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E-Bayesian Estimation for The Parameter of Inverse Weibull Distribution Based on Lower Records

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ABSTRACT

A lower record sample is utilized to derive E-Bayesian (EB) estimates for the rate parameter of the inverse Weibull distribution. These estimates are developed under two different error loss functions: the scaled squared error loss (SSE) function and the linear exponential error loss (LINEX) function. The expected mean squared errors (E-MSEs) of these EB estimates are computed in order to evaluate the accuracy and dependability of these estimates. An exhaustive Monte Carlo simulation research is carried out in order to carry out a detailed comparison of the performance of different estimators. This simulation can be used to better understand how the estimators behave and how resilient they are under different scenarios and sample sizes. The analysis of two real-world data sets offers a further illustration of how the presented approaches can be used in practice. These examples further validate the usefulness of the EB estimates in statistical inference and decision-making processes by demonstrating how well they simulate real-life data.

1. INTRODUCTION

The Weibull distribution is frequently employed for lifetime data analysis in reliability inference and survival analysis because the hazard function (HF) and probability density function (PDF) are flexible. More study has been conducted using the Weibull distribution from both frequentist and Bayesian perspectives. Johnson et al. [1] offer a superb analysis, while reliability sampling plans and Bayesian inference are examined by

Kundu [2] for Weibull distribution. Depending on the value of the shape parameter, the HF may be decreasing or increasing, and the PDF may be declining or uni-modal. When a mortality study based on data sets indicates that the lifetime distribution may have a non-monotone hazard function, the Weibull distribution might not be appropriate for data analysis (Kundu and Howlader [3]; Singh et al. [4]). Consequently, selecting a suitable probability model for examining such data sets is essential. Another possibility for a probability model is the inverse Weibull distribution (IWD). The IWD's PDF and cumulative distribution function (CDF) are given, respectively, by

$$f(x) = \delta\gamma x^{-(\gamma+1)} e^{-\delta x^{-\gamma}}, \quad x \geq 0, \quad \delta > 0, \quad \gamma > 0, \tag{1}$$

and

$$F(x) = e^{-\delta x^{-\gamma}}, \quad x \geq 0, \quad \delta > 0, \quad \gamma > 0, \tag{2}$$

where the rate parameter is denoted by δ and the shape parameter by γ.

The IWD has been investigated and implemented in a variety of industries, including engineering and medicine. The IWD, for instance, was developed by Keller and Kamath [5] and Keller et al. [6] as a useful model to explain mechanical component deterioration phenomena, such as the dynamic components of diesel engines. Erto [7] showed how the IWD matches a number of data sets found in the literature, such as the breakdown durations of an insulating fluid under the influence of a constant tension. Using fundamental mathematics' derivative, it is possible to demonstrate that although IWD hazard function is unimodal, it is not monotone.

In Bayesian estimation process, the joint prior distribution of population parameters frequently depends on the choice of hyper-parameters. In order to address this problem, Lindley and Smith [8] first introduced the hierarchical Bayesian technique. Han [9] conducted research on the hierarchical Bayesian technique and introduced the concept of EB estimation. The hierarchical Bayesian technique requires the establishment of a joint prior in two separate steps. As a result, the Bayesian estimate is less affected by the choice of hyper-parameters. By using a joint prior distribution that is well-suited to the hyper-parameters, the EB estimate method reduces the impact of arbitrary choice. Numerous researchers have examined the EB estimation method. For instance, Han [10] employed the EB estimation in the case of exponential distribution to calculate the failure rate estimator. Based on the binomial distribution, Han [11] developed the formulas for EB and hierarchical Bayesian dependability assessments. Han [12] examined the EB and hierarchical Bayesian estimations of the shape parameter of the Pareto distribution when the scale parameter is known. In order to obtain the estimates of the exponential distribution parameter and the corresponding expected mean square errors (E-MSEs) with a conjugate prior distribution under the scaled square error (SSE) loss function, Han [13] looked into the EB estimation approach. Based on Pareto model, Han [14] examined EB estimates and E-MSEs under squared error (SE), weighted squared error (WSE), and precautionary loss functions. Okasha et al. [15] formulated the EB estimates for the parameter, using a progressive-type-II censored sample from the Weibull distribution.

Gupta and Gupta [16] conducted a comparison between Bayesian and EB estimators for the exponentiated IWD rate parameter. They used gamma prior and evaluated the estimators using Degroot, Al-Bayyati, and lowest expected loss functions. It was assumed that the shape parameter was known. The EB estimate employed a uniform prior distribution throughout the interval (0,1) for the shape hyper-parameter, and three distinct priors for the rate hyper-parameter. Nevertheless, the uniform prior distribution has limited flexibility when applied to random variables inside the range of (0,1). Basheer et al. [17] used the LINEX loss function and exponential prior to examine E-Bayesian and hierarchical Bayesian estimates for the rate parameter of IWD.

The Bayes estimator of the parameter δ and its related MSE is displayed in Section 2. The EB estimator of the IWD rate parameter will be defined in Section 3. Applying a number of loss functions and three joint priors for two hyper-parameters, Section 3 will discuss the E-MSEs and provide closed-form formulas for the EB estimators of the IWD rate parameter. Section 4 discusses the simulation technique and outcomes. Section 5 provides two application examples to demonstrate how the proposed approaches can be applied. Section 6 addresses conclusions and remarks.

2. BAYEIAN ESTIMATION

Record statistics, according to Chandler [18], is a model for successive extremes in a series of independent random variables with the same distributions. A specific dependent structure is taken into consideration by the record statistics model. This implies that with each component failure, the system's distribution of component life-span may change. Only the voltages lower than the previous one can be recorded when a variety of equipment voltages are taken into account; this is an example of a lower record. In this study, the lower record statistics are taken into account. From the PDF $f\,(x)$, let $X_1, X_2, \ldots$ be a series of independent random variables with identical distributions. The formula is $Y_n = max(min) X_1, X_2, \ldots, X_n, n > 1$. In this series, X_l is considered an upper (lower) record if $Y_l >(<) Y_{l-1}$ and $l > 1$. By definition, X_1 is both an upper and lower value. The notations X_{L_n} and X_{U_n} stand for the nth lower and upper records, respectively. Thus, we can observe that several attempts are attempted to obtain records, and the record is created after one of them is successful. Consequently, we do not obtain data with every effort. Records are the source of the information we possess. An interest in record values has grown among some academics, who have examined statistical findings for several models based on lower and upper record values; for instance, Jaheen [19], Baklizi [20] and Mousa et al. [21], among others.

Based on n lower record values $X_{L_1} = x_1, X_{L_2} = x_2, \ldots, X_{L_n} = x_n$ from IWD(δ, γ) distribution with PDF given by Eq. (1), consequently, the likelihood function can be shown as

$$L(\delta, \gamma | \underline{x}) = f(x_n; \delta, \gamma) \prod_{i=1}^{n-1} \frac{f(x_i; \delta, \gamma)}{F(x_i; \delta, \gamma)},$$

$$= \delta^n \tau(\gamma; \underline{x}) e^{-\delta \xi}, \qquad (3)$$

where

$$\tau(\gamma; \underline{x}) = \gamma^n \prod_{i=1}^{n} x_i^{-(\gamma+1)} \quad and \quad \xi \equiv \xi(\gamma; x_n) = x_n^{-\gamma},$$

where $\underline{x} = (x_1, \dots, x_n)$.

The natural logarithm of the likelihood function given by Eq. (3) is expressed as

$$\ell \equiv \ln L(\delta, \gamma | \underline{x}) = n \ln \delta + \ln \tau(\gamma; \underline{x}) - \delta\xi.$$

When the parameter γ is known, the likelihood equation for the parameter δ can be written as

$$\frac{d\ell}{d\delta} = \frac{n}{\delta} - \xi = 0,$$

then, the maximum likelihood estimate (MLE) of the parameter δ can be obtained as

$$\hat{\delta}_{MLE} = \frac{n}{\xi}. \qquad (4)$$

The following gamma conjugate prior density can be used to calculate the Bayes estimator of the parameter δ

$$g(\delta|a,b) = \frac{b^a}{\Gamma(a)} \delta^{a-1} e^{-b\delta}, \qquad \delta > 0, \qquad (5)$$

where $a > 0$ and $b > 0$ are two hyper-parameters. The following posterior distribution of δ given $\underline{x}$ is obtained from Eq. (3) and Eq. (5) as

$$q(\delta|\underline{x}) = \frac{(b+\xi)^{n+a}}{\Gamma(n+a)} \delta^{n+a-1} e^{-(b+\xi)\delta}, \quad \delta > 0. \qquad (6)$$

Using the SSEL function, $L(\delta, \hat{\delta}) = \frac{(\delta-\hat{\delta})^2}{\delta^k}$, which was suggested by Lehmann and Casella [22], the Bayes estimator of δ can be shown to be

$$\hat{\delta}_{BSS}(a,b) = \frac{E(\delta^{1-k}|\xi)}{E(\delta^{-k}|\xi)} = \frac{n+a-k}{b+\xi}, \qquad (7)$$

where $k \geq 0$ and

$$E(\delta^{\nu}|\xi) = \int_0^{\infty} \delta^{\nu} q(\delta|a,b) d\delta = \frac{\Gamma(n+a+\nu)}{(b+\xi)^{\nu}\Gamma(n+a)}, \qquad \nu = -k, 1-k. \qquad (8)$$

In real applications, $k = 0{,}1$ and 2, are typically used. When k equals 0, the SSE loss function is known as SE loss function, and the Bayes estimator under the SE loss function is denoted by (7) as $E(\delta|\xi)$. For $k = 1$, the SSE loss function is known as the WSE loss function, and (7) equals $[E(\delta^{-1}|\xi)]^{-1}$. The SSE loss function becomes the quadratic squared error (QSE) loss function, and (7) equals $E(\delta^{-1}\,|\xi)/E(\delta^{-2}\,|\xi)$, if $k = 2$.

Varian [23] developed an asymmetric loss function,
$L(\hat{\theta}, \theta) = exp(w(\hat{\theta} - \theta)) - w(\hat{\theta} - \theta) - 1$, for a given real integer, $w \neq 0$, which is commonly known as the LINEX function. The Bayes estimate of δ can be illustrated using the LINEX loss function, following the same technique as Varian [23]

$$\hat{\delta}_{BL} = \frac{-1}{w}\ln(E(exp(-w\delta|\xi)) = \frac{-(n+a)}{w}\ln(\frac{b+\xi}{b+\xi+w}), \tag{9}$$

where $w \neq 0$ and $b + \xi > -w$ to ensure $\frac{b+\xi}{b+\xi+w} > 0$ in the domain of the logarithm function based on e. Furthermore, it can be proven that $\hat{\delta}_{BL} \to \hat{\delta}_{\mathrm{BSS}}$ with $k = 0$ when $w \to 0$ by definition and basic calculus; that is, $\lim_{w\to 0}\hat{\delta}_{BL} = \hat{\delta}_{SE}$. When $w = 0$, $\hat{\delta}_{BL}$ can be defined as $\hat{\delta}_{SE}$, resulting in $\hat{\delta}_{BL}$ being a continuous function with respect to w over $w > -b - \xi$.

2.1. Theoretical mean, variance and mean squared errors

Expectation, variance, and mean square error (MSE) are three often used measures for evaluating estimator performance. The transformation approach demonstrates that ξ has gamma density function with parameters (n, δ). As a result, the expectations and mean square errors for the MLE of δ we have

$$E(\hat{\delta}_{MLE})(\delta) = E(\hat{\delta}_{MLE}|\delta) = \int_0^\infty \frac{n}{\xi}\frac{\delta^n}{\Gamma(n)}\xi^{n-1}e^{-\delta\xi}\,d\xi = \frac{n}{n-1}\delta,$$

and

$$MSE(\hat{\delta}_{MLE})(\delta) = E\left(\left(\delta - \hat{\delta}_{MLE}|\xi\right)^2|\delta\right)$$

$$= \delta^2 - 2\delta E\left(\hat{\delta}_{MLE}|\delta\right) + E\left(\left(\hat{\delta}_{MLE}\right)^2|\delta\right)$$

$$= \frac{n+2}{(n-1)(n-2)}\delta^2. \tag{10}$$

And for the Bayes estimation of δ when γ is known, we can write:

(i) Based on SSE loss function

$$MSE\left(\hat{\delta}_{BSS}(a,b)|\xi\right) = E\left(\left(\delta - \hat{\delta}_{BSS}(a,b)\right)^2|\xi\right)$$

$$= E(\delta^2|\xi) - 2\hat{\delta}_{BSS}(a,b)E(\delta|\xi) + \left(\hat{\delta}_{BSS}(a,b)\right)^2$$

$$= \frac{(n+a+1)(n+a)}{(b+\xi)^2} - 2\left(\frac{n+a-k}{b+\xi}\right)\left(\frac{n+a}{b+\xi}\right) + \left(\frac{n+a-k}{b+\xi}\right)^2$$

$$= \frac{n+a+k^2}{(b+\xi)^2}, \tag{11}$$

(ii) Based on LINEX loss function

$$MSE\left(\hat{\delta}_{BL}(a,b)|\xi\right) = E\left(\left(\delta - \hat{\delta}_{BL}(a,b)\right)^2|\xi\right)$$

$$= E(\delta^2|\xi) - 2\hat{\delta}_{BL}(a,b)E(\delta|\xi) + \left(\hat{\delta}_{BL}(a,b)\right)^2$$

$$= \frac{(n+a+1)(n+a)}{(b+\xi)^2} - 2\left(\frac{n+a}{w}\right)\ln\left(1+\frac{w}{b+\xi}\right)\frac{n+a}{b+\xi} + (n+a)^2\left(\frac{1}{w}\ln\left(1+\frac{w}{b+\xi}\right)\right)^2$$

$$= \frac{n+a}{(b+\xi)^2} + \left(\frac{n+a}{b+\xi}\right)^2\left\{1 - \frac{(b+\xi)}{w}\ln\left(1+\frac{w}{(b+\xi)}\right)\right\}^2. \tag{12}$$

It is clear that Eq. (12) is true for $w \neq 0$ and $b+\xi > -w$. Notice that $b+\xi > -w$ is equivalent to $-1 < w/(b+\xi)$. When $-1 < w/(b+\xi) < 1$, Eq. (12) can be represented as

$$MSE\left(\hat{\delta}_{BL}(a,b)|\xi\right) = \frac{n+a}{(b+\xi)^2} + \left(\frac{n+a}{b+\xi}\right)^2\left\{\sum_{i=2}^{\infty}\frac{w^{(i-1)}(-1)^{(i-1)}}{i(b+\xi)^{(i-1)}}\right\}^2,$$

where the series $\left\{\sum_{i=2}^{\infty} \frac{w^{(i-1)}(-1)^{(i-1)}}{i(b+\xi)^{(i-1)}}\right\}^2$ is convergent in every way. When $w = 0$, it is definable as $MSE\left(\hat{\delta}_{BL}(a,b)|\xi\right) = MSE\left(\hat{\delta}_{BSS}(a,b)|\xi\right)$ with $k = 0$ based on the asymptotic relationship between $\hat{\delta}_{BL}$ and $\hat{\delta}_{BSS}$ with $k = 0$ when $w \to 0$. When $w/(b+\xi) = 1$, (12) can be represented as

$$MSE\left(\hat{\delta}_{BL}(a,b)|\xi\right) = \frac{n+a}{(b+\xi)^2} + \left(\frac{n+a}{b+\xi}\right)^2 (1 - \ln(2))^2.$$

Eq. (12) is a continuous function of w when $w/(b+\xi) > 1$. Therefore, using a simple calculus method, it can be demonstrated that (12) is a continuous function of w over $w > -(b+\xi)$ for the specified ξ and hyperparameters $a > 0$ and $b > 0$.

Additionally, it should be noted that $E\left(\hat{\delta}_{MLE}\right)(\delta)$ and $MSE\left(\hat{\delta}_{MLE}\right)(\delta)$ don't depend on ξ and rely on δ, and $MSE\left(\hat{\delta}_{BSS}(a,b)|\xi\right)$ and $MSE\left(\hat{\delta}_{BL}(a,b)|\xi\right)$ don't depend on δ but rely on hyper-parameters and ξ.

3. EB AND E-MSE ESTIMATIONS

Han [9] suggests that selecting the prior parameters a and b ensures that the prior $g(\delta|a,b)$ in (5) is a decreasing function of δ. The derivative of $g(\delta|a,b)$ with regard to δ is

$$\frac{dg(\delta|a,b)}{d\delta} = \frac{b^a}{\Gamma(a)} \delta^{a-2} e^{-b\delta} [(a-1) - b\delta].$$

Thus, for $0 < a < 1, b > 0$, the prior $g(\delta|a,b)$ is a decreasing function of δ because $\frac{dg(\delta|a,b)}{d\delta} < 0$ when $0 < a < 1$ and $b > 0$. Assuming independent hyper-parameters a and b with density functions $\pi_1(a)$ and $\pi_2(b)$, respectively, consequently the following is an expression for the bivariate density function of a and b:

$$\pi(a,b) = \pi_1(a)\pi_2(b).$$

The following is the EB estimate of δ for a given ξ based on the SSE loss function:

$$\hat{\delta}_{EBSS} = \int_{\varrho}\int \hat{\delta}_{BSS}(a,b)\pi(a,b)dadb, \tag{13}$$

and the related E-MSE, given ξ, is

$$\text{E-MSE}\left(\hat{\delta}_{EBSS}|\xi\right) = \int_{\varrho}\int MSE\left(\hat{\delta}_{BSS}(a,b)|\xi\right)\pi(a,b)dadb. \tag{14}$$

The expression $\hat{\delta}_{BSS}(a,b)$ represents the Bayes estimator of δ , as defined in Eq. (7). The term $MSE\left(\hat{\delta}_{BSS}(a,b)|\xi\right)$ refers to the MSE of the Bayes estimate of δ given by (11). The domain ϱ represents the range of values for a and b, within which the prior density decreases with respect to δ. The EB estimate of δ, given ξ, is defined as when the LINEX loss function is utilized.

$$\hat{\delta}_{EBL} = \int_{\varrho}\int \hat{\delta}_{BL}(a,b)\pi(a,b)dadb, \tag{15}$$

and the associated E-MSE, indicated by ξ, is described as

$$\text{E-MSE}\left(\hat{\delta}_{EBL}|\xi\right) = \int_{\varrho}\int MSE\left(\hat{\delta}_{BL}(a,b)|\xi\right)\pi(a,b)dadb, \tag{16}$$

where $\hat{\delta}_{BL}(a,b)$ is the Bayes estimator of δ given by $(9), MSE\left(\hat{\delta}_{BL}(a,b)|\xi\right)$ is MSE of Bayes estimator of δ given by (12) and ϱ is the domain of a and b for which the prior density is decreasing in δ.

3.1. EB estimations of δ

The features of EB estimations of δ depend on the distributions of two hyper-parameters (a and b) in this study. Consider beta density with parameters $u > 0$ and $v > 0$.

$$\pi_1(a) = \frac{1}{B(u,v)}a^{u-1}(1-a)^{v-1}, 0 < a < 1,$$

and three distributions for b will be provided as follows,

$$\pi_{21}(b) = \frac{1}{s},\ \ 0 < b < s,$$
$$\pi_{22}(b) = \frac{2}{s^2}(s-b), 0 < b < s,$$

and

$$\pi_{23}(b) = \frac{2b}{s^2}, 0 < b < s,$$

where $s > 0$, and $B(u,v)$ represents the beta function. To examine the EB estimations of δ, the following three joint distributions, $\pi_i(a,b) = \pi_1(a)\pi_{2i}(b)$ of $0 < a < 1$ and $0 < b < s$ for $i = 1,2,3$, for which the gamma prior, $g(\delta|a,b)$ of (5), is a decreasing function of δ, as shown below

$$\left.\begin{aligned} \pi_1(a,b) &= \frac{1}{sB(u,v)} a^{u-1}(1-a)^{v-1}, \\ \pi_2(a,b) &= \frac{2}{s^2 B(u,v)} (s-b)a^{u-1}(1-a)^{v-1}, \\ \pi_3(a,b) &= \frac{2b}{s^2 B(u,v)} a^{u-1}(1-a)^{v-1}. \end{aligned}\right\} \tag{17}$$

Using (7), (13) and (17), we can calculate EB estimates of δ given $\underline{x}$ and ξ based on SSE loss function. Then EB estimates of δ based on $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$ are as follows,

$$\hat{\delta}_{EBSS1} = \int_Q \int \hat{\delta}_{BSS}(a,b)\pi_1(a,b)\,da\,db$$

$$= \frac{1}{sB(u,v)} \int_0^s \int_0^1 \left(\frac{n+a-k}{b+\xi}\right) a^{u-1}(1-a)^{v-1}\,da\,db$$

$$= \frac{1}{s}\left(n-k+\frac{u}{u+v}\right)\ln\left(\frac{s+\xi}{\xi}\right), \quad k = 0,1,2, \tag{18}$$

$$\hat{\delta}_{EBSS2} = \frac{2}{s}\left(n-k+\frac{u}{u+v}\right)\left(\frac{\xi+s}{s}\ln\left(\frac{s+\xi}{\xi}\right)-1\right), \quad k = 0,1,2, \tag{19}$$

and

$$\hat{\delta}_{EBSS3} = \frac{2}{s}\left(n-k+\frac{u}{u+v}\right)\left(1-\frac{\xi}{s}\ln\left(\frac{s+\xi}{\xi}\right)\right), \quad k = 0,1,2. \tag{20}$$

Using (9), (15), and (17), we can calculate the EB estimates of the parameter δ under the LINEX loss function, given $\underline{x}$ and ξ. Given ξ and the LINEX loss function, the EB estimates of δ based on $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$ are, respectively, as follows

$$\hat{\delta}_{EBL1} = \int_{Q}\int \hat{\delta}_{BL}(a,b)\pi_1(a,b)dbda$$

$$= \frac{1}{wsB(u,v)}\int_0^1\int_0^s (n+a)a^{u-1}(1-a)^{v-1}\ln\left(1+\frac{w}{b+\xi}\right)dbda$$

$$= \frac{1}{ws}\left(n+\frac{u}{u+v}\right)\int_0^s \ln\left(1+\frac{w}{b+\xi}\right)db$$

$$= \frac{1}{ws}\left(n+\frac{u}{u+v}\right)\left\{s\ln\left(1+\frac{w}{s+\xi}\right)+(\xi+w)\ln\left(1+\frac{s}{w+\xi}\right)-\xi\ln\left(\frac{\xi+s}{\xi}\right)\right\}, \qquad (21)$$

$$\hat{\delta}_{EBL2} = \int_{Q}\int \hat{\delta}_{BL}(a,b)\pi_2(a,b)dbda$$

$$= \frac{2}{ws^2B(u,v)}\int_0^1\int_0^s (n+a)(s-b)a^{u-1}(1-a)^{v-1}\ln\left(1+\frac{w}{b+\xi}\right)dbda$$

$$= \left(n+\frac{u}{u+v}\right)\left[\frac{1}{w}\ln\left(1+\frac{w}{\xi}\right)-\frac{(s+\xi)^2}{s^2w}\ln\left(1+\frac{s}{\xi}\right)+\frac{(s+\xi+w)^2}{s^2w}\ln\left(1+\frac{s}{\xi+w}\right)-\frac{1}{s}\right], \qquad (22)$$

and

$$\hat{\delta}_{\mathrm{EBL3}} = \int_{Q}\int \hat{\delta}_{BL}(a,b)\pi_3(a,b)dbda$$

$$= \frac{2}{ws^2B(u,v)}\int_0^1\int_0^s (n+a)a^{u-1}(1-a)^{v-1}b\ln\left(1+\frac{w}{b+\xi}\right)dbda$$

$$= \left(n+\frac{u}{u+v}\right)\left[\frac{1}{w}\ln\left(1+\frac{w}{\xi+s}\right)+\frac{\xi^2}{s^2w}\ln\left(1+\frac{s}{\xi}\right)-\frac{(\xi+w)^2}{s^2w}\ln\left(1+\frac{s}{\xi+w}\right)+\frac{1}{s}\right]. \qquad (23)$$

3.2. E-MSE estimations of δ

The closed forms of the E-MSE estimators for the IWD rate parameter are covered in this section. We use (11), (14) and (17), to obtain E-MSE estimates of the parameter δ, based on SSE loss function and $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$, respectively, by

$$\text{E-MSE}\left(\hat{\delta}_{EBSS1}|\xi\right) = \int_0^s\int_0^1 MSE\left[\hat{\delta}_{BSS}(a,b)|\xi\right]\pi_1(a,b)dadb$$

$$= \frac{1}{sB(u,v)}\int_0^s\int_0^1 \frac{(n+a+k^2)}{(b+\xi)^2}a^{u-1}(1-a)^{v-1}dadb$$

$$=\frac{1}{s}\int_0^s \frac{db}{(b+\xi)^2}\int_0^1 \frac{(n+a+k^2)}{B(u,v)}a^{u-1}(1-a)^{v-1}da$$

$$=\frac{1}{\xi(\xi+s)}\left(n+k^2+\frac{u}{u+v}\right),\quad k=0,1,2, \tag{24}$$

$$\text{E-MSE}\left(\hat{\delta}_{EBSS2}|\xi\right)=\frac{2}{s^2}\left(n+k^2+\frac{u}{u+v}\right)\left[\frac{s}{\xi}-\ln\left(1+\frac{s}{\xi}\right)\right],\quad k=0,1,2, \tag{25}$$

and

$$\text{E-MSE}\left(\hat{\delta}_{EBSS3}|\xi\right)=\frac{2}{s^2}\left(n+k^2+\frac{u}{u+v}\right)\left[\ln\left(1+\frac{s}{\xi}\right)-\frac{s}{\xi+s}\right],\quad k=0,1,2. \tag{26}$$

From (12), (16) and (17), we obtain E-MSE estimates of the parameter δ, based on LINEX loss function with $-1<w/\xi$ and for $\pi_1(a,b),\pi_2(a,b)$ and $\pi_3(a,b)$, respectively by

$$\text{E-MSE}\left(\hat{\delta}_{EBL1}|\xi\right)=\int_0^s\int_0^1 MSE\left[\hat{\delta}_{BL}(a,b)|\xi\right]\pi_1(a,b)dadb$$

$$=\int_0^s\int_0^1\left(\frac{n+a}{(b+\xi)^2}+\left(\frac{n+a}{b+\xi}\right)^2\left\{1-\frac{(b+\xi)}{w}\ln\left(1+\frac{w}{(b+\xi)}\right)\right\}^2\right)$$

$$\times\,\pi_1(a,b)dadb$$

$$=\int_0^s\int_0^1\left(\frac{n+a}{b+\xi}\right)^2\left\{1-\frac{(b+\xi)}{w}\ln\left(1+\frac{w}{(b+\xi)}\right)\right\}^2\pi_1(a,b)dadb$$

$$+\text{E-MSE}\left(\hat{\delta}_{EBSS1}|\xi\right)(k=0), \tag{27}$$

$$\text{E-MSE}\left(\hat{\delta}_{EBL2}|\xi\right)=\int_0^s\int_0^1 MSE\left[\hat{\delta}_{BL}(a,b)|\xi\right]\pi_2(a,b)dadb$$

$$=\int_0^s\int_0^1\left(\frac{n+a}{(b+\xi)^2}+\left(\frac{n+a}{b+\xi}\right)^2\left\{1-\frac{(b+\xi)}{w}\ln\left(1+\frac{w}{(b+\xi)}\right)\right\}^2\right)$$

$$\times\,\pi_2(a,b)dadb$$

$$=\int_0^s\int_0^1\left(\frac{n+a}{b+\xi}\right)^2\left\{1-\frac{(b+\xi)}{w}\ln\left(1+\frac{w}{(b+\xi)}\right)\right\}^2\pi_2(a,b)dadb$$

$$+\text{E-MSE}\left(\hat{\delta}_{EBSS2}|\xi\right)(k=0), \tag{28}$$

and

$$\text{E-MSE}\left(\hat{\delta}_{EBL3}|\xi\right) = \int_0^s \int_0^1 MSE\left[\hat{\delta}_{BL}(a,b)|\xi\right]\pi_3(a,b)\,da\,db$$

$$= \int_0^s \int_0^1 \left(\frac{n+a}{(b+\xi)^2} + \left(\frac{n+a}{b+\xi}\right)^2 \left\{1 - \frac{(b+\xi)}{w}\ln\left(1+\frac{w}{(b+\xi)}\right)\right\}^2\right)$$

$$\times \pi_3(a,b)\,da\,db$$

$$= \int_0^s \int_0^1 \left(\frac{n+a}{b+\xi}\right)^2 \left\{1 - \frac{(b+\xi)}{w}\ln\left(1+\frac{w}{(b+\xi)}\right)\right\}^2 \pi_3(a,b)\,da\,db$$

$$+\text{E-MSE}\left(\hat{\delta}_{EBSS3}|\xi\right)(k=0). \tag{29}$$

4. MONTE CARLO SIMULATION

This section outlines Monte Carlo simulation methodology that will be employed to evaluate the efficacy of the suggested estimation techniques across the full population.

In order to assess the effectiveness of each EB estimator, E-MSE$(\hat{\delta}_{EBSSj}\,|\xi)(k=j)$, for $j = 0,1,2$ and E-MSE$(\hat{\delta}_{EBLi}\,|\ \xi)$ for $i = 1,2,3$ will be utilized over the sampling distribution of sample $\underline{x}$. Simultaneously, the comparison between Bayesian estimators and MLE will be examined. To achieve this objective, MSE$(\hat{\delta}_{BSS}\,(a,b)|\xi)$ and MSE$\left(\hat{\delta}_{BL}\,(a,b)|\xi\right)$ for all Bayesian estimators and the expected MSE$(\hat{\delta}_{MLE}\,)(\delta)$ for MLE computed across the sampling distribution of ξ and across overall population of δ, utilizing Beta distribution for a and the three distributions of b as stated in Section 3.1.

We consider for Monte Carlo simulation, the sample size $n = 3,5,7,9$, and $\gamma = 0.9, 1.5, 3.0$, three joint distributions, $\pi_i(a,b)$, $i = 1,2,3$ from Eq. (17) with $s = 5, 10, 25, 50, 100, 500$, $u = 3$ or 4 and $v = 4$ or 5, using Matlab, perform the following procedures to obtain three SSE loss functions with $k = 0,1$, or 2 and a specified w for the LINEX loss function:

(1) Assign $i = 1$.

(2) If $i \leq 3$, a joint distribution, $\pi_i(a,b) = \pi_1(a)\pi_{2i}(b)$ is selected as specified in Eq. (17), utilizing the provided values of (u,v) and s; if not, proceed to Step 6.

(3) Give a and b random values by utilizing the beta prior, $\pi_1(a)$, and prior, $\pi_{2i}(b)$, respectively.

(4) For a given value of δ, randomly produce a sample, $\underline{x}$, of size n from IW(δ, γ) as defined in Eq. (1).

(5) For $i \leq 3$, compute the values of E-MSE($\hat{\delta}_{EBSSi}\,|\xi)(k)$, for k = 0, 1, 2, respectively, and

E-MSE($\hat{\delta}_{EBLi}\,|\;\xi$).

(6) For $i = 4$,

(I) assess the value of MSE($\hat{\delta}_{M\,LE}$)(δ), utilizing δ from Step 5 and sample size n, and

(II) assess the value of MSE($\hat{\delta}_{BSS}\,(a,b)|\xi$) for $k = 0,1,2$ and determine the value of MSE$\left(\hat{\delta}_{BL}\,(a,b)|\xi\right)$ utilizing a,b from Step 3 and the sample $\underline{x}$.

(7) Repeat Step 3 to Step 6 for 10,000 times. Calculate and label the average of 10,000 calculated values for E-MSE($\hat{\delta}_{EBSSi}\,|\xi)(k)$, for $k = 0,1,2$, and E-MSE($\hat{\delta}_{EBLi}\,|\;\xi$) as E-MSE($\hat{\delta}_{EBSSi}$), $k = 0,1,2$, and E-MSE($\hat{\delta}_{EBLi}$), respectively. The average of 10,000 calculated values for MSE($\hat{\delta}_{BSS}\,(a,b)|\xi$) for $k = 0,1,2$, MSE($\hat{\delta}_{BL}\;|\xi$) and MSE($\hat{\delta}_{M\,LE}$)(δ) are calculated and labeled as MSE($\hat{\delta}_{BSS}$) for k = 0, 1, 2,

MSE($\hat{\delta}_{BL}\,|\xi$) and MSE($\hat{\delta}_{M\,LE}$), respectively.

(8) Set $i = i + 1$ and repeat Steps 2 through 7 until $i > 4$.

The simulation results utilizing $\pi_1(a,b), \pi_2(a,b)$ and $\pi_3(a,b)$ for $\gamma = 3, s = 5$, $u = 4$ and $v = 5$ are displayed in Tables 1 and 2. To see the impact from the range of

b, Tables 3 and 4 also provide the simulation results over a wide range, s, of b for the same sample sizes 5 and 7. Additional simulation results are included in the Appendix Section.

Table 1: E-MSE of $\hat{\delta}$ with $s = 5, \gamma = 3, u = 4$ and $v = 5$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
n	i	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
3	1	0.0566	0.0637	0.0731	0.1224
	2	0.0730	0.0846	0.0943	0.1579
	3	0.0402	0.0428	0.0519	0.0869
5	1	0.0286	0.0300	0.0339	0.0496
	2	0.0330	0.0348	0.0390	0.0572
	3	0.0243	0.0251	0.0287	0.0421
7	1	0.0183	0.0189	0.0208	0.0281
	2	0.0203	0.0211	0.0230	0.0312
	3	0.0163	0.0168	0.0185	0.0251
9	1	0.0124	0.0127	0.0137	0.0177
	2	0.0133	0.0137	0.0148	0.0190
	3	0.0115	0.0117	0.0127	0.0163

Table 2: MSE of $\hat{\delta}$ with $\gamma = 3, u = 4$ and $v = 5$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
n	MSE($\hat{\delta}_{ML}$)	MSE($\hat{\delta}_{BSS}$)	MSE($\hat{\delta}_{BL}$)	MSE($\hat{\delta}_{BSS}$)	MSE($\hat{\delta}_{BSS}$)
3	0.6529	0.0893	0.1188	0.1147	0.1909
5	0.1523	0.0493	0.0748	0.0583	0.0853
7	0.0752	0.0404	0.0690	0.0458	0.0619
9	0.0406	0.0226	0.0428	0.0249	0.0320

Table 3: E-MSE of $\hat{\delta}$ with $n = 5$, $\gamma = 3$, $u = 3$ and $v = 4$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
s	i	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
25	1	0.0011289	0.0011316	0.0013368	0.0019607
	2	0.0012998	0.0013036	0.0015392	0.0022575
	3	0.0009580	0.0009596	0.0011344	0.0016639
50	1	0.0002822	0.0002824	0.0003342	0.0004902
	2	0.0003249	0.0003251	0.0003848	0.0005644
	3	0.0002395	0.0002396	0.0002836	0.0004160
100	1	0.00007055	0.00007057	0.0000836	0.0001225
	2	0.0000812348	0.00008125	0.00009619	0.0001411
	3	0.00005987	0.00005988	0.00007090	0.0001039
500	1	0.00000282216	0.00000282218	0.000003342040	0.00000490165
	2	0.00000324939	0.00000324941	0.00000384796	0.00000564
	3	0.000002394	0.000002395	0.000002836	0.00000415

Table 4: E-MSE of $\hat{\delta}$ with $n = 7, \gamma = 3, u = 3$ and $v = 4$.

	i	$k = 0$	$w = 2$	$k = 1$	$k = 2$
s		E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
25	1	0.0007223	0.0007235	0.0008195	0.0011112
	2	0.00079939	0.0008009	0.0009070	0.0012298
	3	0.00064523	0.00064604	0.00073208	0.00099265
50	1	0.00018058	0.00018066	0.00020489	0.00027781
	2	0.00019985	0.00019995	0.00022675	0.00030746
	3	0.00016131	0.00016136	0.00018302	0.00024816
100	1	0.00004514	0.00004515	0.00005122	0.00006945
	2	0.00004996	0.00004997	0.00005669	0.00007686
	3	0.00004034	0.00004033	0.00004576	0.00006204
500	1	0.00000181	0.00000181	0.00000205	0.00000278
	2	0.00000199	0.00000199	0.00000227	0.00000307
	3	0.00000161	0.00000161	0.00000183	0.00000248

The comparison of EB estimators, as evidenced by Tables 1, 2, 3 and 4 yields the following conclusions:

(1) As the sample size n increases, the simulated average of E-MSE diminishes.

(2) For any choice of k from 0, 1, 2, all simulated averages of E-MSEs over the population of δ under the SSE loss function, E-MSE($\hat{\delta}_{EBSS3}|\xi$) < E-MSE($\hat{\delta}_{EBSS1}|\xi$) < E-MSE($\hat{\delta}_{EBSS2}|\xi$) for any specified sample ξ.

(3) Based on the prior $\pi_i(a, b)$, where i = 1, 2, 3, derived from Eq. (17), all four simulated averages of E-MSEs associated with three distinct SSE and LINEX loss functions, achieve that

E-MSE($\hat{\delta}_{EBSSi}|\xi$)(k = 0) < E-MSE($\hat{\delta}_{EBLi}|\xi$)< E-MSE($\hat{\delta}_{EBSSi}|\xi$)(k = 1)<E-MSE($\hat{\delta}_{EBSSi}|\xi$)(k = 2) for every specified sample ξ.

(4) All three simulated averages of E-MSEs under the LINEX loss function achieve that

E-MSE($\hat{\delta}_{EBL3}|\xi$) < E-MSE($\hat{\delta}_{EBL1}|\xi$) < E-MSE($\hat{\delta}_{EBL2}|\xi$) for any specified ξ.

(5) Tables 1 and 2 demonstrate that the sample size does not influence the comparative results among all E-MSEs.

(6) Tables 3 and 4 demonstrate that the s value does not influence the comparative results among all

E-MSEs.

5. ILLUSTRATIVE EXAMPLES

This section will utilize two real data sets to demonstrate the previously discussed estimation methodologies for the IWD rate parameter, δ.

5.1. Example 1. Breakdown Times at Voltage 34 KV

As supplied by Nelson [24], the 19 documented breakdown times, expressed in minutes, for an insulating fluid between electrodes at a voltage of 34 kV are as follows: 0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89. Abd Ellahal [25] examined the model's applicability to the provided real data set and demonstrated how well the IWD fits it. The following lower records values are obtained using these data and are represented as 4.15, 3.16, 2.78, 1.31, 0.96, 0.78, 0.19. The numerical data presented in Tables 5 to 8 are derived from the previous lower records.

Table 5: Estimates of δ, where $s = 10, \gamma = 0.6434, u = 3$ and $v = 4$.

$\hat{\delta}_{MLE}$	$\hat{\delta}_{BL}$	$\hat{\delta}_{BSS}$	$\hat{\delta}_{BSS}$	$\hat{\delta}_{BSS}$	i	$\hat{\delta}_{EBLi}$	$\hat{\delta}_{EBSSi}$	$\hat{\delta}_{EBSSi}$	$\hat{\delta}_{EBSSi}$
	$w = 2$	$k = 0$	$k = 1$	$k = 2$		$w = 2$	$k = 0$	$k = 1$	$k = 2$
2.4046	1.7303	2.1987	1.9056	1.6124	1	0.9502	1.1065	0.9576	0.8086
					2	1.1487	1.3716	1.1870	1.0023
					3	0.7517	1.2032	1.14123	1.07930

Table 6: Calculated MSE results for $\hat{\delta}$.

MSE($\hat{\delta}_{MLE}$)(δ)	MSE($\hat{\delta}_{BL}\|\xi$)	MSE($\hat{\delta}_{BSS}\|\xi$)		
1.1175	$w = 2$	$k = 0$	$k = 1$	$k = 2$
	1.0998	0.6446	0.7305	0.9884

Table 7: Calculated E-MSE results for $\hat{\delta}$ using $s = 10, \gamma = 0.6434$, $u = 3$, and $v = 4$.

i	E-MSE($\hat{\delta}_{EBLi}\|\xi$)	E-MSE($\hat{\delta}_{EBSSi}\|\xi$)		
	$w = 2$	$k = 0$	$k = 1$	$k = 2$
1	0.2396	0.1976	0.2243	0.3041
2	0.3605	0.2891	0.3280	0.4447
3	0.1186	0.1062	0.1205	0.1634

Table 8: Calculated E-MSE results of $\hat{\delta}$ for different s with u = 3 and v = 4.

		$k=0$	$w=2$	$k=1$	$k=2$
s	i	E-MSE($\hat{\delta}_{EBSSi}\|\xi$)	E-MSE($\hat{\delta}_{EBLi}\|\xi$)	E-MSE($\hat{\delta}_{EBSSi}\|\xi$)	E-MSE($\hat{\delta}_{EBSSi}\|\xi$)
25	1	0.0914	0.1085	0.1037	0.1407
	2	0.1504	0.1822	0.1707	0.2314
	3	0.0324	0.0347	0.0368	0.0499
50	1	0.0482	0.0568	0.0547	0.0742
	2	0.0848	0.1013	0.0963	0.1305
	3	0.0116	0.0122	0.0132	0.0179
100	1	0.0248	0.0291	0.0281	0.0381
	2	0.0457	0.0541	0.0519	0.0704
	3	0.0039	0.0040	0.0044	0.0059
500	1	0.0051	0.0059	0.0058	0.0078
	2	0.0099	0.0116	0.0112	0.0152
	3	0.00024709	0.00025301	0.00028035	0.00038013

5.2. Example 2. Repair Times for An Airborne Communication Transceiver

The data was initially examined by Von Alven [26]. The data is displayed in Table 9. Prior to data analysis, we applied the IW model to this dataset. We employed the Kolmogorov-Smirnov (K-S) distance between the fitted empirical distribution functions and their related p-values. The K-S statistic and the accompanying p-value for this data are 0.0815 and 0.9197, respectively. The IWD demonstrates a strong fit to this data set, see Shahrastani and Makhdoom [27]. The highest likelihood estimates for parameters γ and δ are 1.011941 and 1.125229, respectively. We regard the lower record values from this data as expressed as 0.6, 0.5, 0.5, 0.5, 0.5, 0.3, 0.2. The numerical values are presented in Tables 10 to 13.

Table 9: 72 Repair Times for an airborne communication transceiver.

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.7
0.8	1.0	1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5	1.5
2.0	2.0	2.2	2.5	2.7	3.0	3.3	3.3	4.0	4.0	4.5	4.7
5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3	7.5	8.8	9.0	10.3

Table 10: Estimates of δ, where $s = 10, \gamma = 1.011941$, $u = 3$, and $v = 4$.

$\hat{\delta}_{MLE}$	$\hat{\delta}_{BL}$	$\hat{\delta}_{BSS}$	$\hat{\delta}_{BSS}$	$\hat{\delta}_{BSS}$	i	$\hat{\delta}_{EBLi}$	$\hat{\delta}_{EBSSi}$	$\hat{\delta}_{EBSSi}$	$\hat{\delta}_{EBSSi}$
	$w = 2$	$k = 0$	$k = 1$	$k = 2$		$w = 2$	$k = 0$	$k = 1$	$k = 2$
1.3734	1.1457	1.3400	1.1613	0.9827	1	0.7241	0.8066	0.6980	0.5895
					2	0.8403	0.9498	0.8220	0.6941
					3	0.6079	0.6634	0.5741	0.4848

Table 11: Calculated MSE results for $\hat{\delta}$.

MSE($\hat{\delta}_{MLE}$)(δ)	MSE($\hat{\delta}_{BL}\|\xi$)	MSE($\hat{\delta}_{BSS}\|\xi$)		
	$w = 2$	$k = 0$	$k = 1$	$k = 2$
0.3798	0.4085	0.2394	0.2713	0.3671

Table 12: Calculated E-MSE results for $\hat{\delta}$ using $s = 10, u = 3$, and $v = 4$.

i	E-MSE($\hat{\delta}_{EBLi}\|\xi$)	E-MSE($\hat{\delta}_{EBSSi}\|\xi$)		
	$w = 2$	$k = 0$	$k = 1$	$k = 2$
1	0.1059	0.0965	0.1095	0.1485
2	0.1450	0.1302	0.1477	0.2002
3	0.0668	0.0629	0.0714	0.0968

Table 13: Calculated E-MSE results of $\hat{\delta}$ for different s with $u = 3$ and $v = 4$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
s	i	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
25	1	0.0484	0.0523	0.0549	0.0745
	2	0.0744	0.0814	0.0844	0.1144
	3	0.0225	0.0233	0.0255	0.0346
50	1	0.0265	0.0284	0.0300	0.0407
	2	0.0442	0.0479	0.0501	0.0679
	3	0.0088	0.0090	0.0099	0.0135
100	1	0.0139	0.0149	0.0157	0.0213
	2	0.0247	0.0266	0.0280	0.0379
	3	0.0031	0.0031	0.0035	0.0047
500	1	0.0029	0.0031	0.0033	0.0044
	2	0.0056	0.0059	0.0063	0.0085
	3	0.00021431	0.00021655	0.00024316	0.00032971

6. CONCLUSION

The EB estimators of the rate parameter of the IWD have been analyzed under the SSE and LINEX loss functions. The formulas for the E-MSEs of EB estimators were derived. For the purpose of comparing a particular data set, a number of theoretical characteristics of E-MSEs were developed. The simulation study further validates the properties throughout the entire populations of δ. Two practical examples were employed to illustrate the applications. All significant findings are detailed in Sections 4 and 5. When the shape parameter, γ, is indeterminate, the maximum likelihood estimate of γ cannot be derived in a closed form. There exists no conjugate prior for b, rendering all Bayesian estimators of γ complex and intractable in the research. The simultaneous application of the EB estimation approach to both parameters of the IWD remains an unresolved issue currently under investigation.

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APPENDIX

Table 14: Simulated E-MSE of $\hat{\delta}$ with $s = 5, \gamma = 1.5, u = 3$ and $v = 4$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
n	i	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
3	1	0.0558	0.0627	0.0721	0.1209
	2	0.0719	0.0832	0.0929	0.1559
	3	0.0397	0.0423	0.0512	0.0860
5	1	0.0282	0.0295	0.0334	0.0490
	2	0.0325	0.0343	0.0385	0.0564
	3	0.0239	0.0248	0.0284	0.0416
7	1	0.0181	0.0187	0.0205	0.0278
	2	0.0200	0.0208	0.0227	0.0307
	3	0.0161	0.0166	0.0183	0.0248
9	1	0.0122	0.0125	0.0135	0.0174
	2	0.0132	0.0135	0.0145	0.0187
	3	0.0113	0.0115	0.0125	0.0161

Table 15: MSE of $\hat{\delta}$ with $\gamma = 1.5, u = 3$ and $v = 4$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
n	MSE($\hat{\delta}_{ML}$)	MSE($\hat{\delta}_{BSS}$)	MSE($\hat{\delta}_{BL}$)	MSE($\hat{\delta}_{BSS}$)	MSE($\hat{\delta}_{BSS}$)
3	0.4798	0.0763	0.1018	0.0978	0.1623
5	0.1178	0.0489	0.0743	0.0577	0.0842
7	0.0656	0.0349	0.0598	0.0396	0.0535
9	0.0371	0.0212	0.0402	0.0234	0.0300

Table 16: Calculated E-MSE results of $\hat{\delta}$ for different s with $n = 5, \gamma = 1.5, u = 2$ and $v = 3$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
s	i	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
25	1	0.00109613	0.00109873	0.00129912	0.00190808
	2	0.00126033	0.00126395	0.00149373	0.00219391
	3	0.00093193	0.00093350	0.00110451	0.00162224
50	1	0.00027403	0.00027420	0.00032478	0.00047702
	2	0.00031508	0.00031532	0.00037343	0.00054848
	3	0.00023298	0.00023308	0.00027613	0.00040556
100	1	0.00006851	0.00006852	0.00008119	0.00011925
	2	0.00007878	0.00007879	0.00009336	0.00013712
	3	0.00005825	0.00005825	0.00006903	0.00010139
500	1	0.00000274	0.00000274	0.00000325	0.00000477
	2	0.00000315	0.00000315	0.00000373	0.00000548
	3	0.00000233	0.00000233	0.00000276	0.00000406

Table 17: E-MSE of $\hat{\delta}$ with $s = 5, \gamma = 0.9, u = 2$ and $v = 3$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
	i	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
3	1	0.0542	0.0607	0.0701	0.1179
	2	0.0697	0.0804	0.0903	0.1518
	3	0.0386	0.0411	0.0500	0.0841
5	1	0.0274	0.0286	0.0325	0.0477
	2	0.0315	0.0332	0.0373	0.0548
	3	0.0233	0.0241	0.0276	0.0406
7	1	0.0175	0.0181	0.0199	0.0270
	2	0.0194	0.0201	0.0220	0.0299
	3	0.0157	0.0161	0.0178	0.0242
9	1	0.0119	0.0121	0.0131	0.0169
	2	0.0128	0.0131	0.0141	0.0182
	3	0.0110	0.0112	0.0122	0.0157

Table 18: MSE of $\hat{\delta}$, with $\gamma = 0.9, u = 2$ and $v = 3$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
n	MSE($\hat{\delta}_{ML}$)	MSE($\hat{\delta}_{BSS}$)	MSE($\hat{\delta}_{BL}$)	MSE($\hat{\delta}_{BSS}$)	MSE($\hat{\delta}_{BSS}$)
3	0.5359	0.0886	0.1180	0.1138	0.1893
5	0.1208	0.0428	0.0653	0.0505	0.0736
7	0.0628	0.0294	0.0503	0.0333	0.0449
9	0.0372	0.0233	0.0442	0.0257	0.0330

Table 19: E-MSE of $\hat{\delta}$ with $n = 7, \gamma = 0.9, u = 2$ and $v = 3$.

		$k = 0$	$w = 2$	$k = 1$	$k = 2$
s	i	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBLi}$)	E-MSE($\hat{\delta}_{EBSSi}$)	E-MSE($\hat{\delta}_{EBSSi}$)
25	1	0.00070143	0.00070257	0.00079622	0.00108059
	2	0.00077545	0.00077696	0.00088024	0.00119462
	3	0.00062742	0.00062819	0.00071220	0.00096656
50	1	0.00017536	0.00017543	0.00019906	0.0002701
	2	0.00019386	0.00019396	0.00022006	0.00029865
	3	0.00015685	0.00015690	0.00017805	0.00024164
100	1	0.00004384	0.00004384	0.00004976	0.00006754
	2	0.00004847	0.00004847	0.00005502	0.00007466
	3	0.00003921	0.00003922	0.00004451	0.00006041
500	1	0.00000175	0.00000175	0.00000199	0.00000270
	2	0.00000194	0.000001939	0.00000220	0.00000299
	3	0.00000157	0.00000157	0.00000178	0.00000243