THE EGYIPATIAN STATISTICAL JOURNAL
ISSR, CAIRO UNIV., VOL., 43, NO. 2, 1999

On Record Values From Linear Exponential Distribution

S. E. Abu-Youssef and A. S. Al-Ruzaiza
Department of Statistics and Operations Research
King Saud University
P. O. Box 2455, Riyadh 11451
Saudi Arabia

Abstract

In this paper single and product moments of upper record values are studied. Some recurrence relations for both single and product moments of upper records from linear exponential distribution are derived. Two results for characterization of the linear exponential through the properties of upper records are also presented.

1 Introduction

Chandler (1952) formulated the theory of record values arising from a sequence of independently and identically distributed continuous random variables. Feller (1966) presented some examples of record values in gambling problems, Resnick (1973) and Shorrock (1973) discussed the asymptotic theory of records. Interested readers may refer to the works of Glick (1978), Nevzorov (1987), Nagaraja (1988), Ahsanullah (1988), Arnold and Balakrishnan (1989) and Arnold, Balakrishnan and Nagraga (1992) for reviews on various developments in the area of record values.

Balakrishnan, Malik and Ahmed (1988) derived exact and explicit expressions for the means and product moments of order statistics from the

This Work Was Supported by the Research Center Project No.(Stat/1420/30)

linear exponential distribution. They also established some recurrence relations for both single and product moments of order statistics for the same model. Mohie El-Din et al (1997) derived expressions for the moments and product moments of the order statistics from the doubly truncated linear exponential distribution. They study some recurrence relation satisfied by single and product moments of order statistics for the same distribution. They also presented two theorems for characterizing the linear exponential through the properties of order statistics.

In this paper we establish some recurrence relations satisfied by the single and product moments of upper record values from linear exponential distribution. These recurrence relation will enable one to obtain all the single and product moments of all record values in a single recursive manner. Similar results have been obtained by Balakrishnan, Ahsanullah and Chan (1992), Balakrishnan, Chan and Ahsanullah (1993), Balakrishnan and Ahsanullah ((1994),(1995)) and Balakrishnan and Chan (1993) for Gumble, generalized extreme value, generalized Pareto, exponential and Rayleigh and Weibull distribution. It is shown here that some results of the exponential and Rayleigh are deduced from the results given in this paper.

The p.d.f. and c.d.f. of linear exponential distribution are

$$f(x) = (\lambda + \nu x)e^{-(\lambda x + \frac{\nu x^2}{2})}, \quad \lambda > 0, \nu > 0, \ x \geq 0 \tag{1.1}$$

and

$$1 - F(x) = e^{-(\lambda x + \frac{\nu x^2}{2})}, \tag{1.2}$$

hence

$$f(x) = (\lambda + \nu x)[1 - F(x)]. \tag{1.3}$$

2 Relation for single moments of upper records

Let $X_{U(1)} < X_{U(2)} < \ldots$ be the sequence of upper records values from (1.1). For convenience $X_{U(0)} = 0$. Then the p.d.f. of $X_{U(n)}, n = 1, 2, \ldots$ is

$$f_n(x) = \frac{1}{\Gamma(n)}[-\log[1 - F(x)]]^{n-1} f(x), \tag{2.1}$$

and the $k\underline{th}$ moment of upper record $X_{U(n)}$ is

$$\mu_n^{(k)} = \int_0^\infty x^k f_n(x)dx. \tag{2.2}$$

Then

$$\mu_1 = \sqrt{\frac{2\pi}{\nu}} e^{\frac{\lambda^2}{2\nu}} \left[1 - \Phi(\frac{\lambda}{\sqrt{\nu}})\right], \tag{2.3}$$

where $\Phi(.)$ is a standard normal c.d.f. .

Relation (2.1):

$$\mu_n^{(k)} = \frac{\lambda}{k+1}\left[\mu_n^{(k+1)} - \mu_{n-1}^{(k+1)}\right] + \frac{\nu}{k+2}\left[\mu_n^{(k+2)} - \mu_{n-1}^{(k+2)}\right]. \tag{2.4}$$

$$\mu_1^{(k)} = \frac{\lambda}{k+1}\mu_1^{(k+1)} + \frac{\nu}{k+2}\mu_1^{(k+2)}. \tag{2.5}$$

If $\nu = 0$, we have a recurrence relation between moments of upper records from exponential distribution (Balakrishnan and Ahsanullah (1995)).

If $\lambda = 0$, we obtain the recurrence relations between moments of upper records from Rayleigh distribution.

Proof:

Using (1.3) in (2.2), we get

$$\begin{aligned}\mu_n^{(k)} &= \frac{1}{\Gamma(n)}\int_0^\infty x^k[-\log[1-F(x)]]^{n-1}(\lambda+\nu x)[1-F(x)]dx \\ &= \frac{\lambda}{(k+1)\Gamma(n)}\int_0^\infty [-\log[1-F(x)]]^{n-1}[1-F(x)]dx^{k+1} \\ &\quad +\frac{\nu}{(k+2)\Gamma(n)}\int_0^\infty [-\log[1-F(x)]]^{n-1}[1-F(x)]dx^{k+2}. \end{aligned} \tag{2.6}$$

Integrating (2.6) by parts, the relation (2.4) is obtained.

Putting $n = 1$ in (2.4), (2.5) is obtained.

Relation (2.2):

$$\mu_{n+1}^{(k)} = \frac{\lambda}{n}\mu_n^{(k+1)} + \frac{\nu}{2n}\mu_n^{(k+2)} \tag{2.7}$$

Proof:

From (2.2), we have

$$\begin{aligned}\mu_{n+1}^{(k)} &= \frac{1}{\Gamma(n+1)}\int_0^\infty x^k[-\log[1-F(x)]^{n-1}(\lambda x+\frac{\nu x^2}{2})f(x)dx\\ &= \frac{\lambda}{\Gamma(n+1)}\int_0^\infty x^{k+1}[-\log[1-F(x)]^{n-1}f(x)dx\\ &\quad+\frac{\nu}{2\Gamma(n+1)}\int_0^\infty x^{k+2}[-\log[1-F(x)]^{n-1}f(x)dx.\end{aligned}$$

Then the relation 2 is proved.

Relation (2.3):

$$\mu_{n+1}^{(k)} = (1+\frac{k}{n})\mu_n^{(k)} - \frac{\nu k}{2n(k+2)}\left[\mu_n^{(k+2)} - \mu_{n-1}^{(k+2)}\right]. \qquad (2.8)$$

$$\mu_2^{(k)} = (1+k)\mu_1^{(k)} - \frac{\nu k}{2(k+2)}\mu_1^{(k+2)}. \qquad (2.9)$$

Proof:

For any continuous distribution function, we have

$$\mu_{n+1}^{(k)} - \mu_n^{(k)} = \frac{k}{\Gamma(n+1)}\int_0^\infty x^{k-1}[-\log[1-F(x)]]^n[1-F(x)]dx. \qquad (2.10)$$

Using (1.1) in (2.10), we obtain

$$\begin{aligned}\mu_{n+1}^{(k)} - \mu_n^{(k)} &= \frac{k}{\Gamma(n+1)}\int_0^\infty x^{k-1}[-\log[1-F(x)]]^{n-1}(\lambda x+\frac{\nu x^2}{2})[1-F(x)]dx\\ &= \frac{k}{\Gamma(n+1)}\int_0^\infty x^k[-\log[1-F(x)]]^{n-1}(\lambda+\nu x-\frac{\nu x}{2})[1-F(x)]dx\\ &= \frac{k}{n}\mu_n^{(k)} - \frac{\nu k}{2\Gamma(n+1)}\int_0^\infty x^{k+1}[-\log[1-F(x)]]^{n-1}[1-F(x)]dx.\end{aligned} \qquad (2.11)$$

Using (2.10) in (2.11), (2.8) is obtained.

Putting $n = 1$ in (2.8) we get (2.9).

Relation (2.4):

$$\mu_n^{(k)} = \frac{1}{\Gamma(n)} \sum_{r=0}^{n-1} \binom{n-1}{r} (\lambda)^{n-1-r} \left(\frac{\nu}{2}\right)^r \mu_1^{n+r+k-1}, \quad n > 1. \qquad (2.12)$$

Proof:

From (1.1) and (2.2), we have

$$\begin{aligned} \mu_n^{(k)} &= \frac{1}{\Gamma(n)} \int_0^\infty x^k (\lambda x + \frac{\nu x^2}{2})^{n-1} f(x) dx \\ &= \frac{1}{\Gamma(n)} \sum_{r=0}^{n-1} \binom{n-1}{r} \lambda^{n-1-r} \left(\frac{\nu}{2}\right)^r \int_0^\infty x^{n+k+r-1} f(x) dx \\ &= \frac{1}{\Gamma(n)} \sum_{r=0}^{n-1} \binom{n-1}{r} \lambda^{n-1-r} \left(\frac{\nu}{2}\right)^r \mu_1^{n+k+r-1}. \end{aligned}$$

Theorem 2.1: If k is even, $k \geq 2$

$$\mu_n^{(k)} - \mu_{n-1}^{(k)} = \frac{k\lambda^{k-2}}{\gamma^{k-1}} \sum_{i=0}^{k-2} (-1)^i \left(\frac{\gamma}{\lambda}\right)^i \mu_n^{(i)} + k\left(\frac{\lambda}{\gamma}\right)^{k-1} (-1)^{k-1} (\mu_n - \mu_1) \quad (2.13)$$

and

If k is odd, $k \geq 2$

$$\mu_n^{(k)} - \mu_{n-1}^{(k)} = \frac{k\lambda^{k-2}}{\gamma^{k-1}} \sum_{i=0}^{k-2} (-1)^{i-1} \left(\frac{\gamma}{\lambda}\right)^i \mu_n^{(i)} + k\left(\frac{\lambda}{\gamma}\right)^{k-1} (-1)^{k-1} (\mu_n - \mu_1) \quad (2.14)$$

Proof:

By repeatedly applying the recurrence relation in (2.4), we simply derive the recurence relation in (2.13) and (2.14).

Theorem 2.2:

$$\mu_n^{(k+1)} - \mu_{n-1}^{(k+1)} = \frac{2n(k+1)}{\lambda k} \mu_{n+1}^{(k)} - \frac{k+1}{\lambda} \left(\frac{2n}{k} - 1\right) \mu_n^{(k)} \qquad (2.15)$$

Proof:

From (1.1) and (1.2), it is clear that

$$xf(x) = \{2[-\log[1-F(x)] - \lambda x\}[1-F(x)] \tag{2.16}$$

Using (2.16) and (2.1), we get

$$\begin{aligned}\mu_n^{(k)} &= \frac{1}{\Gamma(n)}\int_0^\infty x^{k-1}[-\log[1-F(x)]]^{n-1}[2[-\log[1-F(x)] - \lambda x][1-F(x)]dx \\ &= \frac{2}{k\Gamma(n)}\int_0^\infty [-\log[1-F(x)]]^{n}[1-F(x)]dx^k \\ &\quad -\frac{\lambda}{(k+1)\Gamma(n)}\int_0^\infty [-\log[1-F(x)]]^{n-1}[1-F(x)]dx^k \end{aligned} \tag{2.17}$$

Integrating (2.17) by parts, we obtain

$$\mu_n^{(k)} = \frac{2n}{k}[\mu_{n+1}^{(k)} - \mu_n^{(k)}] - \frac{\lambda}{k+1}[\mu_n^{(k+1)} - \mu_{n-1}^{(k+1)}].$$

Hence, the theorem is proved.

3 Relation For Product Moments of Upper Records

The joint density of $X_{U(m)}$ and $X_{U(n)}, 1 \leq m < n, m = 0, 1, \ldots$ is given by

$$\begin{aligned} f_{m,n}(x,y) &= \frac{1}{\Gamma(m)\Gamma(n-m)}[-\log[1-F(x)]]^{m-1} \\ &\quad [-\log[1-F(y) + \log[1-F(x)]]^{n-m-1} \\ &\quad \times \frac{f(x)}{1-F(x)}f(y), \qquad x < y. \end{aligned} \tag{3.1}$$

Then, the product moments of upper records is

$$\mu_{m,n}^{(j,k)} = \int_0^\infty \int_x^\infty x^j y^k f_{m,n}(x,y)dydx. \tag{3.2}$$

We derive some recurrence relations for product moments of record values.

Relation (3.1):

$$\mu_{1,n}^{(j,k)} = \frac{j}{\Gamma(n)} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \sum_{L_1=0}^{n-1-i} \binom{n-1-i}{L_1} \sum_{L_2=0}^{i} \binom{i}{L_2} \times \lambda^{L_1+L_2} \left(\frac{\nu}{2}\right)^{n-1-L_1-L_2} \times \frac{\mu_1^{(2n+j+k-L_1-L_2-2)}}{j+2i-L_2}, \qquad n \geq 1. \tag{3.3}$$

Proof.

From (3.1) and (3.2), we have

$$\begin{aligned}\mu_{1,n}^{(j,k)} &= \frac{1}{\Gamma(n-1)} \int_0^\infty \int_0^y x^j y^k [-\log[1-F(y)+\log[1-F(x)]^{n-2} \\ &\quad \times \frac{f(x)}{1-F(x)} f(x) f(y)\, dx\, dy \\ &= \frac{-1}{\Gamma(n)} \int_0^\infty y^k \int_0^y x^j d[-\log[1-F(y)+\log[1-F(x)]^{n-1} f(y)\, dy\end{aligned} \tag{3.4}$$

Integrating (3.4) by parts, (3.3) is obtained.

For $j = k = 1, n = 2$, we get

$$\mu_{1,2} = \frac{\lambda}{2}\mu_1^{(3)} + \frac{\nu}{3}\mu_1^{(4)} \tag{3.5}$$

Relation (3.2):

$$\mu_{m,m+1}^{(j,k)} = \frac{1}{\Gamma(m)} \sum_{i=0}^{m-1} \binom{m-1}{i} \lambda^i \left(\frac{\nu}{2}\right)^{m-1-i} \left[\frac{\lambda\mu_1^{(2m+j+k-i-1)}}{2m+j-i-1} + \frac{\nu\mu_1^{(2m+j+k-i)}}{2m+j-i}\right], \quad m \geq 1 \tag{3.6}$$

Proof.

From (3.2), we have

$$\mu_{m,m+1}^{(j,k)} = \frac{1}{\Gamma(m)} \int_0^\infty \int_0^y x^j y^k [-\log[1-F(x)]^{m-1} \frac{f(x)}{1-F(x)} f(y)\, dx\, dy \tag{3.7}$$

From (1.3) and (3.7), we get

$$\mu_{m,m+1} = \frac{1}{\Gamma(m)} \int_0^\infty \int_0^y x^j y^k (\lambda x + \frac{\nu x^2}{2})^{m-1} (\lambda + \nu x) f(y) dx dy$$

Then, it easy to obtain (3.6).

Relation (3.3):

For the existence of the jth and $(k+2)$th product moments, are derived assuming their existence

$$\mu_{m,n}^{(j,k)} = \frac{\lambda}{k+1}[\mu_{m,n}^{(j,k+1)} - \mu_{m,n-1}^{(j,k+1)}] + \frac{\nu}{k+2}[\mu_{m,n}^{(j,k+2)} - \mu_{m,n-1}^{(j,k+2)}], \quad 1 < m < n, n > 1. \tag{3.8}$$

Proof:

Using (1.3) in (3.2), we have

$$\begin{aligned} \mu_{m,n}^{(j,k)} &= \frac{\lambda}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_x^\infty x^j y^k [-\log[1-F(x)]]^{m-1} \frac{f(x)}{1-F(x)} \\ &\times [-\log[1-F(y)] + \log[1-F(x)]]^{n-m} [1-F(y)] dy dx \\ &+ \frac{\nu}{\Gamma(n)\Gamma(n-m)} \int_0^\infty \int_x^\infty x^j y^{k+1} [-\log[1-F(x)]]^{m-1} \frac{f(x)}{1-F(x)} \\ &\times [-\log[1-F(y)] + \log[1-F(x)]]^{n-m} [1-F(y)] dy dx. \end{aligned} \tag{3.9}$$

The first integral is computed as follows:

$$\begin{aligned} I_1 &= \frac{\lambda}{(k+1)\Gamma(m)\Gamma(n-m)} \int_0^\infty x^j [-\log[1-F(x)]]^{m-1} \frac{f(x)}{1-F(x)} \\ & \int_x^\infty [-\log[1-F(y)] + \log[1-F(x)]^{n-m} [1-F(y)] dy^{k+1} dx. \end{aligned}$$

Integrating by parts w.r.t. y we obtain

$$I_1 = \frac{\lambda}{(k+1)} \left[\mu_{m,n}^{(j,k+1)} - \mu_{m,n-1}^{(j,k+1)}\right]. \tag{3.10}$$

Also, the second integral is given by

$$I_2 = \frac{\nu}{(k+2)} \left[\mu_{m,n}^{(j,k+2)} - \mu_{m,n-1}^{(j,k+1)}\right]. \tag{3.11}$$

Substituting (3.10) and (3.11) in (3.9), relation (3.3) is obtained.

Relation (3.4):

$$\mu_{m+1,n}^{(j,k)} = \frac{\lambda}{m}\mu_{m,n-1}^{(j+1,k)} + \frac{\gamma}{2m}\mu_{m,n-1}^{(j+1,k)}, \quad 1 \leq m < n, n > 1. \tag{3.12}$$

Proof:

It is clear that

$$\begin{aligned}\mu_{m,n}^{(j,k)} &= \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_x^\infty (\lambda x^{j+1} + \frac{\nu}{2}x^{(j+2)})y^k \\ &\times[-\log[1-F(x)]]^{m-2}\frac{f(x)}{1-F(x)} \\ &\times[-\log[1-F(y)+\log[1-F(x)]]^{n-m-1}f(y)dydx, \quad (3.13)\end{aligned}$$

which gives

$$\mu_{m,n}^{(j,k)} = \frac{\lambda}{m}\mu_{m-1,n-1}^{(j+1,k)} + \frac{\nu}{2}\mu_{m-1,n-1}^{(j+2,k)}, \; n > m > 1.$$

Theorem (3.1):

$$\mu_{m+1,n}^{(j+1,k)} - \mu_{m,n}^{(j+1,k)} = \frac{2j}{j+1}\mu_{m+1,n}^{(j,k)} - \frac{2j(j+1)}{\lambda j}(1+\frac{j}{2m})\mu_{m,n}^{(j,k)}. \tag{3.14}$$

Proof:

From (3.2) and (2.16), we have

$$\begin{aligned}\mu_{m,n}^{j,k} &= \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_0^y x^{j-1}y^k[-\log[1-F(x)]^{m-1}[-\log[1-F(x) \\ &+\log[1-F(x)]^{n-m-1}[2\{-\log[1-F(x)\}-\lambda x]f(y)dy \\ &= \frac{2}{j\Gamma(m)\Gamma(n-m)}\int_0^\infty y^k\int_0^y[-\log[1-F(x)]]^{m-1} \\ &\times[-\log[1-F(y)+\log[1-F(x)]^{n-m-1}\times dx^j f(y)dy \\ &+\frac{\lambda}{(j+1)\Gamma(n-m)\Gamma(m)}\int_0^\infty y^k\int_0^y[-\log[1-F(x)]^{m-1} \\ &\times[-\log[1-F(y)+\log[1-F(x)]^{n-m}dx^{j+1}f(y)dy.\end{aligned}$$

Integrating by parts, we obtain

$$\mu_{m,n}^{(j,k)} = \frac{2m}{j}[\mu_{m+1,n}^{j,k} - \mu_{m,n}^{j,k}] - \frac{\lambda j}{j+1}[\mu_{m+1,n}^{j+1,n)} - \mu_{m,n}^{j+1,k}],$$

and then the theorem is proved.

For $\lambda = 0$ in above theorem, we obtain

$$\mu_{m+1,n}^{(j,k)} = (1 + \frac{j}{2m})\mu_{m,n}^{(j,k)}$$

(Balakrishnan and Chan (1993), Theorem 3).

4 Characterization of the Linear Exponential Distribution

In this section, we state and prove two theorems for characterizing the linear exponential distribution through the properties of upper record values.

For the left truncation at x, it is clear that

$$f(x_{U(r+1)}|X_{U(r)} = x) = \frac{f(y)}{1 - F(x)}, \qquad x < y, \tag{4.1}$$

and

$$E[X_{U(r+1)}|x_{U(r)} = x] = x + \frac{e^{\frac{\lambda^2}{2\nu}}}{1 - F(x)}\sqrt{\frac{2\pi}{\nu}}[1 - \phi(\sqrt{\nu}x + \frac{\lambda}{\sqrt{\nu}})], \tag{4.2}$$

where $\phi(.)$ is the standard normal distribution.

Also, from right truncation at y, it is clear that

$$f(x_{U(1)}|X_{U(2)} = y] = \frac{f(x)}{[1 - F(x)][-\log[1 - F(y)]]}, \qquad x < y, \tag{4.3}$$

and

$$E[X_{U(1)}|X_{U(2)} = y] = \frac{y^2(3\lambda + 2\nu y)}{6[-\log[1 - F(y)]]}. \tag{4.4}$$

Theorem 4.1

If $F(x) < 1$, be a.c.d.f. of the random variable X, then

$$F(x) = 1 - e^{-(\lambda x + \frac{\nu x^2}{2})}, \quad \lambda > 0,\ \nu > 0,\ 0 < x < \infty \text{ iff}$$

$$E(X_{U(r+1)} | X_{U(r)} = x] = x + \frac{e^{\frac{\lambda^2}{2\nu}}}{1 - F(x)} \sqrt{\frac{2\pi}{\nu}} \left[1 - \phi(\sqrt{\nu} x + \frac{\lambda}{\sqrt{\nu}})\right]. \quad (4.5)$$

Proof:

The necessity condition is proved from (4.2). To prove the sufficient condition, from (4.1) and (4.5), we have

$$\int_x^\infty y f(y) dy = x[1 - F(x)] + e^{\frac{\lambda^2}{2\nu}} + \sqrt{\frac{2\pi}{\nu}} [1 - \phi(\sqrt{\nu}(x + \frac{\lambda}{\sqrt{\nu}}))]$$

Integrating both sides w. r. t. x, we obtain

$$-x f(x) = -x f(x) + [1 - F(x)] + e^{\frac{\lambda^2}{2\nu}} e^{-\frac{\nu}{2}(x + \frac{\lambda}{\nu})^2}.$$

Then

$$1 - F(x) = e^{-(\lambda x + \frac{\nu x^2}{2})}.$$

The theorem is proved.

Theorem 4.2:

If the random variable X has a c.d.f. $F(x)$ $(0 < F(x) < 1)$, $F(0) = 0$, then

$$F(x) = 1 - e^{(-\lambda x + \frac{\nu x^2}{2})}, \quad x \geq 0, \lambda > 0, \nu > 0 \text{ iff}$$

$$E(X_{U(1)} | X_{U(2)} = y] = \frac{y^2(3\lambda + 2\nu y)}{6[-\log[1 - F(y)]]}, \quad x < y. \quad (4.6)$$

Proof:

The necessity part is proved from (4.4). To prove the sufficient part, from (4.3) and (4.6), we have

$$\int_0^y \frac{xf(x)}{1-F(x)} = \frac{y^2(3\lambda + 2\nu y)}{6}.$$

Integrating both sides w.r.t. y, we give

$$\frac{yf(y)}{1-F(y)} = \frac{6\lambda y + 6\nu y^2}{6} = y(\lambda + \nu y).$$

Then

$$\frac{f(y)}{1-F(y)} = \lambda + \nu y.$$

The theorem is proved.

REFERENCES

Ahsanullah, M. (1988). Introduction to record values, Ginn Press, Needham Heights, MA

Arnold, B. C. and Balakrishnan, N. (1989). Relation bounds and approximation for order statistics, Lecture Notes in Statistics, 53, Springer-Verlage, New York.

Arnold, B. C. and Balakrishnan, N. and Nagraga, H. N. (1992). A First course in order statistics, John Wiley & Sons, New York.

Balakrishnan, N. and Ahsanullah, M. (1995). Relation for single and product moments of record values from exponential distributions. J. Appl. Statist., Vol. 2, No. 1, 73-87.

Balakrishnan, N. and Ahsanullah, M. (1994). Recurrence relations for single and product moments of record values from generalized Pareto distribution. Commun. Statist. Theor., Meth., 23(10), 2841-2852.

Balakrishnan, N., Ahsanullah, M. and Chan, P. S. (1992). Relations for the single and product moments of record values from Gumble distribution. Statist. Prob. Lett. 15, 223-227.

Balakrishnan, N. and Chan, P. S. (1993). Record values from Rayleigh and Weibull distribution and associated inference. National Institute of Standards and technology Journal of Research special publications 866, 41-51.

Balakrishnan, N. Chan, P. S. and Ahsanullah, M. (1993). Recurrence relations for moments of record values from generalized extreme value distribution. Commun. Statist. Theor. Meth., 22(5), 1471-1482.

Balakrishnan, N., Malik, H. and Ahmed, S. E. (1988). Recurrence relations and identities for moments of order statistics, II specific continuous distribution. Commun. Statist. Theor. Meth. 17(8), 2657-2694.

Chandler, K. M. (1952). The distribution and frequency of record values. J. Roy, Statist. Soc. B, 14, 220-228.

Feller, W. (1966). An introduction to probability theory and its applications, Vol. 2, second edition, John Wiley & Sons, New-York.

Glick, N. (1978). Breaking records and breaking boards. Amer. Math. Monthly, 85, 2-26.

Mohie El-Din, M. M., Mahmoud, M. A. W., Abu-Youssef, S. E. and Sultan, K. S. (1997). Order statistics from the doubly truncated linear exponential distribution and its characterization. Commun. Statist. Simula. 26(1), 281-290.

Nagraja, H. N. (1988). Record values and related statistics. Review Commun. Statist. Theory-Meth. 17, 2223-2238.

Nevzorov, V. B. (1987). Records theor. prob. Appl. 32, 201-228.

Resnick, S. I. (1973). Record values and maxima, Ann. Probab. 2, 650-662.

Shorrock, R. W. (1973). Record values and inter-record times. J. Appl. Prob., 10, 543-555.