THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CAIRO UNIV., VOL., 43 NO. 1, 1999

CHARACTERIZATION OF THE THREE PARAMETERS GAMMA DISTRIBUTION WITH MIXING DISTRIBUTIONS

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ABSTRACT

Using Laplace transforms, necessary and sufficient conditions are given for the characterizations of the three parameters gamma distribution using five different mixing distributions. The achieved results generalize some known results in this connection and have its relevance to some practical applications.

Key words: Exponential and gamma distribution; Fourier and Laplace transform; characterization; mixing distributions.

1- INTRODUCTION

The probability density function (pdf) of the two parameters exponential distribution is defined by

$$f(x;\theta,\mu) = (1/\theta) \exp\{-(x-\mu)/\theta\}; \quad x \geq \mu \geq 0 \qquad (1.1)$$

where the scale parameter, θ is itself a random variable and the location parameter, μ is known. The characteristic and reliability functions of (1.1) are

$$\phi_x(t) = \exp(it\mu) / (1 - it\theta) \qquad (1.2)$$

and

$$R(t_o) = \exp\{-(t_o - \mu)/\theta\}; \quad t_o > \mu$$

We assume that n items from the population (1.1) are placed on a life test that and the experiment is continued until all items are failed and let the accumulated observed times, T is equal $\sum_{i=1}^{n} x_i$, where $x_1, x_2, \ldots, x_n$ are an independent random sample of size n.

Let T be a nonnegative continuous random variable (r.v.) with probability density function (pdf), $f(T/\theta)$ where the parameter $\theta > 0$ is a r.v. with pdf., $g(\theta)$. The pdf, $f(T)$ of the mixture distribution is given by

$$f(T) = \int_0^\infty f(T/\theta) g(\theta) d\theta$$

where $f(T/\theta)$ can be obtained using the inversion formula of characteristic function $\phi_T(t)$, where $\phi_T(t) = [\phi_x(t)]^n$ and $\phi_x(t)$ is defined by (1.2). We note that $\phi_T(t)$ is the fourier transform of $f(T/\theta)$, in particular that, $f(T/\theta)$ can be expressed in terms of $\phi_T(t)$ as

$$f(T/\theta) = (1/2\pi) \int_{-\infty}^{\infty} \exp(-itT)\, \phi_T(t) dt$$

$$= \frac{\theta^{-n}}{\Gamma(n)} (T-n\mu)^{n-1} \exp\{-(T-n\mu)/\theta\}; \; T \geq n\mu \qquad (1.3)$$

which is the three parameters gamma distribution, $g(\theta)$ is usually called the mixing distribution, and

$$\Gamma(n) = \int_0^\infty y^{n-1} e^{-y}\, dy \text{ be the complete gamma function}$$

$= (n-1)!$ for the positive integer values of n.

$= n\,\Gamma(n-1)$ for the real values of n.

The problem of characterizations of probability distributions has been discussed by many authors, see for example, Galambos and Kotz (1978). The main results concerning the characterization problems associated with the exponential distribution are surveyed by Azlarov and Volodin (1986). Mixing distributions of the exponential distribution with other continuous distributions are used in many practical problems, see for example, Blischke (1978), Gupta (1981) and Gharib (1996).

Characterization of the gamma distribution by the negative binomial distribution has been discussed by Engel and el (1980). Finally Block and Savits (1980) has used the Laplace transform to give necessary and sufficient conditions for a classes of life distributions which have increasing and decreasing failure rate.

In this paper some characterization results for the three parameters gamma distribution (1.3) are given using five different mixing distributions, $g(\theta)$. The achieved results incorporate the identification of this distribution.

2- THE MIXING DISTRIBUTIONS

Let $\theta > 0$ be a random variable with pdf $g(\theta)$, and the problem of characterization will be examined for five types of mixing distributions, $g(\theta)$, namely

$$\text{(i)}\quad g_1(\theta) = \frac{H^{\nu-1}}{\Gamma(\nu-1)}\ \theta^{-\nu}\ e^{-H/\theta};\ \theta > 0,\ H > 0,\ \nu > 1 \qquad (2.1)$$

which is the inverted gamma density of Raiffa and Schlaifer (1961).

$$\text{(ii)}\quad g_2(\theta) = \frac{H^{\nu-1}}{\Gamma(\nu-1)}\ \theta^{\nu-2}\ e^{-H\theta};\ \theta > 0,\ \nu > 1 \qquad (2.2)$$

which is the gamma density and for $\nu = 2$, the exponential density is obtained.

$$\text{(iii)}\quad g_3(\theta) = \sqrt{\frac{\lambda}{2\pi}}\ \theta^{-3/2}\ \exp\left\{\frac{-\lambda}{2m^2\theta}\ (\theta-m)^2\right\};\ \theta,\lambda,m>0 \qquad (2.3)$$

which is the inverse Gausian density. As $m \to \infty$, the distribution of $y = \theta^{-1}$ tends to the gamma distribution [see Wasan and Roy (1967).

(iv) $g_4(\theta) = \frac{\alpha^{1-p-q}}{\beta(p,q)}\ \theta^{p-1}\ (\alpha-\theta)^{q-1};$

$$0 < \theta < \alpha;\ p,\ q > 0 \tag{2.4}$$

which be the beta density over the interval $[0,\ \alpha]$ and for $\alpha = 1$, the beta density is obtained over the interval $[0,1]$. Here $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ be the beta function.

(v) $g_5(\theta) \propto \theta^{-\alpha}\ ;\ \theta > 0$ (2.5)

which is the quasi-density of Bhattacharya (1967). For the case $\alpha = 0$, (2.5) reduces to a constant improper density $g(\theta) = 1$ over the positive real line.

In Bayesian analysis, the first four cases of $g(\theta)$ represent, the full prior information about the parameter θ while the last case used when the experimenter's has no prior information about the parameter θ.

3- THE RESULTS

Let T be a nonnegative continuous random variable with pdf , $f(T/\theta)$ defined by (1.3) and the parameter $\theta > 0$ is a random variable with pdf $g_i(\theta)$, $i = 1,2,3,4,5$ defined by (2.1) - (2.5). The pdf $f(T)$ of the mixture distribution is given by

$$f_i(T) = \int_0^\infty f(T/\theta)\ g_i(\theta)\ d\theta;\ i = 1,2,3,4,5 \tag{3.1}$$

The main theorem can now be given.

<u>Theorem</u>:

(i) given that θ has a pdf (2.1), then the conditional distribution of T is (1.3) if and only if the mixture distribution $f_1(T)$ is

$$f_1(T) = \frac{H^{-n}}{\beta(n,\ \nu-1)} (T - n\mu)^{n-1} (1 + \frac{T-n\mu}{H})^{-(n+\nu-1)}; \quad T \geq n\mu$$

where $\beta(n,\nu-1)$ is the well known beta function and $\nu > 1$. The transformed variabe $y = (T-n\mu)/H$ has a beta distribution of second type with parameters n, $\nu - 1$ and $y > 0$.

(ii) given that θ has a pdf (2.2), then the conditional distribution of T is (1.3) if and only if the mixture distribution $f_2(T)$ is

$$f_2(T) = \frac{2.H}{\Gamma(n)\Gamma(\nu-1)} [H \quad (T-n\mu)]^{\frac{1}{2}(n+\nu-1)-1} K_{(\nu-n-1)} (2\sqrt{H(T-n\mu)}); \quad T\geq n\mu$$

which is the four parameters compound gamma distribution defined by Johnson and Kotz (1970). where $K_a(b)$ is the modified Bessel function of the third kind of order a, which is defined in Gradshteyn and Ryzhik (1980).

(iii) given that θ has a pdf (2.3), then the conditional distribution of T is (1.3) if and only if the mixture distribution $f_3(T)$ is

$$f_3(T) = \frac{2\ \exp(\lambda/m)\sqrt{\lambda}}{\sqrt{2\pi}\ \Gamma(n)} (T-n\mu)^{n-1}[\frac{\lambda}{2m^2(T-n\mu+\frac{\lambda}{2})}]^{\frac{1}{2}(n+\frac{1}{2})} \times$$

$$\times K_{(n+\frac{1}{2})} (2 \sqrt{\frac{\lambda(T-n\mu+\lambda/2)}{2m^2}}); \quad T \geq n\mu$$

It is easy to verify that, since

$$\int_{n\mu}^{\infty} f_3(T)dT = \frac{2\ \exp\ (\lambda/m)\sqrt{\lambda}}{\sqrt{2\pi}\ \Gamma(n)} \int_{n\mu}^{\infty} (T - n\mu)^{n-1}$$

$$[\frac{\lambda}{2m^2(T-n\mu+\frac{\lambda}{2})}]^{\frac{1}{2}(n+\frac{1}{2})} K_{(n+\frac{1}{2})} (2 \sqrt{\frac{\lambda(T-n\mu+\lambda/2)}{2m^2}})\ dT$$

$$= \frac{\exp(\lambda/m)\sqrt{\lambda}}{\sqrt{2\pi}\,\Gamma(n)} \int_{n\mu}^{\infty} (T - n\mu)^{n-1} \int_{0}^{\infty} x^{n-1/2} \exp\left\{- \frac{\lambda}{2m^2 x} -\right.$$

$$\left. - x(T - n\mu + \lambda/2)\right\} dx\, dT$$

Since x and T are independent, changing the order of integration on the right-hand side, we have

$$= \frac{\exp(\lambda/m)\sqrt{\lambda}}{\sqrt{2\pi}\,\Gamma(n)} \int_{0}^{\infty} x^{n-1/2} \exp\left(- \frac{\lambda}{2m^2 x} - \frac{x\lambda}{2}\right) \times$$

$$\times \int_{n\mu}^{\infty} (T-n\mu)^{n-1} \exp\{-x(T-n\mu)\}\, dT dx$$

$$= \sqrt{\frac{\lambda}{2\pi}} \int_{0}^{\infty} y^{-3/2} \exp\{- \frac{\lambda}{2m^2 y} (y-m)^2\} dy = 1$$

Therefore, $f_3(T)$ is a probability density function.

(iv) Given that, θ has a pdf (2.4), then the conditional distribution of T is (1.3) if and only if the mixture distribution $f_4(T)$ is

$$f_4(T) = \frac{\alpha^{-n}}{\Gamma(n)\beta(p,q)} (T-n\mu)^{n-1} \exp\left\{- \frac{(T-n\mu)}{\alpha}\right\} \times$$

$$\times L_T[(\frac{y}{1+y})^{p-n-1}(1 - \frac{y}{1+y})^{q-1} e^{Ty} \times$$

$$\times \frac{1}{(1+y)^2} \exp\left\{- \frac{(T-n\mu)}{\alpha}\right\}];\ T \geq n\mu.$$

Where L_T denote the Laplace transform with T the parameter of the transformation.

(v) Given that, θ has a pdf (2.5), then the conditional distribution of T is (1.3) if and only if the mixture distribution $f_5(T)$ is

$$f_5(T) = \frac{\Gamma(n + \alpha - 1)}{\Gamma(n)} (T - n\mu)^{-\alpha}$$

Proof of theorem:

Necessary and sufficient conditions for the above cases will be given as follows:

1. **For the case (i):**

Necessity:

If T has the density (1.3) and θ has the density (2.1); then according to (3.1) we have

$$f_1(T) = \frac{H^{\nu-1}(T-n\mu)^{n-1}}{\Gamma(\nu-1)\ \Gamma(n)} \int_0^\infty \theta^{-(n+\nu)}\ e^{-(T-n\mu+H)/\theta}\ d\theta$$

$$= \frac{H^{-n}}{\beta(n,\nu-1)}\ (T-n\mu)^{n-1}\left(1+\frac{T-n\mu}{H}\right)^{-(n+\nu-1)} \qquad (3.2)$$

where $T \geq n\mu$, μ, $H > 0$, $\nu > 1$ and n is the sample size. The transformed variable $x = T - n\mu$ for the R.H.S. of (3.2) has the density function of the F-distribution with degrees of freedom 2n and $2(\nu-1)$ under the condition that $H = \frac{n}{\nu-1}$ and if we put $x = e^{2Z}$, we can get the pdf of Fisher's Z-distribution with $-\infty < Z < \infty$.

Sufficiency:

By using equation (3.1) with $g(\theta)$ defined by (2.1), we have

$$\frac{H^{\nu-1}}{\Gamma(\nu-1)} \int_0^\infty \theta^{-\nu} f(T/\theta)\ e^{-H/\theta}\ d\theta = \frac{H^{-n}\ (T-n\mu)^{n-1}}{\beta(n,\nu-1)\left(1+\frac{T-n\mu}{H}\right)^{n+\nu-1}}$$

i.e.,

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} \int_0^\infty Z^{\nu-2}\ f(T/Z^{-1})\ e^{-ZH}\ dZ = \frac{\Gamma(n+\nu-1)}{(H+T-n\mu)^{n+\nu-1}},$$

where $Z = \theta^{-1}$.

This can be written as

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} L_H \left\{ Z^{\nu-2} f(T/\theta) \right\} = \frac{\Gamma(n+\nu-1)}{(H+T-n\mu)^{n+\nu-1}} \tag{3.3}$$

where L_H denotes the Laplace transform with H as the parameter of the transformation.

Now the R.H.S. of (3.3) is the Laplace transform of the function $Z^{n+\nu-2} \exp\{-Z(T-n\mu)\}$. Hence, we must have

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} Z^{\nu-2} f(T/\theta) = Z^{n+\nu-2} e^{-Z(T-n\mu)}$$

Consequently, under $Z = \theta^{-1}$, we have

$$f(T/\theta) = \frac{\theta^{-n}}{\Gamma(n)} (T-n\mu)^{n-1} e^{-\frac{T-n\mu}{\theta}}; \; T > n\mu, \; \theta > 0.$$

Since the Laplace transform is a one-to-one mapping and given the uniqueness of the inverse Laplace transform, the proof of the case (i) is complete.

(2) For the case (ii):

Necessity:

If the conditional distribution of T has the density (1.3) and if θ has the density (2.2), then according to (3.1), we have:

$$f_2(T) = \frac{H^{\nu-1}(T-n\mu)^{n-1}}{\Gamma(n)\Gamma(\nu-1)} \int_0^\infty \theta^{\nu-n-2} e^{-\frac{(T-n\mu)}{\theta} - H\theta} d\theta$$

$$= \frac{2H^{\frac{1}{2}(\nu+n-1)}}{\Gamma(n)\Gamma(\nu-1)} (T-n\mu)^{\frac{1}{2}(\nu+n-3)} K_{(\nu-n-1)} \{2\sqrt{H(T-n\mu)}\},$$

where $T \geq n\mu$, $H > 0$ and $\nu > 1$

Sufficiency:

By using (3.1) with $g_2(\theta)$ defined by (2.2), we have

$$\frac{H^{\nu-1}}{\Gamma(\nu-1)} \int_0^\infty \theta^{\nu-2} f(T/\theta) e^{-H\theta} d\theta =$$

$$= 2 \frac{H^{\frac{1}{2}(\nu+n-1)}}{\Gamma(n)\Gamma(\nu-1)} (T-n\mu)^{\frac{1}{2}(\nu+n-3)} K_{(\nu-n-1)} \{2\sqrt{H(T-n\mu)}\},$$

i.e.,

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} \int_0^\infty \Theta^{\nu-2} f(T/\Theta)\ e^{-H\Theta}\, d\Theta = 2\left(\frac{T-n\mu}{H}\right)^{\frac{(\nu-n-1)}{2}} K_{(\nu-n-1)} \{2\sqrt{H(T-n\mu)}\}.$$

This can be written as

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} L_H \left\{\Theta^{\nu-2} f(T/\Theta) \right\} = 2 \left(\frac{T-n\mu}{H}\right)^{\frac{(\nu-n-1)}{2}} K_{(\nu-n-1)} \{2\sqrt{H(T-n\mu)}\} \tag{3.4}$$

Now the R.H.S. of (3.4)is the Lapace transform of the function $\Theta^{\nu-n-2}\ e^{-\frac{(T-n\mu)}{\Theta}}$ [See Doetsch (1971)].

Thus we get

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} \left\{\Theta^{\nu-2} f(T/\Theta) \right\} = \Theta^{\nu-n-2}\ e^{-\frac{(T-n\mu)}{\Theta}}$$

or

$$f(T/\Theta) = \frac{\Theta^{-n}}{\Gamma(n)} (T-n\mu)^{n-1}\ e^{-\frac{(T-n\mu)}{\Theta}} ;\ T \geq n\mu.$$

The uniqueness of the inverse Laplace transform completes the proof of the case (ii). [See for example Jaggi and Mathuer (1985)].

(3) <u>For the case (iii)</u>:

<u>Necessity</u>:

If the conditional distribution of T has the density (1.3) and Θ has the density (2.3), then according to (3.1), we get:

$$f_3(T) = \sqrt{\frac{\lambda}{2\pi}} \cdot \frac{(T-n\mu)^{n-1}}{\Gamma(n)} \int_0^\infty \theta^{-(n+3/2)} e^{-1/\theta[(T-n\mu) + \frac{\lambda(\theta-m)^2}{2m^2}]}$$

$$= \sqrt{\frac{\lambda}{2\pi}} \; e^{\lambda/m} \; (2) \frac{(T-n\mu)^{n-1}}{\Gamma(n)} \left[\frac{\lambda}{2m^2(T-n\mu+\lambda/2)}\right]^{\frac{1}{2}(n+\frac{1}{2})}$$

$$K_{(n+1/2)} \{\frac{2\sqrt{\lambda(T-n\mu+\lambda/2)}}{2m^2}\}, \quad T \geq n\mu, \; \lambda, \; m > 0. \qquad (3.5)$$

[See for example Gradshteyn and Ryzhik (1980)].

<u>Sufficiency</u>:

If the conditional distribution of T has the density $f(T/\theta)$ and θ has the density (2.3), then we have

$$\int_0^\infty \theta^{-3/2} f(T/\theta) \; e^{-\lambda(\theta-m)^2/2\theta m^2} \, d\theta =$$

$$= 2. \; e^{\lambda/m} \frac{(T-n\mu)^{n-1}}{\Gamma(n)} \left[\frac{\lambda}{2m^2(T-n\mu+\lambda/2)}\right]^{\frac{1}{2}(n+\frac{1}{2})}$$

$$K_{(n+1/2)} \{\frac{2\sqrt{\lambda(T-n\mu+\lambda/2)}}{2m^2}\}.$$

i.e.,

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} \int_0^\infty [\theta^{-3/2} f(T/\theta) \; e^{-\lambda/2\theta}] \; e^{-\beta\theta} \, d\theta$$

$$= 2 \left[\frac{\lambda}{2m^2(T-n\mu+\lambda/2)}\right]^{\frac{1}{2}(n+\frac{1}{2})} K_{(n+1/2)} \{\frac{2\sqrt{\lambda(T-n\mu+\lambda/2)}}{2m^2}\},$$

where $\beta = \lambda/2m^2 > 0$.

This can be written as

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} L_\beta \left\{\Theta^{-3/2} f(T/\Theta)\ e^{-\lambda/2\Theta}\right\}$$

$$= 2\left[\frac{2m^2(T-n\mu+\lambda/2)}{\lambda}\right]^{-\frac{1}{2}(n+\frac{1}{2})} K_{(n+1/2)}\{2\sqrt{\frac{\lambda(T-n\mu+\lambda/2)}{2m^2}}\}, \tag{3.6}$$

where $K_{-a}(.) = K_a(.)$

Now the R.H.S. of (3.6) is the Laplace transform of the function $\Theta^{-n-3/2}\ e^{-\frac{(T-n\mu+\lambda/2)}{\Theta}}$ [See Doetsch (1971)].

Therefore, we have

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} \left\{\Theta^{-3/2} f(T/\Theta)\ e^{-\lambda/2\Theta}\right\} = \Theta^{-n-3/2}\ e^{-\frac{(T-n\mu+\lambda/2)}{\Theta}}$$

or

$$f(T/\Theta) = \frac{\Theta^{-n}}{\Gamma(n)} (T-n\mu)^{n-1}\ e^{-\frac{(T-n\mu)}{\Theta}}\ ;\ T \geq n\mu.$$

Since the Laplace transform is a one-to-one mapping, this completes the proof of case (iii).

(4) <u>For the case (iv)</u>:

<u>Necessity</u>:

If the conditional distribution of T has the density (1.3) and Θ has the density (2.4), then according to (3.1), we have

$$f_4(T) = \frac{\alpha^{1-p-q}}{\Gamma(n)\beta(p,q)} (T-n\mu)^{n-1} \int_0^\alpha \Theta^{p-n-1} (\alpha-\Theta)^{q-1} \times$$

$$\times \exp\left\{-\ \frac{(T-n\mu)}{\Theta}\right\} d\Theta$$

$$= \frac{\alpha^{-n}}{\Gamma(n)\beta(p,q)} (T-n\mu)^{n-1} \int_0^1 z^{p-n-1} (1-z)^{q-1} \times$$

$$\times \exp\left\{- \frac{(T-n\mu)}{\alpha z}\right\} dz$$

$$= \frac{\alpha^{-n}}{\Gamma(n)\beta(p,q)} (T-n\mu)^{n-1} \exp\left\{- \frac{(T-n\mu)}{\alpha}\right\} \times$$

$$\times \int_0^\infty [(\frac{y}{1+y})^{p-n-1} (1- \frac{y}{1+y})^{q-1} \frac{1}{(1+y)^2} \times$$

$$\times \exp\left\{- \frac{(T-n\mu)}{\alpha y}\right\} e^{Ty}] e^{-Ty} dy .$$

$$= \frac{\alpha^{-n}}{\Gamma(n)\beta(p,q)} (T-n\mu)^{n-1} \exp\left\{- \frac{(T-n\mu)}{\alpha}\right\} \times$$

$$\times L_T [(\frac{y}{1+y})^{p-n-1} (1- \frac{y}{1+y})^{q-1} \frac{1}{(1+y)^2} \times$$

$$\times \exp\left\{- \frac{(T-n\mu)}{\alpha y}\right\} e^{Ty}], \quad T \geq n\mu. \qquad (3.7)$$

Sufficiency:

If the conditional distribution of T has the density $f(T/\theta)$ and θ has the density (2.4), then we get

$$\frac{\alpha^{1-p-q}}{\beta(p,q)} \int_0^\alpha \theta^{p-1} (\alpha-\theta)^{q-1} f(T/\theta) \, d\theta$$

$$= \frac{\alpha^{-n}}{\Gamma(n)\beta(p,q)} (T-n\mu)^{n-1} \exp\left\{- \frac{(T-n\mu)}{\alpha}\right\} \times$$

$$\times L_T [(\frac{y}{1+y})^{p-n-1} (1- \frac{y}{1+y})^{q-1} \frac{1}{(1+y)^2} \times$$

$$\times \exp(Ty) \exp\left\{- \frac{(T-n\mu)}{\alpha y}\right\}].$$

i.e.,

$$\frac{1}{\beta(p,q)} L_T \left[\left(\frac{y}{1+y}\right)^{p-1} \left(1-\frac{y}{1+y}\right)^{q-1} \frac{1}{(1+y)^2} \exp(Ty) \, . \, f\left(T / \frac{\alpha y}{1+y}\right)\right.$$

$$= \frac{\alpha^{-n}}{\Gamma(n)\beta(p,q)} (T-n\mu)^{n-1} \exp\left\{-\frac{(T-n\mu)}{\alpha}\right\} \times$$

$$\times L_T \left[\left(\frac{y}{1+y}\right)^{p-n-1} \left(1-\frac{y}{1+y}\right)^{q-1} \frac{1}{(1+y)^2} \times\right.$$

$$\times \exp(Ty) \exp\left\{-\frac{(T-n\mu)}{\alpha y}\right\}. \qquad (3.8)$$

Taking the inverse Laplace transform for both sides of (3.8), we get

$$\left(\frac{y}{1+y}\right)^{p-1} f\left(T / \frac{\alpha y}{1+y}\right)$$

$$= \frac{\alpha^{-n}}{\Gamma(n)} (T-n\mu)^{n-1} \exp\left\{-\frac{(T-n\mu)}{\alpha}\right\} \times$$

$$\times \left(\frac{y}{1+y}\right)^{p-n-1} \exp\left\{-\frac{(T-n\mu)}{\alpha y}\right\}.$$

Then after taking $z = y/(1+y)$

$$f(T/\alpha z) = \frac{z^{-n}}{\alpha^n \Gamma(n)} (T-n\mu)^{n-1} \exp\left\{-\frac{(T-n\mu)}{\alpha}\right\} \times$$

$$\times \exp\left\{-\left(\frac{T-n\mu}{\alpha y}\right)(1-z)\right\}.$$

Since $z = \frac{\theta}{\alpha}$, we have

$$f(T/\theta) = \frac{\theta^{-n}}{\Gamma(n)} (T-n\mu)^{n-1} \exp\left\{-\frac{(T-n\mu)}{\theta}\right\}.$$

Hence, by the uniqueness property of the inverse Laplace transform, the case (iv) is proved.

(5) For the case (v):

Necessity:

If the conditional distribution of T has the density (1.3) and Θ has the density (2.5), then according to (3.1), we have

$$f_5(T) = \frac{(T-n\mu)^{n-1}}{\Gamma(n)} \int_0^\infty \Theta^{-(n+\alpha)} e^{-(T-n\mu)/\Theta} d\Theta$$

$$= \frac{\Gamma(n+\alpha-1)}{\Gamma(n)} (T-n\mu)^{-\alpha}; \qquad (3.9)$$

which is the quasi-mixture ditribution of T.

Sufficiency:

If the conditional distribution of T has the density $f(T/\Theta)$ where Θ has the density (2.5) and the mixture distribution defined by (3.9), then we get

$$\int_0^\infty (\Theta^{-\alpha} f(T/\Theta)\ e^{T/\Theta})\ e^{-T/\Theta}\ d\Theta = \frac{\Gamma(n+\alpha-1)}{\Gamma(n)\,(T-n\mu)^{\alpha}}$$

i.e.,

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} \int_0^\infty (Z^{\alpha-2}\ f(T/Z^{-1})\ e^{TZ})\ e^{-TZ}\ dz = \frac{\Gamma(n+\alpha-1)}{(T-n\mu)^{n+\alpha-1}}$$

where $Z = \Theta^{-1}$

This can be written as

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} L_T(Z^{\alpha-2}\ f(T/\Theta)\ e^{TZ}) = \frac{\Gamma(n+\alpha-1)}{(T+(-n\mu))^{n+\alpha-1}} \qquad (3.10)$$

Now the R.H.S. of (3.10) is the Laplace transform of the function $Z^{n+\alpha-2}\ e^{Z(n\mu)}$

Therefore,

$$\frac{\Gamma(n)}{(T-n\mu)^{n-1}} \left(\Theta^{-\alpha+2} f(T/\Theta)\, e^{T/\Theta}\right) = \Theta^{-n-\alpha+2} e^{n\mu/\Theta}$$

or

$$f(T/\Theta) = \frac{\Theta^{-n}}{\Gamma(n)} (T-n\mu)^{n-1} e^{-\frac{(T-n\mu)}{\Theta}} \quad ; \; T \geq n\mu \;, \; \Theta > 0$$

Hence, by the uniqueness property of the inverse Laplace transform, the case (v) is proved.

In Bayesian analysis, the density function of the mixture distribution, f(T) is used for finding the Bayesian risk of the Bayesian estimator of Θ, while the density function of Θ, $g(\Theta)$ is used to define the Bayesian risk for the non-Bayesian estimator of Θ. The mixture distribution of $g(\Theta)$ with the likelihood function of $x_1, x_2, \ldots, x_n$ gives the posterior density of Θ conditional on data T, $g(\Theta/T)$.

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