and upper bounds of W, Pearson and

Table 1
The value convergence of β_1 and β_2 for several sample sizes

n	10	30	50	100	200	500
β_1	0.093	0.016	0.009	0.004	0.002	0.0008
β_2	2.879	2.967	2.982	2.991	2.996	2.999

Table 2
The approximate percentage points of W

				Area			
n	0.01	0.05	0.10	0.50	0.90	0.95	0.99
4	0.800	0.822	0.844	0.971	1.198	1.276	1.433
5	0.744	0.775	0.802	0.972	1.239	1.320	1.468
6	0.693	0.742	0.779	0.978	1.253	1.335	1.482
7	0.658	0.722	0.767	0.982	1.258	1.339	1.486
8	0.636	0.712	0.762	0.985	1.258	1.338	1.485
9	0.623	0.707	0.760	0.987	1.257	1.335	1.481
10	0.616	0.706	0.761	0.989	1.254	1.331	1.475
15	0.619	0.718	0.774	0.993	1.234	1.304	1.437
20	0.640	0.737	0.791	0.995	1.215	1.280	1.401
25	0.661	0.754	0.805	0.996	1.200	1.259	1.371
30	0.681	0.769	0.817	0.997	1.187	1.242	1.346
40	0.712	0.792	0.836	0.998	1.167	1.216	1.308
50	0.736	0.810	0.851	0.998	1.152	1.196	1.280
100	0.804	0.860	0.890	0.999	1.111	1.143	1.204
150	0.837	0.884	0.909	0.999	1.092	1.118	1.168
200	0.858	0.899	0.921	1.000	1.080	1.103	1.146

Stephens (1964) found that

$$2\sqrt{\frac{n-1}{n}} \leq \frac{Y_n - Y_1}{SV} \leq \sqrt{2(n-1)}$$

if n is even, and

$$2\sqrt{\frac{n}{n+1}} \leq \frac{Y_n - Y_1}{SV} \leq \sqrt{2(n-1)}$$

if n is odd, where

$$(SV)^2 = \frac{\sum_{i=1}^{n} [X_i - \overline{X}]^2}{n-1} .$$

Using this result together with the formula of W, the bounds are

$$\frac{6n}{(n+1)(n+2)} \leq w \leq \frac{3n^2}{(n+1)(n+2)}$$

if n is even, and

$$\frac{6n}{(n+1)(n+2)} \le w \le \frac{3(n-1)}{n+2}$$

if n is odd.

4 The Distribution Function of W

The technique used here to derive the exact distribution function of W under the null hypothesis of uniformity is similar to those used by Currie (1978;1980) and Samanta (1985). The derivation of the distribution function takes advantage of the location and scale invariance property of W and the possibility of assuming that X~U(0,1).

By using the following set of variable transformations

$$G_1 = Y_1 \ , \ G_i = Y_i - Y_{i-1} \ , \ i=2,\ldots,n \ ,$$
$$U_1 = \sum_{i=2}^{n} G_i \ , \ U_i = \frac{iG_i}{U_1} \ , \ i=2,\ldots,n-1 \ ,$$
$$Z_i = \frac{1}{\sqrt{i(i+1)}}(1-U_2-\ldots-U_i) \ , \ i=2,\ldots,n-1 \ ,$$

the test statistic W and its distribution function can be given as

$$W = \frac{12}{(n+1)(n+2)}\left[\frac{n}{2} + n\sum_{i=2}^{n-1} Z_i^2\right] \ , \quad where \ -\infty < z_i < \infty \ , \ i=2,\ldots,n-1 \ .$$

$$F(w) = Pr(W \le w) = \frac{1}{(n-2)!} Pr\left(\sum_{i=2}^{n-1} Z_i^2 \le v^2\right)$$

$$= \frac{1}{(n-2)!} \underset{\sum_{i=2}^{n-1} Z_i^2 \le v^2}{\int \ldots \int \ldots \int} f(z_2,\ldots,z_{n-1})\, dz_2 \ldots dz_{n-1} \ .$$

Then, after performing several probability integral transformations, the resulting exact distribution function is given as follows:

$$F(w)=Q=\frac{\sqrt{\frac{n}{2}\Pi^{n-2}\ v^{n-2}}}{\Gamma(\frac{n}{2})},$$

$$\text{when } \frac{6n}{(n+1)(n+2)}\leq w\leq\frac{8n}{(n+1)(n+2)}.$$

$$F(w)=Q-\sqrt{2n\Pi^{n-3}}\ \left[\frac{2I_n}{\Gamma(\frac{n-3}{2})}-\frac{\sqrt{b^{n-3}}}{\sqrt{6}\ \Gamma(\frac{n-1}{2})}\right],$$

$$\text{when } \frac{8n}{(n+1)(n+2)}\leq w\leq\frac{9n}{(n+1)(n+2)}.$$

$$F(w)=1-R,$$

$$\text{when } n=4,\ \frac{9n}{(n+1)(n+2)}\leq w\leq\frac{3n^2}{(n+1)(n+2)}.$$

In addition,

$$R=\sqrt{2h^2v^2-2h^4}-\sqrt{\frac{(6v^2-1)}{2}}+\sqrt{2}\ v^2\left[\sin^{-1}(\frac{h}{v})-\sin^{-1}\frac{1}{\sqrt{6}v}\right]-\frac{h^2}{2}+\frac{3}{4},$$

$$h=\frac{-\sqrt{6}+\sqrt{288\ v^2-48}}{18},$$

$$I_n=\int\sqrt{v^2-r^2}\ r^{n-4}\ dr\ ,\ \text{where } n\geq4,$$

$$b=v^2-\frac{1}{6},$$

$$v^2=\frac{(n+1)(n+2)\ w}{12\ n}-\frac{1}{2},$$

$$r^2=\sum_{i=3}^{n-1} z_i^2.$$

After deriving the exact distribution function for some regions of W, it is possible to obtain some exact percentage points using the values in Table 2 together with the application of Newton-Raphson method. Table 3 shows the exact percentage points for some sample sizes. It is easy to notice the closeness between the values in Tables 2 and 3.

Table 3
The exact percentage points of W

			Area				
n	0.01	0.05	0.10	0.50	0.90	0.95	0.99
4	0.804	0.818	0.836	0.980	1.189	1.289	1.769
5	0.733	0.769	0.802	0.970			
6	0.687	0.741	0.782				
7	0.657	0.724	0.769				
8	0.639	0.714	0.761				
9	0.627	0.707					
10	0.620						

5 The Power Comparison

Power comparison of several tests designed all to test for a null distribution depend on sample size, alternative distributions, and the existence and method of estimation of the unknown parameters contained in the null distribution. In addition, the modified versions of some tests can play a role in power improvement. Considering these points, the W test proposed in this article was compared in terms of power with some other well known powerful tests for uniformity. These known tests used here include some modified EDF goodness-of-fit tests for uniformity when limits are unknown. They were recommended by Green & Hegazy (1976).

Again, let $Y_1, Y_2, \ldots, Y_n$ denote the ordered sample. Put

$$X_{(i)}=\frac{Y_i-a^*}{b^*-a^*}\ ,\quad i=1,\ldots,n\ ,$$

where a^* and b^* are the best linear unbiased estimators (BLUE) of the unknown parameters a and b respectively of the null U(a,b) distribution. These two BLUEs have respectively the following formulas:

$$a^*=Y_1-\frac{Y_n-Y_1}{n-1}\ ,$$
$$b^*=Y_n+\frac{Y_n-Y_1}{n-1}\ .$$

The modified EDF tests used in this article are [see Durbin & Knott (1972) and Green & Hegazy (1976)]:

1. The Anderson-Darling test statistic:

$$A=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log X_{(i)}+\log\left(1-X_{(n-i+1)}\right)\right].$$

2. The Cramer-Von Mises test statistic:

$$W^*=\sum_{i=1}^{n}\left[X_{(i)}-\frac{2i-1}{2n}\right]^2 .$$

3. The Kolmogorov-Smirnov test statistic:

$$D_1=\sum_{i=1}^{n}\left|X_{(i)}-\frac{i}{n}\right|\ ,$$

$$D_2=\sum_{i=1}^{n}\left|X_{(i)}-\frac{i+\frac{1}{2}}{n+1}\right|$$

According to the transformation from Y_i to $X_{(i)}$ above, together with the fact that the distributions of A, W^*, D_1, and D_2 do not depend on a and b under H_0, there is no loss in generality by assuming that a=0 and b=1.

The power comparison here includes some non-EDF tests. According to Schader & Schmid (1997), let $X^*_{(i)}=Y_{i+1}/(Y_n-Y_1)$, where i=1,...,n-2. Define VR as $VR=Y_1/(Y_n-Y_1)$. Let $Y_{(i+1)}=X^*_{(i)}-VR$, where i=1,...,n-2, and $Y_{(2)},Y_{(3)},\ldots,Y_{(n-1)}$ are jointly distributed as an ordered sample of size (n-2) from U(0,1). Using this ordered sample, Hegazy & Green (1975) proposed the following two test statistics for uniformity:

$$T_1=\frac{\sum_{i=2}^{n-1}\left|Y_{(i)}-\frac{i-1}{m-1}\right|}{m},$$

and

$$T_2=\frac{\sum_{i=2}^{n-1}[Y_{(i)}-\frac{i-1}{m-1}]^2}{m},$$

where m=n-2.

The power comparison performed here used a broad range of possible alternative families of distributions. They were proposed by Quesenberry & Miller (1977). These families are:

1. The J-shaped family of distributions, such as the exponential distribution.
2. The Bell-shaped family of distributions, such as the normal distribution.
3. The U-shaped family of distributions.
4. The V-shaped family of distributions.

To compare the power of the W test against the other tests mentioned in this article, a Monte Carlo simulation study was done in which 1000 samples were generated for the sample size n=5, 10, 20, 40, and 80. The significance level considered was 5 percent. Random numbers following a U(0,1) were generated and then the alternative distributions were obtained by transforming the uniform distribution to them. The W test has two tails, lower and upper. Hence, it seemed better to consider its two tails separately in the power study. The results, the proportions of 1000 Monte Carlo samples declared significant, are given in Table 4. The numbers given in this table become powers if they are just divided by 1000. The most extreme case away from U(0,1) is reported here for every family of alternative distributions.

Table 4 shows that in regard to the first family of alternatives, the W test in its two tails is dominated by all the other tests included in the table, in particular, D_1 and D_2 in the case of small samples. For the second and third families, the W test outperforms the other tests in its lower tail for all sample sizes. For the fourth family, the W test seems to be more powerful in its upper tail than the other tests especially for the small sample sizes such as n=5, 10, and 20, while for n=40 and 80 the performance of the other tests is comparable to that of W. According to Table 4, in the case of testing for uniformity with unknown parameters and the availability of a complete sample, the non-EDF tests such as W, T_1, and T_2 seem to be more powerful than the EDF tests. But it is worth it to mention that the most comparable EDF test here is A.

Another point in favour of the W test statistic is that it can be easily calculated compared to all the other tests considered here. Its calculation does not depend on any parameters which have to be estimated. As a conclusion, the W test represents a powerful and easily calculated test for examining the composite hypothesis of uniformity for a wide class of alternative distributions.

Table 4
1000 x Power for testing H_0: Uniform at the 5-percent level

H_1	n	$W_{0.05}$	$W_{0.95}$	A	W^*	D_1	D_2	T_1	T_2
Family 1	5	8	106	479	450	565	565	386	408
	10	10	101	921	889	934	932	876	869
	20	213	75	999	999	999	998	996	997
	40	376	27	1000	1000	1000	1000	1000	1000
	80	595	10	1000	1000	1000	1000	1000	1000
Family 2	5	79	26	51	49	51	51	51	47
	10	193	8	62	63	55	52	75	81
	20	562	1	144	154	128	114	178	185
	40	921	0	523	495	427	360	537	548
	80	1000	0	948	900	900	874	930	935
Family 3	5	94	28	50	49	55	55	56	51
	10	181	10	73	71	66	62	81	82
	20	543	2	147	169	138	120	180	192
	40	910	0	498	470	423	352	523	523
	80	1000	0	955	926	918	895	942	945
Family 4	5	2	416	311	312	157	157	303	312
	10	1	807	545	375	219	272	210	285
	20	0	986	949	892	824	914	678	576
	40	0	1000	1000	1000	1000	1000	1000	1000
	80	0	1000	1000	1000	1000	1000	1000	1000

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