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Recurrence Relations and Characterizations for the Exponentiated Family of Distributions via Dual Generalized Order Statistics

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Abstract: In this paper, recurrence relations for single, conditional moment-generating functions and product moments for the exponentiated family of distributions are established using dual generalized order statistics. These recurrence relations are derived as special cases for moments of lower record values and reversed order statistics. Depending on recurrence relations for single moments, conditional moment generating functions, product moments, and the failure rate function, we establish characterizations of the exponentiated family of distributions. The study illustrates the application of general findings concerning EFDs to specific distributions in this family. These particular distributions include the extended erlang-truncated exponential, exponentiated Weibull, and exponentiated additive Weibull distributions.

Keywords: Recurrence relations; Characterizations; Dual generalized order statistics; Exponentiated family of distribution.

2020 AMS Subject Classifications: 62E10, 62G30, 60E05.

1 Introduction and Preliminaries

Kamps [1] first described the concept of generalized order statistics (GOSs), which encompass nearly all significant models of random variables (RVs) sorted in ascending order of magnitude, including records, Pfeifer's records, order statistics, and progressive type II censored order statistics. Numerous authors have utilized GOSs to derive recurrence relations. For instance, A-Rahman et al. [2] derived recurrence relations for the single and product moments of GOSs for the Weibull-Weibull distribution. However, this paradigm cannot be used to study ordered RVs given in descending order of magnitude. Dual generalized order statistics (DGOSs), which incorporate the arrangement of RVs in decreasing order of magnitude, were first described by Pawlas and Szynal [3]. Burkschat et al. [4] investigated the concept of DGOSs by using these properties to express models related to the decreasing

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order of magnitude. Dual GOSs include lower record values (LRVs) and reversed order statistics (ROSs). DGOSs were described by Burkschat et al. [4] as follows:

Suppose $n \in N$, $l \geq 1$ and $p \in \Re$ be the parameters where $\gamma_s = l + (n-s)(p+1) > 0$, $\forall\, s \in [1, n]$. The RVs, $U_D(1, n, p, l), U_D(2, n, p, l), ..., U_D(n, n, p, l)$ are n DGOSs from an absolutely continuous cumulative distribution function (CDF) with probability density function (PDF) whether their combined PDF contains the following form

$$l \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [F(u_i)]^p f(u_i) \right) \left[F(u_n) \right]^{l-1} f(u_n), \tag{1}$$

for $F^{-1}(1) > u_1 \geq u_2 \geq ... \geq u_n > F^{-1}(0)$.

The PDF of the s^{th} DGOSs, $U_D(s, n, p, l)$ is

$$f_{U_D(s,n,p,l)}(u) = \frac{C_{s-1}}{\Gamma(s)} [F(u)]^{\gamma_s - 1} f(u)\, g_p^{s-1}(F(u)). \tag{2}$$

The joint PDF of $U_D(s_1, n, p, l)$ and $U_D(s_2, n, p, l)$, $1 \leq s_1 \leq s_2 \leq n$, is

$$\begin{aligned} f_{U_D(s_1,n,p,l),U_D(s_2,n,p,l)}(u,v) = \frac{C_{s_2-1}}{\Gamma(s_1)\Gamma(s_2-s_1)} & f(u)\,[F(u)]^p\, g_p^{s_1-1}(F(u))\,[F(v)]^{\gamma_{s_2}-1} \\ & \times f(v)\,[h_p(F(v)) - h_p(F(u))]^{s_2-s_1-1}, \qquad v < u, \end{aligned} \tag{3}$$

where,

$$C_{s_1-1} = \prod_{i=1}^{s_1} \gamma_i \quad , \quad h_p(u) = \begin{cases} -\frac{1}{p+1} u^{p+1}, & \text{p} \neq \text{-1}, \\ -\ln u, & \text{p = -1}, \end{cases}$$

and

$$g_p(u) = h_p(u) - h_p(1), \quad 0 \leq u < 1.$$

We get ROSs when $p = 0$ and $l = 1$, $U_D(s_1, n, p, l) \equiv U_{n-s_1+1:n}$. Also, if $p = -1$ and $l = 1$, we obtain LRVs, $U_D(s, n, p, l) \equiv U_{L(s)}$.

In probability and statistics, the characterization of probability distributions is crucial. Numerous techniques, such as conditional expectancies, recurrence relations, and truncation moments, can be used to describe a probability distribution. DGOSs have been applied by several authors to characterize distributions. Among these authors are Galambos and Kotz [5], Ahsanullah [6], Khan et al. [7], Khan and Kumar [8], Abdul-Moniem [9], Khan et al. [10], Mahmoud et al. [11], Khan and Khan [12], El-Adll [13], Khan and Iqrar [14], Khan [15], Anwar et al. [16], Alam et al. [17], and Singh et al. [18]. Mahmoud et al. [11] constructed recurrence relations for single and product moments of DGOSs from Weibull gamma distribution, and they provided characterizations of Weibull gamma distribution using hazard function, truncated moments of a specific function of the variable, and recurrence relations for single moments. Anwar et al. [16] derived explicit formulas and some recurrence relations for the single and product moments of DGOSs using the extended erlang-truncated exponential (EETE) distribution. Alam et al. [17] studied various theorems and some relations using ratio and inverse moment for the exponentiated Weibull distribution based on DGOSs.

In recent years, some authors have studied moments of DGOSs from a general class of distributions. The moment-generating functions (MGFs) of DGOSs were derived by Khan and Kumar [19] within a class of doubly truncated distributions. Additionally, using the MGFs of DGOSs, they provided two theorems for characterizing the generic form of the distribution. By using DGOSs and truncated moments, Domma and Hamedani [20] established

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characterizations of a wide class of distributions. From a general class of distributions, Saran et al. [21] establish some general recurrence relations between single and product moments of DGOSs. The exponentiated generalized class of distributions described by Cardeiro et al. [22] was used by Athar et al. [23] to investigate the moment properties of DGOSs. Nayabuddin and Akhter [24] established the moment recurrence relations for functions of single and DGOSs from the Marshall-Olkin extended general class of distributions. Mahmoud and Ghazal [25] studied the exponentiated family of distributions (EFDs) to characterize a mixture of two exponentiated families of distribution components utilizing conditional MGFs and recurrence relations for moments of GOSs. The $F(u)$ of the EFDs is provided by

$$F(u) = (1-e^{-\delta(u)})^{\varphi}, \qquad u \geq 0, \tag{4}$$

where $\varphi > 0$ and $\delta(u)$ is a non-negative, differentiable function of u where $\delta(u) \to 0$ as $u \to 0^+$ and $\delta(u) \to \infty$ as $u \to \infty$. The EFDs contain many exponentiated distributions such as the exponentiated Rayleigh [26], the exponentiated Weibull (EW) [27], the exponentiated exponential [28], the exponentiated Lomax [9], the exponentiated modified Weibull [29], the exponentiated linear failure rate [30], the exponentiated Pareto [8], exponentiated Burr type XII [31], the exponentiated Gamma [32], the exponentiated Gompertz [33], the exponentiated generalized linear exponential [34], the EETE [35], and the exponentiated additive weibull (EaddW) [36] distributions,...etc. In this study, we establish recurrence relations of single, conditional MGF and product moments of the EFDs utilizing DGOSs. ROSs and LRVs are obtained as special cases. We offer characterizations of the EFDs using recurrence relation for single, conditional MGF, product moments and failure rate function.

The PDF of the EFDs is provided by

$$f(u) = \varphi \lambda'(u) e^{-\delta(u)} [1-e^{-\delta(u)}]^{\varphi-1}. \tag{5}$$

From (5), we obtain

$$F(u) = -\frac{(1-e^{\delta(u)})}{\varphi \delta'(u)} f(u). \tag{6}$$

In this paper, we present a unified framework for characterizing the EFDs by integrating various moment properties and the failure rate function. Our approach employs recurrence relations for single moments, conditional MGFs, product moments, and the failure rate function to comprehensively understand EFDs. By focusing on DGOS, we derive specific recurrence relations and characterizations tailored to EFDs. We demonstrate the practical utility of our method by applying it to the EW, EETE, and EaddW distributions, showcasing its effectiveness in handling diverse characteristics. This framework has the potential to be extended to other distribution classes and complex data scenarios.

The structure of the paper is as follows: We establish the recurrence relations for single, conditional MGF, and product moments for the EFDs using DGOSs in Section 2. In Section 3, we construct characterizations for the EFDs using recurrence relations for single moments, conditional MGF, product moments, and the FR function. The recurrence relations of the EW, the EETE, and the EaddW distributions are derived as examples of the EFDs in Section 4. Concluding remarks are summarized in Section 5.

2 Recurrence Relations

This section uses DGOSs to deduce the recurrence relations of single, conditional MGF, and product moments for the EFDs.

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2.1 Recurrence relations for single moments of EFDs depending on DGOSs

Theorem 1. *For integers a where $a \geq 1$, the recurrence relation (7) holds.*

$$M^{(a)}_{U_D(s,n,p,l)}(t) - M^{(a)}_{U_D(s-1,n,p,l)}(t) = \frac{at}{\varphi\,\gamma_s}\,E\left[\frac{U_D^{(a-1)}(s,\,n,\,p,\,l)\,e^{t\,U_D^{(a)}(s,n,p,l)}\,(1-e^{\delta(U_D(s,n,p,l))})}{\delta'(U_D(s,\,n,\,p,\,l))}\right], \tag{7}$$

where $2 \leq s \leq n$, $n \geq 2$, $p \geq -1$, $l = 1,2,\ldots$ and $\varphi > 0$.

Proof . Using (2), we get

$$M^{(a)}_{U_D(s,n,p,l)}(t) = \frac{C_{s-1}}{\gamma_s\,\Gamma(s)}\int_0^\infty e^{t\,u^a}\,g_p^{s-1}(F(u))\,d\Big[[F(u)]^{\gamma_s}\Big]. \tag{8}$$

Integrating (8), we obtain

$$M^{(a)}_{U_D(s,n,p,l)}(t) - M^{(a)}_{U_D(s-1,n,p,l)}(t) = -\frac{at\,C_{s-1}}{\gamma_s\,\Gamma(s)}\int_0^\infty u^{a-1}\,e^{t\,u^a}\,[F(u)]^{\gamma_s}\,g_p^{s-1}(F(u))\,du. \tag{9}$$

We apply (6) to (9), we obtain

$$\begin{aligned} M^{(a)}_{U_D(s,n,p,l)}(t) - M^{(a)}_{U_D(s-1,n,p,l)}(t) &= \frac{at\,C_{s-1}}{\gamma_s\,\Gamma(s)}\int_0^\infty u^{a-1}\,e^{t\,u^a}\,[F(u)]^{\gamma_s-1}\,g_p^{s-1}(F(u))\Big[\frac{(1-e^{\delta(u)})}{\varphi\,\delta'(u)}\,f(u)\Big]\,du \\ &= \frac{at\,C_{s-1}}{\varphi\,\gamma_s\,\Gamma(s)}\int_0^\infty \frac{u^{a-1}\,e^{t\,u^a}\,(1-e^{\delta(u)})}{\delta'(u)}[F(u)]^{\gamma_s-1}\,g_p^{s-1}(F(u))\,f(u)\,du. \end{aligned} \tag{10}$$

Thus, we have the result.

Corollary 1. *By differentiating (7) with regard to t and then making $t = 0$, we can derive the recurrence relation for moments of DGOSs as follows given*

$$E[U_D^a(s,\,n,\,p,\,l)] - E[U_D^a(s-1,\,n,\,p,\,l)] = \frac{a}{\varphi\,\gamma_s}\,E\left[\frac{U_D^{(a-1)}(s,\,n,\,p,\,l)\,(1-e^{\delta(U_D(s,n,p,l))})}{\delta'(U_D(s,\,n,\,p,\,l))}\right], \tag{11}$$

where $2 \leq s \leq n$, $n \geq 2$, $p \geq -1$, $l = 1,2,\ldots$ and $\varphi > 0$.

Corollary 2. *We obtain the explicit expression for the moments of ROSs as follows by substituting $p = 0$ and $l = 1$ in (7) and (11).*

$$M^{(a)}_{U_{n-s+1:n}}(t) - M^{(a)}_{U_{n-s+2:n}}(t) = \frac{at}{\varphi\,(n-s+1)}\,E\left[\frac{U^{(a-1)}_{n-s+1:n}\,e^{t\,U^{(a)}_{n-s+1:n}}\,(1-e^{\delta(U_{n-s+1:n})})}{\delta'(U_{n-s+1:n})}\right], \tag{12}$$

and

$$E[U^{(a)}_{n-s+1:n}] - E[U^{(a)}_{n-s+2:n}] = \frac{a}{\varphi\,(n-s+1)}\,E\left[\frac{U^{(a-1)}_{n-s+1:n}\,(1-e^{\delta(U_{n-s+1:n})})}{\delta'(U_{n-s+1:n})}\right], \tag{13}$$

where $2 \leq s \leq n$, $n \geq 2$ and $\varphi > 0$.

Corollary 3. *The explicit expression for the moments of ROSs is obtained as follows by setting $p = -1$ and $l = 1$ in (7) and (11).*

$$M^{(a)}_{U_{L(s)}}(t) - M^{(a)}_{U_{L(s-1)}}(t) = \frac{at}{\varphi}\,E\left[\frac{U^{(a-1)}_{L(s)}\,e^{t\,U^{(a)}_{L(s)}}\,(1-e^{\delta(U_{L(s)})})}{\delta'(U_{L(s)})}\right], \tag{14}$$

and

$$E[U^{(a)}_{U_{L(s)}}] - E[U^{(a)}_{U_{L(s-1)}}] = \frac{a}{\varphi}\,E\left[\frac{U^{(a-1)}_{L(s)}\,(1-e^{\delta(U_{L(s)})})}{\delta'(U_{L(s)})}\right]. \tag{15}$$

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2.2 Recurrence relations of conditional MGF for EFDs using DGOSs

The conditional distribution function of $U_D(s_2, n, p, l)$ given $U_D(s_1, n, p, l) = v$, $1 \leq s_1 \leq s_2 \leq n$, can be obtained using (2) and (3) as follow:

$$f_{U_D(s_2,n,p,l)|U_D(s_1,n,p,l)}(u|v) = \frac{C_{s_2-1}}{\Gamma(s_2-s_1)C_{s_1-1}}\,[F(v)]^{p-\gamma_{s_1}+1}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,[F(u)]^{\gamma_{s_2}-1}\,f(u). \quad (16)$$

Theorem 2. *Suppose that s_1, s_2 be two integers such that $1 \leq s_1 \leq s_2 \leq n$ and $p, l \in \Re$ where $p \geq -1, l \geq 1$. Then the following recurrence relation holds.*

$$\begin{aligned} &M_{U_D^a(s_2,n,p,l)|U_D(s_1,n,p,l)}(t|v) - M_{U_D^a(s_2-1,n,p,l)|U_D(s_1,n,p,l)}(t|v) \\ &= \frac{at}{\varphi\,\gamma_{s_2}}\,E\left[\frac{U_D^{(a-1)}(s_2,n,p,l)\,e^{t\,U_D^{(a)}(s_2,n,p,l)}\left(1-e^{\delta(U_D(s_2,n,p,l))}\right)}{\delta'(U_D(s_2,n,p,l))}|U_D(s_1,n,p,l)=v\right]. \end{aligned} \quad (17)$$

Proof . Considering (16), we obtain

$$\begin{aligned} M_{U_D^a(s_2,n,p,l)|U_D(s_1,n,p,l)}(t|v) &= E[e^{t\,U_D^{(a)}(s_2,n,p,l)}|U_D(s_1,n,p,l)=v] \\ &= \frac{C_{s_2-1}\,[F(v)]^{p-\gamma_{s_1}+1}}{\gamma_{s_2}\,\Gamma(s_2-s_1)\,C_{s_1-1}}\int_0^v e^{t\,u^a}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,d\left[[F(u)]^{\gamma_{s_2}}\right]. \end{aligned} \quad (18)$$

Integrating (18), yields

$$\begin{aligned} M_{U_D^a(s_2,n,p,l)|U_D(s_1,n,p,l)}(t|v) = &-\frac{at\,C_{s_2-1}\,[F(v)]^{p-\gamma_{s_1}+1}}{\gamma_{s_2}\,(s_2-s_1-1)!\,C_{s_1-1}}\int_0^v u^{a-1}\,e^{t\,u^a}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,[F(u)]^{\gamma_{s_2}}\,du \\ &+\frac{C_{s_2-2}\,[F(v)]^{p-\gamma_{s_1}+1}}{(s_2-s_1-2)!\,C_{s_1-1}}\int_0^v e^{t\,u^a}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-2}\,[F(u)]^{\gamma_{s_2-1}-1}\,f(u)\,du. \end{aligned}$$

The second term in the right hand side is $M_{U_D^a(s_2-1,n,p,l)|U_D(s_1,n,p,l)}(t|v)$, so we get

$$\begin{aligned} &M_{U_D^a(s_2,n,p,l)|U_D(s_1,n,p,l)}(t|v) - M_{U_D^a(s_2-1,n,p,l)|U_D(s_1,n,p,l)}(t|v) \\ &= -\frac{at\,C_{s_2-1}\,[F(v)]^{p-\gamma_{s_1}+1}}{\gamma_{s_2}\,\Gamma(s_2-s_1)\,C_{s_1-1}}\int_0^v u^{a-1}\,e^{t\,u^a}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,[F(u)]^{\gamma_{s_2}}\,du. \end{aligned} \quad (19)$$

Utilizing (6) and (19), we obtain

$$\begin{aligned} &M_{U_D^a(s_2,n,p,l)|U_D(s_1,n,p,l)}(t|v) - M_{U_D^a(s_2-1,n,p,l)|U_D(s_1,n,p,l)}(t|v) \\ &= \frac{at\,C_{s_2-1}\,[F(v)]^{p-\gamma_{s_1}+1}}{\varphi\,\gamma_{s_2}\,\Gamma(s_2-s_1)\,C_{s_1-1}}\int_0^v \frac{u^{a-1}\,e^{t\,u^a}(1-e^{\delta(u)})}{\delta'(u)}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,[F(u)]^{\gamma_{s_2}-1}\,f(u)\,du. \end{aligned} \quad (20)$$

From (18), we have the result.

Corollary 4. *By differentiating (17) with regard to t, we have*

$$\begin{aligned} &E[U_D^a(s_2,n,p,l)|U_D(s_1,n,p,l)=v] - E[U_D^a(s_2-1,n,p,l)|U_D(s_1,n,p,l)=v] \\ &= \frac{a}{\varphi\,\gamma_{s_2}}\,E\left[\frac{U_D^{(a-1)}(s_2,n,p,l)\left(1-e^{\delta(U_D(s_2,n,p,l))}\right)}{\delta'(U_D(s_2,n,p,l))}|U_D(s_1,n,p,l)=v\right]. \end{aligned} \quad (21)$$

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Corollary 5. *Putting $p = 0$ and $l = 1$ in (17) and (21), we get*

$$M_{U^a_{n-s_2+1:n}|U_{n-s_1+1:n}}(t|v) - M_{U^a_{n-s_2+2:n}|U_{n-s_1+1:n}}(t|v)$$
$$= \frac{at}{\varphi(n-s_2+1)} E\left[\frac{U^{(a-1)}_{n-s_2+1:n} e^{tU^{(a)}_{n-s_2+1:n}} \left(1-e^{\delta(U_{n-s_2+1:n})}\right)}{\delta'(U_{n-s_2+1:n})} |U_{n-s_1+1:n} = v\right], \tag{22}$$

and

$$E[U^a_{n-s_2+1:n}|U_{n-s_1+1:n} = v] - E[U^a_{n-s_2+2:n}|U_{n-s_1+1:n} = v]$$
$$= \frac{a}{\varphi(n-s_2+1)} E\left[\frac{U^{(a-1)}_{n-s_2+1:n} \left(1-e^{\delta(U_{n-s_2+1:n})}\right)}{\delta'(U_{n-s_2+1:n})} |U_{n-s_1+1:n} = v\right]. \tag{23}$$

Corollary 6. *Putting $p = -1$ and $l = 1$ in (17) and (21), we obtain*

$$M_{U^a_{L(s_2)}|U_{L(s)}}(t|v) - M_{U^a_{L(s_2-1)}|U_{L(s)}}(t|v) = \frac{at}{\varphi} E\left[\frac{U^{(a-1)}_{L(s_2)} e^{tU^{(a)}_{L(s_2)}} (1-e^{\delta(U_{L(s_2)})})}{\delta'(U_{L(s_2)})} |U_{L(s)} = v\right], s = s_1, s_1+1, \tag{24}$$

and

$$E[U^a_{L(s_2)}|U_{L(s_1)} = v] - E[U^a_{L(s_2-1)}|U_{L(s_1)} = v] = \frac{a}{\varphi} E\left[\frac{U^{(a-1)}_{L(s_2)} (1-e^{\delta(U_{L(s_2)})})}{\delta'(U_{L(s_2)})} |U_{L(s_1)} = v\right], s = s_1, s_1+1. \tag{25}$$

2.3 Recurrence relations of product moments for EFDs using DGOSs

Lemma 1. Khan et al. [7]. For $1 \le s_1 < s_2 \le n-1$, $n \ge 2$ and $l = 1, 2, \ldots$.

$$E\left[U^i_D(s_1, n, p, l)U^j_D(s_2, n, p, l)\right] - E\left[U^i_D(s_1, n, p, l)U^j_D(s_2-1, n, p, l)\right]$$
$$= -\frac{jC_{s_2-1}}{\gamma_{s_2}\Gamma(s_1)\Gamma(s_2-s_1)} \int_0^\infty \int_0^u u^i v^{j-1} [F(u)]^p f(u) g_p^{s_1-1}(F(u))$$
$$\times [h_p(F(v)) - h_p(F(u))]^{s_2-s_1-1} [F(v)]^{\gamma_{s_2}} dv\, du, \qquad v < u. \tag{26}$$

Theorem 3. *Let U be a RVs with distribution $F(u)$, thence the recurrence relation (27) holds.*

$$E\left[U^i_D(s_1, n, p, l)U^j_D(s_2, n, p, l)\right] - E\left[U^i_D(s_1, n, p, l)U^j_D(s_2-1, n, p, l)\right]$$
$$= \frac{j}{\varphi\gamma_{s_2}} E\left[\left[\frac{(1-e^{\delta(U_D(s_2, n, p, l))})}{\delta'(U_D(s_2, n, p, l))}\right] U^i_D(s_2, n, p, l) U^{j-1}_D(s_2, n, p, l)\right]. \tag{27}$$

Proof . From (26) and (6), we have

$$E\left[U^i_D(s_1, n, p, l)U^j_D(s_2, n, p, l)\right] - E\left[U^i_D(s_1, n, p, l)U^j_D(s_2-1, n, p, l)\right]$$
$$= \frac{jC_{s_2-1}}{\varphi\gamma_{s_2}\Gamma(s_1)\Gamma(s_2-s_1)} \int_0^\infty \int_0^u u^i v^{j-1} [F(u)]^p f(u) g_p^{s_1-1}(F(u))$$
$$\times [h_p(F(v)) - h_p(F(u))]^{s_2-s_1-1} [F(v)]^{\gamma_{s_2}-1} \left[\frac{(1-e^{\delta(v)})}{\delta'(v)}\right] f(v)\, dv\, du. \tag{28}$$

Using (3) in (28), the result is proved.

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Corollary 7. *Setting $p=0$ and $l=1$ in (27), we get*

$$E\left[U^{i}_{n-s_1+1:n}U^{j}_{n-s_2+1:n}\right]-E\left[U^{i}_{n-s_1+1:n}U^{j}_{n-s_2+2:n}\right]=\frac{j}{\varphi\,(n-s_2+1)}E\left[\left[\frac{(1-e^{\delta(U_{n-s_2+1:n})})}{\delta'(U_{n-s_2+1:n})}\right]U^{i}_{n-s_1+1:n}U^{j-1}_{n-s_2+1:n}\right].\tag{29}$$

Corollary 8. *Putting $p=-1$ and $l=1$ in (27), we have*

$$E[U^{(i)}_{U_{L(s_1)}}U^{(j)}_{U_{L(s_2)}}]-E[U^{(i)}_{U_{L(s_1)}}U^{(j)}_{U_{L(s_2-1)}}]=\frac{j}{\varphi}E\left[\left[\frac{(1-e^{\delta(U_{L(s_2)})})}{\delta'(U_{L(s_2)})}\right]U^{i}_{L(s_1)}U^{j-1}_{L(s_2)}\right].\tag{30}$$

3 Characterization

In this Section, we construct characterizations for the EFDs using recurrence relations for single moments, conditional MGF, product moments, and the FR function.

3.1 Characterization for EFDs using a recurrence relation of single moments

Theorem 4. *Let U be a RVs. Then for integers a. Eq. (7) holds, iff U has the CDF (4).*

Proof . If U has the CDF (4), then (7) holds from Theorem 2.1.1. Further, if (7) is satisfied, then from (7) and (8), we obtain

$$\frac{C_{s-1}}{\Gamma(s)}\int_0^\infty e^{t\,u^a}[F(u)]^{\gamma_s-1}f(u)\,g_p^{s-1}(F(u))\,du-\frac{C_{s-2}}{(s-2)!}\int_0^\infty e^{t\,u^a}[F(u)]^{\gamma_s+p}f(u)\,g_p^{s-2}(F(u))\,du$$
$$=\frac{a\,t\,C_{s-1}}{\varphi\,\gamma_s\,\Gamma(s)}\int_0^\infty\frac{u^{a-1}e^{t\,u^a}(1-e^{\delta(u)})}{\delta'(u)}[F(u)]^{\gamma_s-1}g_p^{s-1}(F(u))\,f(u)\,du.\tag{31}$$

Using the second integral on the left side of (31), we get

$$\frac{C_{s-1}}{\Gamma(s)}\int_0^\infty e^{t\,u^a}[F(u)]^{\gamma_s-1}f(u)\,g_p^{s-1}(F(u))\,du$$
$$-\frac{a\,t\,C_{s-2}}{\Gamma(s)}\int_0^\infty u^{a-1}e^{t\,u^a}[F(u)]^{\gamma_s}g_p^{s-1}(F(u))\,du$$
$$-\frac{\gamma_s C_{s-2}}{\Gamma(s)}\int_0^\infty e^{t\,u^a}[F(u)]^{\gamma_s-1}f(u)\,g_p^{s-1}(F(u))\,du$$
$$=\frac{a\,t\,C_{s-1}}{\varphi\,\gamma_s\,\Gamma(s)}\int_0^\infty\frac{u^{a-1}e^{t\,u^a}(1-e^{\delta(u)})}{\delta'(u)}[F(u)]^{\gamma_s-1}g_p^{s-1}(F(u))\,f(u)\,du,$$

which reduces to

$$-\frac{a\,t\,C_{s-1}}{\gamma_s\Gamma(s)}\int_0^\infty u^{a-1}e^{t\,u^a}[F(u)]^{\gamma_s}g_p^{s-1}(F(u))\,du$$
$$=\frac{a\,t\,C_{s-1}}{\varphi\,\gamma_s\,\Gamma(s)}\int_0^\infty\frac{u^{a-1}e^{t\,u^a}(1-e^{\delta(u)})}{\delta'(u)}[F(u)]^{\gamma_s-1}g_p^{s-1}(F(u))\,f(u)\,du.\tag{32}$$

Simplify (32), we get

$$\frac{a\,t\,C_{s-1}}{\gamma_s\Gamma(s)}\int_0^\infty u^{a-1}e^{t\,u^a}[F(u)]^{\gamma_s-1}g_p^{s-1}(F(u))\left[F(u)+\frac{f(u)\,(1-e^{\delta(u)})}{\varphi\,\delta'(u)}\right]du=0.\tag{33}$$

Using the Müntz-Szàsz theorem's extension Hwang and Lin [37], we get

$$F(u)=-\frac{(1-e^{\delta(u)})}{\varphi\,\delta'(u)}f(u).$$

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3.2 Characterization for EFDs using a recurrence relation of conditional MGF

Theorem 5. *Let U be a RVs. Then for $a \geq 1$. Eq.(17) holds, iff U has the CDF (4).*

Proof . If U has the CDF (4), then (17) holds from Theorem 2.2.1. Otherwise, if Eq.(17) is satisfied, then from (18) and (19), we get

$$-\frac{a\,t\,C_{s_2-1}\,[F(v)]^{p-\gamma_{s_1}+1}}{\gamma_{s_2}\,\Gamma(s_2-s_1)\,C_{s_1-1}}\int_0^v u^{a-1}\,e^{t\,u^a}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,[F(u)]^{\gamma_{s_2}}\,du$$
$$=\frac{a\,t\,C_{s_2-1}\,[F(v)]^{p-\gamma_{s_1}+1}}{\varphi\,\gamma_{s_2}\,\Gamma(s_2-s_1)\,C_{s_1-1}}\int_0^v \frac{u^{a-1}\,e^{t\,u^a}\,(1-e^{\delta(u)})}{\delta'(u)}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,[F(u)]^{\gamma_{s_2}-1}\,f(u)\,du.$$

That can be written as

$$\frac{a\,t\,C_{s_2-1}\,[F(v)]^{p-\gamma_{s_1}+1}}{\gamma_{s_2}\,\Gamma(s_2-s_1)\,C_{s_1-1}}\int_0^v u^{a-1}\,e^{t\,u^a}\,[h_p(F(u))-h_p(F(v))]^{s_2-s_1-1}\,[F(u)]^{\gamma_{s_2}-1}\left[F(u)+\frac{f(u)\,(1-e^{\delta(u)})}{\varphi\,\delta'(u)}\right]du=0.$$

Utilizing the Müntz-Szàsz theorem's extension, we get

$$F(u)=-\frac{(1-e^{\delta(u)})}{\varphi\,\delta'(u)}\,f(u).$$

3.3 Characterization of EFD based on a recurrence relation for product moments

Theorem 6. *Let U be a RVs. For positive integers a. Eq. (27) holds, iff U has the CDF (4).*

Proof . If U has the CDF (4), then (28) holds from Theorem 2.3.1. Moreover, if (27) is satisfied, then from (26) and (28), we have

$$-\frac{j\,C_{s_2-1}}{\gamma_{s_2}\,\Gamma(s_1)\,\Gamma(s_2-s_1)}\int_0^\infty\int_0^u u^i\,v^{j-1}\,[F(u)]^p\,f(u)\,g_p^{s_1-1}(F(u)\,[h_p(F(v))-h_p(F(u))]^{s_2-s_1-1}\,[F(v)]^{\gamma_{s_2}}\,dv\,du$$
$$=\frac{j\,C_{s_2-1}}{\varphi\,\gamma_{s_2}\,\Gamma(s_1)\,\Gamma(s_2-s_1)}\int_0^\infty\int_0^u u^i\,v^{j-1}\,[F(u)]^p\,f(u)\,g_p^{s_1-1}(F(u))$$
$$\times[h_p(F(v))-h_p(F(u))]^{s_2-s_1-1}\,[F(v)]^{\gamma_{s_2}-1}\left[\frac{(1-e^{\delta(v)})}{\delta'(v)}\right]f(v)\,dv\,du. \tag{34}$$

which can be expressed as

$$\int_0^\infty\int_0^u u^i\,v^{j-1}\,[F(u)]^p\,f(u)\,g_p^{s_1-1}(F(u))\,[h_p(F(v))-h_p(F(u))]^{s_2-s_1-1}\,[F(v)]^{\gamma_{s_2}-1}\left[F(v)+\frac{(1-e^{\delta(v)})}{\varphi\,\delta'(v)}\,f(v)\right]dv\,du=0.$$

Upon simplification, we conclude

$$F(v)=-\frac{(1-e^{\delta(v)})}{\varphi\,\delta'(v)}\,f(v).$$

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3.4 Characterizations based on FR function

The FR function is described by

$$h(u) = \frac{f(u)}{1-F(u)}. \tag{35}$$

It is easy to show that the following differential equation is satisfied by the FR function of a twice differentiable function.

$$\frac{h'(u)}{h(u)} - h(u) = w(u), \tag{36}$$

where $w(u)$ is a suitable integrable function. Despite the fact that this differential equation has an obvious form

$$\frac{f'(u)}{f(u)} = \frac{h'(u)}{h(u)} - h(u). \tag{37}$$

Deriving a differential equation in terms of the FR function is the aim of the characterization, depending on the FR function.

Theorem 7. *Suppose $U : \Omega \to (0,\infty)$ be a continuous RV. The PDF of U is (5) iff its $h(u)$ satisfies*

$$h'(u) - \left(\frac{\delta''(u)}{\delta'(u)} - \delta'(u)\right) h(u) = \frac{\varphi\,\delta'^2(u)\,e^{-2\delta(u)}\,(1-e^{-\delta(u)})^{\varphi-2}}{1-(1-e^{-\delta(u)})^{\varphi}} \left(\varphi - 1 + \frac{\varphi\,(1-e^{-\delta(u)})^{\varphi}}{1-(1-e^{-\delta(u)})^{\varphi}}\right), \quad 0 < u < \infty \tag{38}$$

besides the initial condition

$$h(u_0) = \frac{\varphi\,\delta'(u_0)\,e^{-\delta(u_0)}\,[1-e^{-\delta(u_0)}]^{\varphi-1}}{1-(1-e^{-\delta(u_0)})^{\varphi}}.$$

Proof . When U has PDF (5), it is obvious that (38) is true. Now, if (38) is valid, then

$$\frac{e^{\delta(u)}}{\delta'(u)} h(u) = \int \frac{\varphi\,\delta'(u)\,e^{-\delta(u)}\,(1-e^{-\delta(u)})^{\varphi-2}}{1-(1-e^{-\delta(u)})^{\varphi}} \left(\varphi - 1 + \frac{\varphi\,(1-e^{-\delta(u)})^{\varphi}}{1-(1-e^{-\delta(u)})^{\varphi}}\right) du. \tag{39}$$

Integrating (39), we have

$$\frac{e^{\delta(u)}}{\delta'(u)} h(u) = \frac{\varphi\,(1-e^{-\delta(u)})^{\varphi-1}}{1-(1-e^{-\delta(u)})^{\varphi}}. \tag{40}$$

Eq. (40) can be written as

$$h(u) = \frac{f(u)}{1-F(u)} = \frac{\varphi\,\delta'(u)\,e^{-\delta(u)}\,(1-e^{-\delta(u)})^{\varphi-1}}{1-(1-e^{-\delta(u)})^{\varphi}}. \tag{41}$$

Integrating (41) from 0 to u, we get

$$F(u) = (1-e^{-\delta(u)})^{\varphi}, \qquad u \geq 0.$$

4 Special Cases from EFDs

In this section, as examples of the EFDs, we establish the recurrence relations of the EW, EETE, and EaddW distributions.

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4.1 EW distribution

Let $\delta(u) = (\lambda\, u)^{\beta}$, then $\delta'(u) = \beta\, \lambda^{\beta}\, u^{\beta-1}$, and subsequently, the EW distribution's CDF is

$$F(u) = (1 - e^{-(\lambda\, u)^{\beta}})^{\varphi}.$$

Hence, by using (7) and (11), we obtain

$$M^{(a)}_{U_D(s,n,p,l)}(t) - M^{(a)}_{U_D(s-1,n,p,l)}(t) = \frac{a\,t}{\varphi\,\beta\,\lambda^{\beta}\,\gamma_s}\, E\left[U_D^{a-\beta}(s,\, n,\, p,\, l)\, e^{t\, U_D^{a}(s,n,p,l)} \left(1 - e^{(\lambda\, U_D(s,n,p,l))^{\beta}}\right)\right], p \geq -1, \quad (42)$$

and

$$E[U_D^{a}(s,\, n,\, p,\, l)] - E[U_D^{a}(s-1,\, n,\, p,\, l)] = \frac{a}{\varphi\,\beta\,\lambda^{\beta}\,\gamma_s}\left[E\left[U_D^{a-\beta}(s,\, n,\, p,\, l)\right] - E\left[\psi\Big(U_D(s,\, n,\, p,\, l)\Big)\right]\right], p \geq -1, \quad (43)$$

where $\psi(u) = u^{a-\beta}\, e^{(\lambda\, u)^{\beta}}$

Utilizing (12), (13), (14), and (15), we thus obtain

$$M^{(a)}_{U_{n-s+1:n}}(t) - M^{(a)}_{U_{n-s+2:n}}(t) = \frac{a\,t}{\varphi\,\beta\,\lambda^{\beta}\,(n-s+1)}\, E\left[U^{a-\beta}_{n-s+1:n}\, e^{t\, U^{a}_{n-s+1:n}} \left(1 - e^{(\lambda\, U_{n-s+1:n})^{\beta}}\right)\right]. \quad (44)$$

In view of (44), we obtain the same result obtained by Khan et al. [7].

$$E[U^{(a)}_{n-s+1:n}] - E[U^{(a)}_{n-s+2:n}] = \frac{a}{\varphi\,\beta\,\lambda^{\beta}\,(n-s+1)}\left[E\left[U^{a-\beta}_{n-s+1:n}\right] - E\left[\psi\Big(U_{n-s+1:n}\Big)\right]\right].$$

$$M^{(a)}_{U_{L(s)}}(t) - M^{(a)}_{U_{L(s-1)}}(t) = \frac{a\,t}{\varphi\,\beta\,\lambda^{\beta}}\, E\left[U^{(a-\beta)}_{L(s)}\, e^{t\, U^{(a)}_{L(s)}}\, (1 - e^{(\lambda\, U_{L(s)})^{\beta}})\right].$$

$$E[U^{(a)}_{U_{L(s)}}] - E[U^{(a)}_{U_{L(s-1)}}] = \frac{a}{\varphi\,\beta\,\lambda^{\beta}}\left[E\left[U^{(a-\beta)}_{L(s)}\right] - E\left[\psi(U_{L(s)})\right]\right].$$

4.2 EETE distribution

Let $\delta(u) = \beta\,(1 - e^{-\lambda})\, u$, then $\delta'(u) = \beta\,(1 - e^{-\lambda})$, and thence the EETE distribution's CDF is

$$F(u) = (1 - e^{-\beta\,(1-e^{-\lambda})\, u})^{\varphi}.$$

Hence, by using (7) and (11), we obtain

$$M^{(a)}_{U_D(s,n,p,l)}(t) - M^{(a)}_{U_D(s-1,n,p,l)}(t) = \frac{a\,t}{\varphi\,\beta\,(1 - e^{-\lambda})\,\gamma_s}\, E\left[U_D^{a-1}(s,\, n,\, p,\, l)\, e^{t\, U_D^{a}(s,n,p,l)} \left(1 - e^{(\lambda\, U_D(s,n,p,l))^{\beta}}\right)\right], p \geq -1,$$

and

$$E[U_D^{a}(s,\, n,\, p,\, l)] - E[U_D^{a}(s-1,\, n,\, p,\, l)] = \frac{a}{\varphi\,\beta\,(1 - e^{-\lambda})\,\gamma_s}\left[E\left[U_D^{a-1}(s,\, n,\, p,\, l)\right] - E\left[\phi\Big(U_D(s,\, n,\, p,\, l)\Big)\right]\right], p \geq -1,$$

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where $\phi(u) = u^{a-1} e^{\beta(1-e^{-\lambda})u}$

Using (12), (13), (14), and (15), we get

$$M^{(a)}_{U_{n-s+1:n}}(t) - M^{(a)}_{U_{n-s+2:n}}(t) = \frac{at}{\varphi\beta(1-e^{-\lambda})(n-s+1)} E\left[U^{a-1}_{n-s+1:n} e^{t U^{a}_{n-s+1:n}} \left(1 - e^{(\lambda U_{n-s+1:n})^{\beta}}\right)\right].$$

$$E[U^{(a)}_{n-s+1:n}] - E[U^{(a)}_{n-s+2:n}] = \frac{a}{\varphi\beta(1-e^{-\lambda})(n-s+1)} \left[E\left[U^{a-1}_{n-s+1:n}\right] - E\left[\phi\left(U_{n-s+1:n}\right)\right]\right]. \tag{45}$$

In view of (45), we get the same result obtained by Anwar et al. [16].

$$M^{(a)}_{U_{L(s)}}(t) - M^{(a)}_{U_{L(s-1)}}(t) = \frac{at}{\varphi\beta(1-e^{-\lambda})} E\left[U^{(a-1)}_{L(s)}\, e^{t U^{(a)}_{L(s)}} (1 - e^{(\lambda U_{L(s)})^{\beta}})\right].$$

$$E[U^{(a)}_{U_{L(s)}}] - E[U^{(a)}_{U_{L(s-1)}}] = \frac{a}{\varphi\beta(1-e^{-\lambda})} \left[E\left[U^{(a-1)}_{L(s)}\right] - E\left[\psi(U_{L(s)})\right]\right].$$

4.3 EaddW distribution

Let $\delta(u) = \alpha u^{\beta} + \gamma u^{\lambda}$, then $\delta'(u) = \alpha\beta u^{\beta-1} + \gamma\lambda u^{\lambda-1}$, and thus the EaddW distribution's CDF is

$$F(u) = (1 - e^{-\alpha u^{\beta} - \gamma u^{\lambda}})^{\varphi}.$$

Thus, using (7) and (11), we get

$$M^{(a)}_{U_D(s,n,p,l)}(t) - M^{(a)}_{U_D(s-1,n,p,l)}(t) = \frac{at}{\varphi\gamma_s} E\left[\frac{U^{(a-1)}_D(s,n,p,l)\, e^{t U^{(a)}_D(s,n,p,l)} (1 - e^{\eta\left(U_D(s,n,p,l)\right)})}{\mu\left(U_D(s,n,p,l)\right)}\right], p \geq -1, \tag{46}$$

where $\eta(u) = \alpha u^{\beta} + \gamma u^{\lambda}$, and $\mu(u) = \alpha\beta u^{\beta-1} + \gamma\lambda u^{\lambda-1}$.

$$E[U^{a}_D(s,n,p,l)] - E[U^{a}_D(s-1,n,p,l)] = \frac{a}{\varphi\gamma_s} E\left[\frac{U^{(a-1)}_D(s,n,p,l)(1 - e^{\eta\left(U_D(s,n,p,l)\right)})}{\mu\left(U_D(s,n,p,l)\right)}\right], p \geq -1. \tag{47}$$

Utilizing (12), (13), (14), and (15), we get

$$M^{(a)}_{U_{n-s+1:n}}(t) - M^{(a)}_{U_{n-s+2:n}}(t) = \frac{at}{\varphi(n-s+1)} = E\left[\frac{U^{(a-1)}_{n-s+1:n}\, e^{t U^{(a)}_{n-s+1:n}} (1 - e^{\eta(U_{n-s+1:n})})}{\mu(U_{n-s+1:n})}\right].$$

$$E[U^{(a)}_{n-s+1:n}] - E[U^{(a)}_{n-s+2:n}] = \frac{a}{\varphi(n-s+1)} E\left[\frac{U^{(a-1)}_{n-s+1:n}(1 - e^{\eta(U_{n-s+1:n})})}{\mu(U_{n-s+1:n})}\right]. \tag{48}$$

$$M^{(a)}_{U_{L(s)}}(t) - M^{(a)}_{U_{L(s-1)}}(t) = \frac{at}{\varphi} E\left[\frac{U^{(a-1)}_{L(s)}\, e^{t U^{(a)}_{L(s)}} (1 - e^{\eta(U_{L(s-1)})})}{\mu(U_{L(s-1)})}\right].$$

$$E[U^{(a)}_{U_{L(s)}}] - E[U^{(a)}_{U_{L(s-1)}}] = \frac{a}{\varphi} E\left[\frac{U^{(a-1)}_{L(s)}(1 - e^{\eta(U_{L(s-1)})})}{\mu(U_{L(s-1)})}\right].$$

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5 Concluding Remarks

The primary objective of this paper is to establish the recurrence relations of single, conditional MGFs, and product moments for the EFDs using DGOSs. The EFDs include many exponentiated distributions, such as the EW, exponentiated lomax, exponentiated modified Weibull, exponentiated linear failure rate, exponentiated Pareto, exponentiated Gompertz, exponentiated generalized linear exponential, and EaddW distributions. The recurrence relations for single, conditional MGFs, and product moments were derived as special cases for moments of LRVs and ROSs. We constructed characterizations for the EFDs using recurrence relations for single moments, conditional MGFs, product moments, and the FR function. The recurrence relations of the EW, the EETE, and the EaddW distributions were established as examples of the EFDs. We observed similar recurrence relations to those found in earlier studies, such as Khan et al. [7] and Anwar et al. [16]. Our work can serve as a foundation for future research exploring characterizations based on DGOSs in diverse statistical settings. Our findings contribute to a deeper understanding of the properties of EFDs and their behavior in various ordered data scenarios. Future research could extend this framework to multivariate EFDs, investigate its application in reliability analysis with censored data, and develop new statistical inference procedures based on these characterizations. Ultimately, these advancements can lead to more accurate and insightful data analysis across a range of fields where EFDs play a significant role. This paper may be useful for researchers working on distribution theory, engineering sciences, life testing, and ordered random variables.

Availability of data and material

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

The author declares that he has no competing interests.

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Authors' contributions

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