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# FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL DYNAMICAL SYSTEMS WITH PERIODIC BVPS

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Abstract. The primary objective of this study is to comprehensively investigate the outcomes concerning the existence and Ulam stability of a fractional dynamic system, specifically one involving a neutral partial integro-differential equation with periodic boundary conditions on time scales, using the Caputo fractional nabla derivative. The study applies standard fixed point methods to derive its results, with a focus on controllability and Ulam stability. Additionally, the practical relevance of the theoretical findings is showcased through an illustrative example, which includes a graphical representation.

## 1. Introduction

In recent years, there has been a surge of interest in fractional differential equations and their diverse applications. This heightened attention can be attributed to the rapid advancements in the theory of fractional calculus, which finds widespread utility across various academic domains, including mathematics, physics, chemistry, biology, medicine, mechanics, control theory, signal and image processing, environmental science, finance, and other interdisciplinary field[1, 2, 3, 4, 6, 7]. Fractional order differential equations, characterized by fractional orders, offer a generalized framework incorporating power-law memory kernels in both time and spatial domains, capturing nonlocal relationships. These equations serve as a robust tool for elucidating the memory characteristics of diverse substances and the inherent nature of inheritance phenomena. The underlying physical motivations behind these studies have paved the way for a burgeoning field of scientific research, encompassing novel theoretical analysis and numerical methodologies for fractional order dynamical systems[8, 9].

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In the realm of practical applications, certain scenarios may manifest wherein a complete representation cannot be achieved through either wholly continuous or wholly discrete phenomena. In such instances, the necessity arises for a shared domain that can concurrently accommodate both conditions. Addressing this need, Stefan Hilger introduced a unifying domain denoted as the time scale (T), serving to reconcile the demands of both continuous and discrete calculus. Formulation in dynamic equations over time scale emerged as a strategic solution for modeling systems that exhibit a synthesis of differential and difference equations. Extensive scholarly efforts have been devoted to the exploration of dynamic equations, encompassing both linear and nonlinear formulations, incorporating considerations of local initial and boundary conditions. Many researchers have extensively investigated dynamic equations using fractional calculus because of its accuracy and significant benefits in providing physical insights[10, 11, 12, 13, 52, 48].

A neutral difference equation (NDE) is characterized by the inclusion of the higher-order difference of an unknown sequence in the equation, featuring both delayed and undelayed (advanced) terms. It is essential to acknowledge that the theoretical framework surrounding neutral difference equations introduces complexities, and conclusions established for non-neutral difference equations may not necessarily apply to neutral equations [14, 15, 16, 24, 49, 5]. Beyond mere mathematical curiosity, the exploration of these equations is propelled by their practical applications[17]. Neutral difference equations are highly significant across numerous applied mathematics fields, including circuit theory, bifurcation analysis, population dynamics, the dynamic behavior of delayed network systems, signal processing, and more. Additionally, these equations manifest in the examination of vibrating masses connected to elastic bars, where, for instance, the Euler equation is extensively utilized in various variational problems and plays a crucial role in the theory of automatic control [18, 19, 20].

Integer neutral integro-differential systems use integer derivatives and fixed delays to model processes with localized memory, suitable for simpler dynamics. In contrast, fractional systems utilize non-integer derivatives, offering continuous memory and greater flexibility, allowing them to model complex systems with long-term dependencies more accurately. Fractional systems also provide improved robustness and stability, making them better suited for real-world applications with extended memory and non-local behaviors, where integer systems are less effective[1, 2, 3, 4, 6, 7, 8].

One of the essential qualitative aspects to contemplate among various solution characteristics is the stability of the solutions[28]. The latest research data's thoroughly covers an extensive array of stability theories applicable for differential and difference equations. Nevertheless, in various mathematical analysis fields, Ulam stability has crucial and far-reaching applications, as it adeptly addresses the existence of solutions in close proximity to each approximation. It becomes advantageous, especially in situations where acquiring the exact solution poses challenges. Hyers [29] and Ulam [30] commenced an in-depth examination of stability type of functional equations during 1940 and 1941. In recent times, numerous scholars have extensively examined Ulam stability for differential, difference, and integral equations, employing a variety of approaches. For reader's convenience, refer to [31, 32, 33] and the cited works therein.

In recent years, substantial research has been conducted on fractional dynamic systems, addressing various aspects such as controllability, stability, and boundary value problems. For instance, [34] rigorously investigated the controllability of fractional neutral differential systems with non-instantaneous impulses, providing a foundational understanding of this complex area. Similarly, V. Kumar and M. Malik in [35] examined the existence, stability, and controllability of fractional dynamic systems over time scales, emphasizing their significance in population dynamics. Furthermore, [36] explored controllability in fractional integro-differential equations, integrating delayed impulses across different time scales.

Picard's operator and dynamic inequalities were utilized by Bohner and Tikare in [37] to obtain Ulam stability results for first-order nonlinear dynamic equations over time scales. The enhanced approach to fractional integrals and Nabla derivatives proposed in [38] has significantly advanced our understanding of these concepts within time scales. Additionally, [40] provided a thorough analysis of boundary value problems with periodic conditions in the context of fractional dynamic equations.

Building on these motivational studies, this paper aims to address a critical gap by exploring the periodic boundary value problem of dynamical systems involving partial neutral integro-differential equations across time scales.

$$\left.\begin{array}{ll} {}^{C}D^{\gamma}[\hbar(\iota) - g(\iota,\hbar)] = \mathcal{L}\big(\iota, \hbar(\iota), \mathcal{N}(\hbar(\iota)), {}^{C}D^{\gamma}\hbar(\iota)\big), & \iota \in \mathbb{T} \\ \hbar(0) = \hbar(\mathbb{T}) = 0, & \mathbb{T} \in \mathcal{R}. \end{array}\right\} \quad (1.1)$$

Here,

$$\mathcal{N}(\hbar(\iota)) = \int_0^{\iota} u(\iota, s, \hbar(\iota))\nabla s$$

where $\iota \in \mathbb{T}, \mathbb{T} > 0$ *and* An ld-continuous function are $\mathcal{L} : \mathbb{T} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$. For $0 < \gamma < 1$, ${}^{C}D^{\gamma}$ is the Caputo fractional derivative and $[0, \mathbb{T}] \in \mathbb{T}$. Also $g(\iota, \hbar)$ and $\mathcal{L}(\iota, \hbar)$ are continuous in $\hbar$ and $\mathfrak{i}$ respectively. $\iota \in \mathbb{T}$, here $\mathbb{T}$ are time scale interval such that $\iota \in \mathbb{T} : 0 \leq \iota \leq \mathbb{T}, \quad \mathbb{T} \in \mathcal{R}$.
The motivation for this research arises from the need to further investigate and refine the theoretical frameworks established in these prior works, ultimately contributing to a deeper understanding of the dynamics involved. This research bridges this gap by incorporating the neutral term into the analysis and utilizing fixed-point methods to assess controllability and Ulam stability. By contrasting our findings with existing literature, we highlight the innovative approach of integrating the neutral aspect with these complex features, thereby providing valuable new theoretical insights and practical advancements in the field.

## 2. Preliminaries

This section provides the fundamental definitions and concepts needed for the analysis of fractional dynamic systems. These foundational elements, including fractional calculus and related theorems, will support the methods and results presented later in the study.
Assume a time scale $\mathbb{T}$ containing a subset i.e., closed, nonempty in a real number system $\mathcal{R}$. In $\iota \in \mathbb{T}$, $\rho(\iota) = \sup\{\varepsilon \in \mathbb{T} : \varepsilon < \iota\}$ is defined for backward jump operator. We define $\rho(\iota) = \iota$ when $\mathbb{T}$ has a minimum element $\iota$, so, we obtain $\inf \mathbb{T} = \sup \phi$. Using $\rho$, we categorize $\iota \in \mathbb{T}$ to be left-scattered if $\rho(\iota) < \iota$ and to be left-dense(ld) if $\rho(\iota) = \iota$. Also, backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ can be determined as $\nu(\iota) = \iota - \rho(\iota)$. We obtain a set $\mathbb{T}_k$ as follows:
Let $\mathbb{T}_k = \mathbb{T}\ \{m\}$, when $\mathbb{T}$ contains right-scattered minimum $m$, Else, $\mathbb{T}_k = \mathbb{T}$. Here $\mathbb{T}_k$ is important in defining $\nabla$- derivative.

**Definition 2.1.** [41] *($\nabla$-derivative). Assume $\hbar : \mathbb{T} \to \mathcal{R}$ is a function and $\iota \in \mathbb{T}_k$. At this juncture $\iota$, let's explicitly define $\hbar$'s $\nabla$-derivative as a number. $\hbar_{\nabla}(\iota)$ is characterized by the feature that for all $\epsilon > 0$ there exists neighborhood $\mathcal{U}$ of $\iota$ such that*

$$|\hbar(\rho(\iota)) - \hbar(\varsigma) - \hbar_{\nabla}(\iota)[\rho(\iota) - \varsigma]| \leq \epsilon|\rho(\iota) - \varsigma| \quad for\ all \quad \varsigma \in \mathcal{U}.$$

*Here $\iota \in \mathbb{T}_k$, $\hbar$ is termed $\nabla$-differentiable.*

**Theorem 2.1.** [42] *Let $\hbar : \mathbb{T} \to \mathcal{R}$ is a function and $\iota \in \mathbb{T}_k$. Hence:*

(i) *At the point $\iota$, assuming $\iota$ is a left-scattered point, the function $\hbar$ exhibits continuity and $\nabla$-differentiability alongside*

$$\hbar_\nabla(\iota) = \frac{\hbar(\iota) - \hbar(\rho(\iota))}{t - \rho(t)}.$$

(ii) *Let $\iota$ denote ld, such that $\hbar$ be $\nabla$-differentiable in $\iota$ iff*

$$\lim_{\varsigma \to \iota} \frac{\hbar(\iota) - \hbar(\varsigma)}{\iota - \varsigma}.$$

*there exists a finite number*

$$\hbar_\nabla(\iota) = \lim_{\varsigma \to \iota} \frac{\hbar(\iota) - \hbar(\varsigma)}{\iota - \varsigma}.$$

**Definition 2.2.** [42] *A function $g : \mathscr{I} \to \mathcal{R}$ is ld-continuous if it is continuous at each ld point in $\mathscr{I}$ and there exists r.h.s limit for all rd point in $\mathscr{I}$.*
*At this juncture, $\mathcal{L}(\mathscr{I}, \mathcal{R})$ denotes set of all ld-continuous functions mapping from $\mathscr{I}$ to $\mathcal{R}$. $\mathscr{I} = \mathcal{L}(\mathscr{I}, \mathcal{R})$ is defined as a Banach space whenever,*

$$||g|| = \sup_{\iota \in \mathscr{I}} |g(\iota)|, \quad for \ all \quad g \in \mathcal{L}. \tag{2.1}$$

**Definition 2.3.** [42] *Let $g : \mathscr{I} \to \mathcal{R}$ are an $\nabla$-integrable function. Thus, for all $\iota \in \mathscr{I}$, which results in*

$$\int_0^{\mathbb{T}} g(\varsigma)\nabla\varsigma = \int_0^{\iota} g(\varsigma)\nabla\varsigma + \int_{\iota}^{\mathbb{T}} g(\varsigma)\nabla\varsigma.$$

**Remark 1.** [43] *Assume $\hbar_\gamma : \mathbb{T} \times \mathbb{T} \to \mathcal{R}$ for $\gamma \geq 0$, such that $\hbar_0(\iota, \iota_0) = 1$ and*

$$\hbar_{\gamma+1}(\iota, \iota_0) = \int_{\iota_0}^{\iota} \hbar_\gamma(\varsigma, \iota_0)\nabla\varsigma \quad for \ all \ \ \iota, \iota_0 \in \mathbb{T}. \tag{2.2}$$

*Also, for $\alpha, \gamma > 1$, we get*

$$\int_{\rho(\vartheta)}^{\iota} \hbar_{\alpha-1}(\iota, \rho(\varsigma))\hbar_{\gamma-1}(\varsigma, \rho(\vartheta))\nabla\varsigma = \hbar_{\alpha+\gamma-1}(\iota, \rho(\vartheta)), \tag{2.3}$$

*for all $\iota, \vartheta \in \mathbb{T}$ with $\vartheta \leq \iota$.*

**Definition 2.4.** [43] *Suppose $g \in \mathcal{L}(\mathbb{T}_k, \mathcal{R})$. On $\mathbb{T}$, the function is integrable with respect to the Lebesgue $\nabla$ measure. Fractional $\nabla$- integral for $0 < \gamma < 1$ may be explicitly formulated as*

$$\mathbb{I}_{\iota_0} g(\iota) = \int_{\iota_0}^{\iota} \hbar_{\gamma-1}(\iota, \rho(\varsigma) g(\varsigma))\nabla\varsigma, \quad for \ all \ \ \iota \in \mathcal{U}, \tag{2.4}$$

*where $\mathcal{U}$ is neighbourhood of $\iota$ such that $\mathcal{U} \subset \mathbb{T}$. Hence $\mathbb{I}^0 g(\iota) = g(\iota)$.*

**Remark 2.** $\nabla$*-power function $\hbar_{\gamma-1}(\iota, \rho(\varsigma))$ varies across various time scales $\mathbb{T}$.*
*Suppose $\mathbb{T} = \mathcal{R}$, results in $\rho(\varsigma) = \varsigma$ and $\hbar_{\gamma-1}(\iota, \rho(\varsigma)) = \dfrac{(\iota - \varsigma)^{\gamma-1}}{\Gamma(\gamma)}$. Here, (2.4) is demonstrated as*

$$\mathbb{I}_{\iota_0} g(\iota) = \int_{\iota_0}^{\iota} \frac{(\iota - \varsigma)^{\gamma-1}}{\Gamma(\gamma)} g(\varsigma) d\varsigma.$$

*Let* $\mathbb{T} = \mathscr{Z}$*, results in* $\rho(\varsigma) = \varsigma - 1$ *and* $\hbar_{\gamma-1}(\iota, \rho(\varsigma)) = \dfrac{(\iota - \rho(\varsigma))^{\overline{\gamma-1}}}{\Gamma(\gamma)} = \begin{pmatrix} \iota - \rho(\iota) \\ \gamma \end{pmatrix}$*, here for all* $\gamma \in \mathcal{R}, \iota^{\overline{\gamma}} = \dfrac{\Gamma(\iota+\gamma)}{\Gamma(\iota)}$*. According to equation (2.4), it is feasible to derive*

$$\begin{aligned}
\mathbb{I}^{\gamma}_{0+}g(\iota) &= \int_{\iota_0}^{\iota} \hbar_{\gamma-1}(\iota, \rho(\varsigma)g(\varsigma))\nabla\varsigma,\\
&= \frac{1}{\Gamma(\gamma)}\int_0^{\iota}(\iota-\rho(\varsigma))^{\overline{\gamma-1}}g(\varsigma)\nabla\varsigma,\\
&= \frac{1}{\Gamma(\gamma)}\sum_{\varsigma=0}^{\iota-1}(\iota-(\varsigma-1))^{\overline{\gamma-1}}g(\varsigma).
\end{aligned}$$

*For* $\mathbb{T} = q^{\mathcal{N}_0}$*, we have* $\hbar_{\gamma-1}(\iota, \rho(\varsigma)) = \Gamma_q(\gamma)\frac{q^{\gamma-1}}{q-1}(\iota - q\varsigma)_q^{\gamma-1}$.

**Definition 2.5.** [44] *(R-L fractional* $\nabla$*- derivative). Let an ld-continuous function are* $g : \mathbb{T}_{k^m} \to \mathcal{R}$*. We establish the R-L fractional* $\nabla$*-derivative for any* $\gamma \in \mathcal{R}$ *as follows*

$$D^{\gamma}_{o+}g(\iota) = D^{m}_{o+}\mathbb{I}^{m-\gamma}_{o+}g(\iota), \qquad \iota \in \mathscr{I}, \gamma > 0.$$

**Definition 2.6.** [44] *(Caputo fractional* $\nabla$*-derivative). Let an ld-continuous function are* $g : \mathbb{T}_{k^m} \to \mathcal{R}$ *such that for* $m \in \mathcal{N}_0$*, in* $\mathbb{T}_{k^m}$ *there exists* $\nabla^m g$*. Thus, CF*$\nabla$*D of* $g$ *is*

$${}^{C}D^{\gamma}_{o+}g(\iota) = \int_{\iota_0}^{\iota}\hbar_{m-1}(\iota, \rho(\iota))\nabla^m g(\varsigma))\nabla\varsigma, \quad \text{for all} \quad \iota \in \mathscr{I}, \gamma \geq 0.$$

**Remark 3.** *According to Definition 2.5, we show* ${}^{C}D^{\gamma}_{o+}g(\iota) = \mathbb{I}^{m-\gamma}_{o+}{}^{C}D^{\gamma}_{o+}$*, here* $m = [\gamma]+1$.

**Theorem 2.2.** [45] *Take into account the collection* $\mathfrak{C}(\mathbb{T}, \mathcal{R})$*, encompassing all continuous functions defined on* $\mathbb{T}$*. A subset* $D$ *of* $\mathfrak{C}(\mathbb{T}, \mathcal{R})$ *exhibits relative compactness iff it exhibits both boundedness and equicontinuity.*

**Definition 2.7.** [46] *Let us suppose* $\mathcal{X}$ *and* $\mathcal{Y}$ *are Banach spaces. For* $\mathcal{B} \subseteq \mathcal{X}$*, the set* $\mathcal{G}(\mathcal{B})$ *is asserted to exhibit relative compactness in* $\mathcal{Y}$ *under the mapping* $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$*, assuming* $\mathcal{G}$ *exhibits complete continuity.*

**Proposition 1.** [47] *Suppose* $g \in \mathcal{L}([0, \mathbb{T}]_{\mathbb{T}}, \mathcal{R})$*, for the group* $g$*, let* $\tilde{g}$ *denote an extension over* $[0, \mathbb{T}]$ *such that*

$$\tilde{g} = \begin{cases} g(\iota) & if \quad \iota \in \mathbb{T} \\ g(\varsigma) & if \quad \iota \in (\rho(\varsigma), \varsigma) \notin \mathbb{T}, \end{cases}$$

*we obtain*

$$\int_0^{\mathbb{T}} g(\iota)\nabla\iota \leq \int_0^{\mathbb{T}} \tilde{g}(\iota)d\iota.$$

**Theorem 2.3.** [50] *(Krasnoselskii). Let* $\mathfrak{C}$ *denote a subset that is nonempty, closed, and convex within* $\mathcal{B}$*. Suppose* $\mathscr{F}_1, \mathscr{F}_2 : \mathfrak{C} \to \mathcal{B}$ *such that*
*1.* $F_1$ *is contraction.*
*2.* $\mathscr{F}_2$ *is continuous and* $F_1(\mathfrak{C})$ *is relatively compact.*
*3.* $F_1[\iota] + F_2[\vartheta] \in \mathfrak{C}$ *for all* $\iota, \vartheta \in \mathfrak{C}$,
*there exists* $\bar{\iota} \in \mathfrak{C}$ *such that* $F_1[\bar{\iota}] + \mathscr{F}_2[\bar{\iota}] = \bar{\iota}$.

## 3. Main Results

**Definition 3.8.** *Suppose $g \in \mathcal{L} \cap \mathcal{L}_{\nabla}(\mathscr{I}, \mathcal{R})$ denote the solution of (1.1) iff $g(\iota) \geq 0, \iota \in \mathscr{I}$, and equations and conditions outlined in (1.1) are strictly followed by $g$, here $\mathcal{L}_{\nabla}(\mathscr{I}, \mathcal{R})$ a function that is Lebesgue $\nabla$- integrable from $\mathscr{I} \to \mathcal{R}$.*

**Lemma 3.1.** *Given $1 < \gamma < 2$. Thus, $g \in \mathcal{L} \cap \mathcal{L}_{\nabla}(\mathbb{T}, \mathcal{R})$ is the solution to PBVP (1.1), iff $g$ solves integral equation as follows*

$$g(\iota) = \int_0^{\mathbb{T}} \mathfrak{G}(\iota, \varsigma) \mathcal{L}(\varsigma, g(\varsigma), \mathcal{N} g(\varsigma)), {}^C D^{\gamma} g(\varsigma)) \nabla \varsigma, \tag{3.1}$$

*where $\mathfrak{G}(\iota, \varsigma)$ is a defined green function as*

$$\mathfrak{G}(\iota, \varsigma) = \begin{cases} \dfrac{\mathbb{T} \hbar_{\gamma-1}(\iota, \rho(\iota), \mathcal{N} g(\varsigma))}{\mathbb{T}} + \dfrac{-\iota \hbar_{\gamma-1}(\mathbb{T}, \rho(\varsigma), \mathcal{N} g(\varsigma))}{\mathbb{T}} & if \quad 0 < \varsigma < \iota \\ \dfrac{-\iota \hbar_{\gamma-1}(\mathbb{T}, \rho(\varsigma), \mathcal{N} g(\varsigma))}{\mathbb{T}} & if \quad \iota \leq \varsigma < \mathbb{T}. \end{cases} \tag{3.2}$$

*Proof.* According to Definition 2.6, for $1 < \gamma < 2$
i.e.,

$${}^C D^{\gamma}_{o+} g(\iota) = \int_0^{\iota} \hbar_{m-\gamma}(\iota, \rho(\iota)) \nabla^m (g(\varsigma)) \nabla \varsigma, \qquad \iota \in \mathbb{T}, \gamma \geq 0.$$

we have,

$${}^C D^{\gamma} g(\iota) = \mathbb{I}^{2-\gamma} g^2 \nabla \iota, \qquad \iota \in \mathbb{T}.$$

Based on Lemma 2.7 in [51], it is evident that,

$$\begin{aligned} {}^C \Delta^{\alpha} \mathfrak{u}(t) &= {}^{\Delta} \mathbb{I}^{2-\alpha} u^{\Delta^2}(t), \\ {}^{\Delta} \mathbb{I}^{\alpha C} \Delta^{\alpha} \mathfrak{u}(t) &= {}^{\Delta} \mathbb{I}^{\alpha} \mathbb{I}^{2-\alpha} u^{\Delta^2}(t). \end{aligned}$$

As a result, we derive

$$\begin{aligned} \mathbb{I}^{\gamma C} D^{\gamma} g(\iota) &= \mathbb{I}^{\gamma} \mathbb{I}^{2-\alpha} g^2_{\nabla}(\iota) \\ &= \mathbb{I}^2 g^2_{\nabla}(\iota) \\ &= g(\iota) + k_o + k_1 \iota. \qquad k_o, k_1 \in \mathcal{R}. \end{aligned}$$

Let ${}^C D^{\gamma} g(\iota) = \mathfrak{r}(\iota), \quad \iota \in \mathbb{T}$.
Thus,

$$g(\iota) = \mathbb{I}^{\gamma} \mathfrak{r}(\iota) - k_o - k_1 \iota. \tag{3.3}$$

By applying the boundary condition from equation (1.1), we obtain $k_o = 0$, and
From Definition (2.4),

$$\mathbb{I}^{\gamma}_{\iota_o} g(\iota) = \int_{\iota_0}^{\iota} \hbar_{\gamma-1}(\iota, \rho(\varsigma)) g(\varsigma) \nabla \varsigma, \qquad \iota \in \mathfrak{u}.$$

Drawing from equation (3.3), we can take $k_1 \iota$

$$k_1 = \frac{1}{T} \int_0^T \hbar_{\gamma-1}(T, \beta(\varsigma)) \mathfrak{r}(\varsigma) \nabla \varsigma.$$

Consequently, equation (3.3) leads us to

$$g(\iota) = \int_0^{\iota} \hbar_{\gamma-1}(\iota, \rho(\varsigma)) \mathfrak{r}(\varsigma) \nabla \varsigma - \frac{\iota}{T} \int_0^T \hbar_{\gamma-1}(T, \rho(\varsigma)) \mathfrak{r}(\varsigma) \nabla \varsigma,$$

Thus,

$$g(\iota) = \begin{cases} \dfrac{\mathbb{T}\hbar_{\gamma-1}(\iota, \rho(\iota), \mathcal{N}g(\varsigma))}{\mathbb{T}} + \dfrac{-\iota\hbar_{\gamma-1}(\mathbb{T}, \rho(\varsigma), \mathcal{N}g(\varsigma))}{\mathbb{T}} \\ \dfrac{[-\iota\hbar_{\gamma-1}(\mathbb{T}, \rho(\varsigma), \mathcal{N}g(\varsigma))]}{\mathbb{T}}\mathfrak{r}(\varsigma)\nabla\varsigma. \end{cases}$$

$$g(\iota) = \int_0^T \mathfrak{G}(\iota,\varsigma)\mathfrak{r}(\varsigma)\mathcal{N}g(\varsigma)\nabla\varsigma,$$

i.e.,

$$g(\iota) = \int_0^T \mathfrak{G}(\iota,\varsigma)^C D^\gamma g(\varsigma)\mathcal{N}g(\varsigma)\nabla\varsigma.$$

One can derive (3.1) in conjunction with the equation given by (1.1). During our analysis, we base our findings on the following assumptions provided:

(H1) The function $\mathcal{L} : \mathcal{I} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ exhibits ld-continuous behavior with respect to each of its three variables independently.

(H2) In (H1), there exists '+'ve constants $\mathcal{E} > 0$ and $\mathcal{F}$ that satisfies $0 < \mathcal{F} < 1$ for a function $\mathcal{L}$ such that

$$|\mathcal{L}(\iota, \varsigma_1, \varphi_1) - \mathcal{L}(\iota, \varsigma_2, \varphi_2)| \leq \mathcal{E}|\varsigma_1 - \varsigma_2| + \mathcal{F}|\varphi_1 - \varphi_2|,$$

for $(\iota, \varsigma_i, \varphi_i) \in \mathcal{I} \times \mathcal{R} \times \mathcal{R}$(i=1,2).

(H3) In (H1), there exists $\mathcal{P} \in \mathcal{L}$ and $\mathcal{R} > 0$ and $\mathcal{Q}$ alongside $0 < \mathcal{Q} < 1$ for a function $\mathcal{L}$ such that

$$|\mathcal{L}(\iota, \varsigma, \varphi)| \leq |\mathcal{P}(\iota)| + \mathcal{R}(\varsigma)| + \mathcal{Q}|(\varphi)|,$$

for $(\iota, \varsigma, \varphi) \in \mathcal{I} \times \mathcal{R} \times \mathcal{R}$.

(H4) On the interval $[0, \mathbb{T}]$, consider $g(\cdot, \cdot)$ as the Green function, characterized by being bounded and piecewise continuous. also, function $\mathfrak{G}$ conforms to

$$\int_0^\iota |\mathfrak{G}(\iota,\varsigma)|\nabla\varsigma \leq k \quad and \quad \int_\iota^{\mathbb{T}} |\mathfrak{G}(\iota,\varsigma)|\nabla\varsigma \leq m,$$

here $k$ and $m$ are '+'ve real constants and $0 < \iota < \mathbb{T}$ and,

$$\int_0^{\mathbb{T}} \mathfrak{G}(\iota,\varsigma)\nabla\varsigma = A \in \mathcal{R}.$$

(H5) Let $\pi : \mathcal{B} \to \mathcal{B}$ be defined as;

$$\begin{aligned}(\pi g)(\iota) = &\int_0^{\mathbb{T}} \mathfrak{G}(\iota,\varsigma)\mathcal{L}(\varsigma, g(\varsigma), \mathcal{N}(g(\varsigma)), {}^C D^\gamma g(\varsigma))\nabla\varsigma \\ &+ \int_0^{\mathbb{T}} \varphi_A(\iota, g(\varsigma))\big[\mathcal{B}(\varsigma)u(\varsigma) + \mathcal{L}(\varsigma, g(\varsigma), \mathcal{N}(g(\varsigma), {}^C D^\gamma g(\varsigma))\nabla\varsigma)\big].\end{aligned}$$

The validation of the resultant existence hinges on the application of Theorem 2.3. Prior to achieving the outcome, it is essential to obtain the following results.

Analyze a specific subset of $\mathcal{L}$, precisely identified as

$$\mathfrak{M}_\alpha = \{g : \mathcal{I} \to \mathcal{R} : g(\iota) \in \mathcal{L}, ||g|| \leq \alpha, \alpha > 0\}. \tag{3.4}$$

Here, $\mathfrak{M}_\alpha$ forms a Banach subspace embedded within the space $\mathcal{L}$. Consequently, a precise definition can be formulated for $\mathscr{F}_1 : \mathfrak{M}_\alpha \to \mathcal{L}$ and $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ by

$$\mathscr{F}_1[g](\iota) = \int_0^\iota \mathfrak{G}(\iota,\varsigma)\mathfrak{L}(\varsigma, g(\varsigma), {}^C D^\gamma g_1(\varsigma))\nabla\varsigma, \tag{3.5}$$

and

$$\mathscr{F}_2[g](\iota) = \int_0^\iota \mathfrak{G}(\iota,\varsigma)\mathfrak{L}(\varsigma, g(\varsigma), {}^C D^\gamma g_1(\varsigma))\nabla\varsigma, \tag{3.6}$$

respectively. □

**Lemma 3.2.** *Assume that hypotheses (H1), (H2), and (H4) are valid. If* $\dfrac{\mathscr{E}K}{1-\mathscr{F}} < 1$, *then* $\mathscr{F}_1 : \mathfrak{M}_\alpha \to \mathcal{L}$ *expression provided in equation (3.5) demonstrates a tendency to contract.*

*Proof.* Suppose ${}^C D^\gamma g_{\mathfrak{i}}(\iota) = \mathfrak{r}_{\mathfrak{i}}(\iota)$, $\iota \in \mathbb{T}, \mathfrak{i} = 1, 2$, here $g_1, g_2 \in \mathfrak{M}_\alpha$. From (3.5), for $\iota \in \mathbb{T}$,

$$\begin{aligned} |\mathscr{F}_1[g_1](\iota) - \mathscr{F}_1[g_2](\iota)| &= \Big|\mathcal{N}g(\varsigma)\Big(\int_0^\iota \mathfrak{G}(\iota,\varsigma)\mathfrak{L}(\varsigma, g_1(\varsigma), {}^C D^\gamma g_1(\varsigma))\nabla\varsigma \\ &\quad - \int_0^\iota \mathfrak{G}(\iota,\varsigma)\mathfrak{L}(\varsigma, g_2(\varsigma), {}^C D^\gamma g_2(\varsigma))\nabla\varsigma\Big)\Big|, \\ &= \Big|\mathcal{N}g(\varsigma)\Big(\int_0^\iota \mathfrak{G}(\iota,\varsigma)\big(\mathfrak{L}(\varsigma, g_1(\varsigma), {}^C D^\gamma g_1(\varsigma))\big) \\ &\quad - \big(\mathfrak{L}(\varsigma, g_2(\varsigma), {}^C D^\gamma g_2(\varsigma))\big)\nabla\varsigma\Big)\Big|, \\ &\le \int_0^\iota \mathcal{N}g(\varsigma)\big|\mathfrak{G}(\iota,\varsigma)\big|\big|\mathfrak{L}(\varsigma, g_1(\varsigma), \mathfrak{r}_1(\varsigma))\big| \\ &\quad - \big|\mathfrak{L}(\varsigma, g_2(\varsigma), \mathfrak{r}_2(\varsigma))\big|\nabla\varsigma, \end{aligned} \tag{3.7}$$

where $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathfrak{M}_\alpha$. But in view of (1.1), for $\varsigma \in \mathbb{T}$.

$$|\mathfrak{r}_1(\varsigma) - \mathfrak{r}_2(\varsigma)| = \mathcal{N}g(\varsigma)\big|\mathfrak{L}(\varsigma, g_1(\varsigma), \mathfrak{r}_1(\varsigma)) - \mathfrak{L}(\varsigma, g_2(\varsigma), \mathfrak{r}_2(\varsigma))\big|.$$

Using the assumption (H2) we get,

$$\le \mathscr{E}|g_1(\varsigma) - g_2(\varsigma)| + \mathscr{F}|\mathfrak{r}_1(\varsigma) - \mathfrak{r}_2(\varsigma)|.$$

This gives

$$|\mathfrak{r}_1(\varsigma) - \mathfrak{r}_2(\varsigma)| \le \frac{\mathscr{E}\mathcal{N}g(\varsigma)}{1-\mathscr{F}}|g_1(\varsigma) - g_2(\varsigma)|. \tag{3.8}$$

Substituting (3.8) in (3.7), we get

$$\big|\big|\mathscr{F}_1[g_1] - \mathscr{F}_1[g_2]\big|\big| \le \frac{\mathscr{E}}{1-\mathscr{F}}\int_0^\iota \mathcal{N}g(\varsigma)\big|\mathfrak{G}(\iota,\varsigma)\big|\big|\big|g_1 - g_2\big|\big|\nabla\varsigma.$$

Using the assumption (H4) we get,

$$\le \frac{\mathscr{E}K}{1-\mathscr{F}}\big|\big|g_1 - g_2\big|\big|.$$

Since $\dfrac{\mathscr{E}K}{1-\mathscr{F}} < 1$, $\mathscr{F}_1 : \mathfrak{M}_\alpha \to \mathcal{L}$ is contractive. □

**Theorem 3.4.** *Given that conditions (H1)-(H4) are satisfied. In (3.6),* $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ *are continuous and* $\mathscr{F}_2(\mathfrak{M}_\alpha)$ *are relatively compact.*

*Proof.* Assume ${}^C D^\gamma g_{\mathcal{N}}(\iota) = \mathfrak{r}_{\mathcal{N}}(\iota)$, $\mathcal{N} \in \mathcal{N}$ and ${}^C D^\gamma g(\iota) = \mathfrak{r}(\iota)$, $\iota \in \mathbb{T}$. Assume $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$, is detrmined in (3.6).

**Step 1:** $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ exhibits continuity.

In the space $\mathfrak{M}\alpha$, let $g\mathcal{N}_{\mathcal{N}\in\mathcal{N}}$ be a sequence converging to $g$. Consider $\iota \in [0, \mathbb{T}]$,

$$\begin{aligned}
\big|\mathscr{F}_2[g_n](\iota) - \mathscr{F}_2[g](\iota)\big| &= \bigg|\mathcal{N}g(\varsigma)\bigg(\int_\iota^T \mathfrak{G}(\iota,\varsigma)\mathfrak{L}(\varsigma, g_{\mathcal{N}}(\varsigma), {}^C D^\gamma g_{\mathcal{N}}(\varsigma))\nabla\varsigma \\
&\quad - \int_\iota^T \mathfrak{G}(\iota,\varsigma)\mathfrak{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))\nabla\varsigma\bigg)\bigg|, \\
&\leq \bigg|\int_\iota^T \mathfrak{G}(\iota,\varsigma)\mathcal{N}g(\varsigma)\bigg[\mathfrak{L}(\varsigma, g_{\mathcal{N}}(\varsigma), {}^C D^\gamma g_{\mathcal{N}}(\varsigma)) \\
&\quad - \mathfrak{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))\bigg]\nabla\varsigma\bigg|, \\
&\leq \bigg|\int_\iota^T \mathfrak{G}(\iota,\varsigma)\bigg|\bigg|\mathcal{N}g(\varsigma)\bigg|\bigg|\bigg[\mathfrak{L}(\varsigma, g_{\mathcal{N}}(\varsigma), {}^C D^\gamma g_{\mathcal{N}}(\varsigma)) \\
&\quad - \mathfrak{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))\bigg]\bigg|\nabla\varsigma.
\end{aligned} \tag{3.9}$$

here $\mathfrak{r}_n, \mathfrak{r} \in \mathfrak{M}_\alpha$. However, considering (1.1), for $\varsigma \in \mathbb{T}$.

$$|\mathfrak{r}_n(\varsigma) - \mathfrak{r}(\varsigma)| = \mathcal{N}g(\varsigma)\big|\mathfrak{L}(\varsigma, g_n(\varsigma), \mathfrak{r}_1(\varsigma)) - \mathfrak{L}(\varsigma, g(\varsigma), \mathfrak{r}(\varsigma))\big|.$$

Using the assumption (H2) we get,

$$\leq \mathscr{E}|g_n(\varsigma) - g(\varsigma)| + \mathscr{F}|\mathfrak{r}_n(\varsigma) - \mathfrak{r}(\varsigma)|.$$

This gives

$$|\mathfrak{r}_n(\varsigma) - \mathfrak{r}(\varsigma)| \leq \frac{\mathscr{E}\mathcal{N}g(\varsigma)}{1-\mathscr{F}}|g_n(\varsigma) - g(\varsigma)|. \tag{3.10}$$

Substituting (3.10) in (3.9), we get

$$\big|\big|\mathscr{F}_2[g_n] - \mathscr{F}_2[g]\big|\big| \leq \frac{\mathscr{E}}{1-\mathscr{F}}\int_0^\iota \big|\mathcal{N}g(\varsigma)\mathfrak{G}(\iota,\varsigma)\big|\big|\big|g_n - g\big|\big|\nabla\varsigma.$$

Using the assumption (H4) we get,

$$\leq \frac{\mathscr{E}K}{1-\mathscr{F}}\big|\big|g_n - g\big|\big|,$$

i.e.,

$$\big|\big|\mathscr{F}_2[g_n] - \mathscr{F}_2[g]\big|\big| \leq \frac{\mathscr{E}K}{1-\mathscr{F}}\big|\mathcal{N}g(\varsigma)\big|\big|\big|g_n - g\big|\big|.$$

Consequence shows that r.h.s of above inequality $\to 0$ as $g_n$ tends towards $g$. Thus $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ is continuous.

**Step 2:** $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ exhibits boundedness.

By employing equation (3.6), it is possible to represent $\iota \in \mathbb{T}$ as

$$\begin{aligned}
\big|\mathscr{F}_2[g](\iota)\big| &\leq \int_\iota^{\mathbb{T}} \big|\mathfrak{G}(\iota,\varsigma)\big|\big|\mathcal{N}g(\varsigma)\big|\big|\mathfrak{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))\big|\nabla\varsigma, \\
&= \int_\iota^{\mathbb{T}} \big|\mathfrak{G}(\iota,\varsigma)\big|\big|\mathcal{N}g(\varsigma)\big|\big|\mathfrak{L}(\varsigma, g(\varsigma), \mathfrak{r}(\varsigma))\big|\nabla\varsigma, \\
&= \int_\iota^{\mathbb{T}} \big|g(\iota,\varsigma)\big|\big|\mathcal{N}g(\varsigma)\big|\big|\mathfrak{r}(\varsigma))\big|\nabla\varsigma,
\end{aligned} \tag{3.11}$$

here $\mathfrak{r} \in \mathfrak{M}$. By (1.1), for $\varsigma \in \mathbb{T}$, we have,

$$\big|\mathfrak{r}(\varsigma))\big| = \big|\mathfrak{L}(\varsigma, g(\varsigma), \mathfrak{r}(\varsigma))\big|.$$

Applying the assumption (H3) results in,

$$\leq |\mathcal{P}(\varsigma)| + \mathcal{R}|g(\varsigma)| + \mathcal{Q}|\mathfrak{r}(\varsigma)|.$$

Thus

$$|\mathfrak{r}(\varsigma)| \leq \frac{|\mathcal{P}(\varsigma)| + \mathcal{R}|g(\varsigma)|}{1-\mathcal{Q}}\mathcal{N}g(\varsigma). \tag{3.12}$$

Substituting equation (3.12) into equation (3.11), and subsequently applying norm from (2.1), leads us to derive

$$||\mathscr{F}_2[g]|| \leq \int_{\iota}^{\mathbb{T}} \big|\mathfrak{G}(\iota,\varsigma)\big|\big|\mathcal{N}g(\varsigma)\big|\frac{||\mathcal{P}|| + \mathcal{R}||g||}{1-\mathcal{Q}}\nabla\varsigma.$$

Using the assumption (H4) we get,

$$\leq \frac{[||\mathcal{P}|| + \mathcal{R}\alpha]}{1-\mathcal{Q}}\big|\mathcal{N}g(\varsigma)\big|,$$

That is,

$$||\mathscr{F}_2[g]|| \leq \frac{[||\mathcal{P}|| + \mathcal{R}\alpha]}{1-\mathcal{Q}}\big|\mathcal{N}g(\varsigma)\big|.$$

Hence, $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ is bounded.
**Step 3:** $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ is equicontinuous.
Let $\iota_1, \iota_2 \in \mathbb{T}$ be such that $\iota_1 < \iota_2$. Then for $g \in \mathfrak{M}_\alpha$, we have

$$\begin{aligned}
\big|\mathscr{F}_2[g](\iota_1) - \mathscr{F}_2[g](\iota_2)\big| &= \bigg|\mathcal{N}g(\varsigma)\bigg(\int_{\iota_1}^{T} \mathfrak{G}(\iota_1,\varsigma)\mathfrak{L}(\varsigma, g(\varsigma), {}^{C}D^{\gamma}g(\varsigma))\nabla\varsigma \\
&\quad - \int_{\iota_2}^{T} \mathfrak{G}(\iota_2,\varsigma)\mathfrak{L}(\varsigma, g(\varsigma), {}^{C}D^{\gamma}g(\varsigma))\nabla\varsigma\bigg)\bigg|, \\
&= \bigg|\int_{\iota_1}^{T} \mathfrak{G}(\iota_1,\varsigma)\nabla\varsigma - \int_{\iota_2}^{T} \mathfrak{G}(\iota_2,\varsigma)\nabla\varsigma\bigg|\bigg|\mathcal{N}g(\varsigma)\bigg| \\
&\quad \bigg|\mathfrak{L}(\varsigma, g(\varsigma), {}^{C}D^{\gamma}g(\varsigma))\bigg|, \\
&= \bigg|\int_{\iota_1}^{T} \mathfrak{G}(\iota_1,\varsigma)\nabla\varsigma - \int_{\iota_2}^{T} \mathfrak{G}(\iota_2,\varsigma)\nabla\varsigma\bigg|\bigg|\mathcal{N}g(\varsigma)\bigg|\big|\mathfrak{r}(\varsigma)\big|.
\end{aligned}$$

Substituting (3.12) in the above equation,

$$\leq \bigg|\int_{\iota_1}^{T} \mathfrak{G}(\iota_1,\varsigma)\nabla\varsigma - \int_{\iota_2}^{T} \mathfrak{G}(\iota_2,\varsigma)\nabla\varsigma\bigg|\bigg|\mathcal{N}g(\varsigma)\bigg|\bigg(\frac{|\mathcal{P}(\varsigma)| + \mathcal{R}|g(\varsigma)|}{1-\mathcal{Q}}\bigg).$$

i.e.,

$$\begin{aligned}
\big|\mathscr{F}_2[g](\iota_1) - \mathscr{F}_2[g](\iota_2)\big| &\leq \frac{[||\mathcal{P}|| + \mathcal{R}\alpha]}{1-\mathcal{Q}}\bigg|\mathcal{N}g(\varsigma)\bigg| \\
&\qquad \bigg|\int_{\iota_1}^{T} \mathfrak{G}(\iota_1,\varsigma)\nabla\varsigma - \int_{\iota_2}^{T} \mathfrak{G}(\iota_2,\varsigma)\nabla\varsigma\bigg|.
\end{aligned} \tag{3.13}$$

Based on (3.2) along with Remark 1, for $\iota_1, \iota_2 \in \mathbb{T}$ results in,

$$\int_{\iota_1}^{\mathbb{T}} \iota_1 \hbar(\mathbb{T}, \rho(\varsigma))\nabla\varsigma = \int_{\iota_1}^{\mathbb{T}} -\iota_1 \hbar_{\gamma-1}(\mathbb{T}, \rho(\varsigma))\nabla\varsigma, \\ = \frac{-\iota_1 \hbar_\gamma(\mathbb{T}, \rho(\varsigma))}{\mathbb{T}}, \tag{3.14}$$

and

$$\int_{\iota_2}^{\mathbb{T}} \iota_2 \hbar(\mathbb{T}, \rho(\varsigma))\nabla\varsigma = \frac{-\iota_2 \hbar_\gamma(\mathbb{T}, \rho(\varsigma))}{\mathbb{T}}. \tag{3.15}$$

Using (3.14) and (3.15) in (3.13), we get

$$\big|\mathscr{F}_2[g](\iota_1) - \mathscr{F}_2[g](\iota_2)\big| \leq \frac{[||\mathcal{P}|| + \mathcal{R}\alpha]}{1-\mathcal{Q}}\Big|\mathcal{N}g(\varsigma)\Big|\frac{\hbar_\gamma(\mathbb{T}, \rho(\varsigma))}{\mathbb{T}}\big(\iota_1 - \iota_2\big). \tag{3.16}$$

As $\iota_1$ approaches $\iota_2$, r.h.s of (3.16) converges to zero . Thus, we obtain $\big|\big|\mathscr{F}_2[g](\iota_1) - \mathscr{F}_2[g](\iota_2)\big|\big| \to 0$. Thus $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$ exhibits equicontinuity. Now, since $\mathscr{F}_2(\mathfrak{M}_\alpha)$ exhibits boundedness and equicontinuous, by Theorem 2.3, it says that $\mathscr{F}_2(\mathfrak{M}_\alpha)$ exhibits relative compactness. ☐

**Theorem 3.5.** [43, 40] *Assume the assumption (H1)-(H4) fulfill the conditions. Suppose* $\mathfrak{M}_\alpha = \{g : \mathbb{T} \to \mathcal{R} : g(\iota) \in \mathcal{L}, ||g|| \leq \alpha\}$ *such that* $\dfrac{(m+k)||\mathcal{P}||}{1-\mathcal{Q}-(m+k)\mathcal{R}} \leq \alpha$. *Hence, equation (1.1) under consideration has a solution belonging to* $\mathfrak{M}_\alpha$.

*Proof.* Using Lemma 3.2, it can be shown that $\mathscr{F}_1 : \mathfrak{M}_\alpha \to \mathcal{L}$, in (3.5), demonstrates contractive behavior. Based on Theorem 3.4, $\mathscr{F}_2 : \mathfrak{M}_\alpha \to \mathcal{L}$, determined in equation (3.6), exhibits continuity and $\mathscr{F}_2(\mathfrak{M}_\alpha)$ are relatively compact. Suppose ${}^C D^\gamma g(\iota) = \mathfrak{r}(\iota) = {}^C D^\gamma \hbar(\iota) = q(\iota)$ for $\iota \in \mathbb{T}$. For $g, \hbar \in \mathfrak{M}_\alpha$, we write

$$|\mathscr{F}_1[g](\iota) + \mathscr{F}_2[\hbar](\iota)| \leq \mathcal{N}g(\varsigma)\bigg(\int_0^\iota |\mathfrak{G}(\iota,\varsigma)||\mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))|\nabla\varsigma \\ + \int_\iota^{\mathbb{T}} |\mathfrak{G}(\iota,\varsigma)||\mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma \hbar(\varsigma))|\nabla\varsigma\bigg), \\ \leq \int_0^\iota \mathcal{N}g(\varsigma)\bigg(|\mathfrak{G}(\iota,\varsigma)||\mathfrak{r}(\varsigma)|\nabla\varsigma + \int_\iota^{\mathbb{T}} |\mathfrak{G}(\iota,\varsigma)||q(\varsigma)|\nabla\varsigma\bigg). \tag{3.17}$$

By utilizing equation (1.1) for all $\varsigma \in \mathbb{T}$, it is possible to derive

$$|\mathfrak{r}(\varsigma)| = |\mathcal{L}(\varsigma, g(\varsigma), \mathfrak{r}(\varsigma))|.$$

From the assumption (H3),

$$|\mathfrak{r}(\varsigma)| \leq |\mathcal{P}(\varsigma)| + \mathfrak{r}|g(\varsigma)| + \mathcal{Q}|\mathfrak{r}(\varsigma)| \\ \leq \frac{|\mathcal{P}(\varsigma)| + \mathfrak{r}|g(\varsigma)|}{1-\mathcal{Q}}, \tag{3.18}$$

and

$$|q(\varsigma)| = |\mathcal{L}(\varsigma, \hbar(\varsigma), q(\varsigma))|.$$

Based on the assumption (H3),

$$|\mathfrak{r}(\varsigma)| \leq |\mathcal{P}(\varsigma)| + \mathcal{R}|\hbar(\varsigma)| + \mathcal{Q}|q(\varsigma)| \\ \leq \frac{|\mathcal{P}(\varsigma)| + \mathcal{R}|\hbar(\varsigma)|}{1-\mathcal{Q}}, \tag{3.19}$$

Substituting (3.18) and (3.19) in (3.17), one can obtain

$$||\mathscr{F}_1[g] + \mathscr{F}_2[h]|| \leq \int_0^\iota |\mathfrak{G}(\iota,\varsigma)|\Big(\frac{||\mathcal{P}|| + \mathcal{R}||g||}{1-\mathcal{Q}}\Big)|\mathcal{N}g(\varsigma)|\nabla\varsigma$$
$$+ \int_0^\iota |\mathfrak{G}(\iota,\varsigma)|\Big(\frac{||\mathcal{P}|| + \mathcal{R}||h||}{1-\mathcal{Q}}\Big)|\mathcal{N}g(\varsigma)|\nabla\varsigma.$$

Using the assumption (H4), we obtain,

$$||\mathscr{F}_1[g] + \mathscr{F}_2[h]|| \leq m\frac{||\mathcal{P}|| + \mathcal{R}\alpha}{1-\mathcal{Q}}|\mathcal{N}g(\varsigma)| + k\frac{||\mathcal{P}|| + \mathcal{R}\alpha}{1-\mathcal{Q}}|\mathcal{N}g(\varsigma)|,$$
$$= \frac{(m+k)(||\mathcal{P}|| + \mathcal{R}\alpha)}{1-\mathcal{Q}}(|\mathcal{N}g(\varsigma)|) \leq \alpha. \tag{3.20}$$

Hence, $\mathscr{F}_1[g] + \mathscr{F}_2[h] \in \mathfrak{M}_\alpha$ for $g, h \in \mathfrak{M}_\alpha$. Thus the criteria outlined in Theorem 2.3 have been entirely satisfied. Thus, there exists $g \in \mathfrak{M}_\alpha$ such that $g = \mathscr{F}_1[g] + \mathscr{F}_2[h]$ demonstrates the solution to PBVP (1.1). □

## 4. Controllability

Examine PBVP concerning the dynamical system featuring a neutral integro-differential equation over time scale incorporating a control component,

$$\left.\begin{aligned} &{}^C D^\gamma[h(\iota) - g(\iota, h))] = \mathcal{L}\big(\iota, h(\iota), \mathcal{N}(h(\iota)), {}^C D^\gamma h(\iota)\big) + \mathcal{B}u(\iota), \iota \in \mathbb{T} \\ &h(0) = h(\mathbb{T}) = 0, \qquad \mathbb{T} \in \mathcal{R}. \end{aligned}\right\} \tag{4.1}$$

Here,

$$\mathcal{N}(h(\iota)) = \int_0^\iota (\iota, s, h(\iota))\nabla s,$$

here $\iota \in \mathbb{T}, \mathbb{T} > 0$ and $\mathcal{L} : \mathbb{T} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ signifies ld-continuous function. ${}^C D^\gamma$ is Caputo fractional derivative of order $0 < \gamma < 1$ and $[0, \mathbb{T}] \in \mathbb{T}$. Also, $g(\iota, h)$ and $\mathcal{L}(\iota, h)$ demonstrates continuity in $h$ and i respectively. $\iota \in \mathbb{T}$ such that $\iota \in \mathbb{T} : 0 \leq \iota \leq \mathbb{T}, \quad \mathbb{T} \in \mathcal{R}$.

**Theorem 4.6.** *Suppose (H1)-(H5) are met and $\mathscr{E}_{\mathscr{F}} = \mathcal{L} + \epsilon\mathbb{T}^\gamma < 1$. Then equation (4.1) is controllable on $\mathbb{I}$.*

*Proof.* For $\beta = \dfrac{k_1}{1-k_1}$, we consider,

$$\mathcal{B} = \{h \in \mathfrak{C}(\mathbb{T}, \mathcal{R}) : ||h||_c\nabla \leq \beta\} \subseteq \mathfrak{C}(\mathbb{T}, \mathcal{R}).$$

Define $\pi : \mathcal{B} \to \mathcal{B}$, results in

$$(\pi g)(\iota) = \int_0^{\mathbb{T}} \mathfrak{G}(\iota,\varsigma)\mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))\nabla\varsigma$$
$$+ \int_0^{\mathbb{T}} \varphi_A(\iota, g(\varsigma))\big[\mathcal{B}(\varsigma)u(\varsigma) + \mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))\mathcal{N}g(\varsigma)\nabla\varsigma\big].$$

where $\pi : \mathcal{B} \to \mathcal{B}$ be well defined. For $\alpha \in \mathbb{T}$ and $g \in \mathcal{B}$ results in,

$$|(\pi g)(\iota)| = \Big|\int_0^{\mathbb{T}} \mathfrak{G}(\iota,\varsigma)\mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma)))\nabla\varsigma\Big|$$
$$+ \Big|\int_0^{\mathbb{T}} \varphi_A(\iota, g(\varsigma))\big[\mathcal{B}(\varsigma)u(\varsigma) + \mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))\mathcal{N}g(\varsigma)\nabla\varsigma\big]\Big|,$$

$$
\begin{aligned}
&\leq \left| \int_0^{\mathbb{T}} \mathfrak{G}(\iota,\varsigma)\mathcal{L}(\varsigma,g(\varsigma),\mathfrak{r}_1(\varsigma)))\nabla\varsigma \right| \\
&\quad + \left| \int_0^{\mathbb{T}} \varphi_A(\iota,g(\varsigma))\big[\mathcal{B}(\varsigma)u(\varsigma)+\mathcal{L}(\varsigma,g(\varsigma),\mathfrak{r}_2(\varsigma))\mathcal{N}g(\varsigma)\nabla\varsigma\big] \right|, \\
&\leq k_1 + k_2\beta, \\
&\leq \frac{k_1}{1-k_2}, \\
&\leq \beta.
\end{aligned}
$$

Hence, $\pi : \mathcal{B} \to \mathcal{B}$ is well defined. Additionally, we demonstrate that $\pi : \mathcal{B} \to \mathcal{B}$ exhibits contractivity and $\iota \in \mathbb{T}$,

$$
\begin{aligned}
|(\pi g)(\iota) - (\pi h)(\iota)| \leq &\Bigg[\left| \int_0^{\mathbb{T}} \mathfrak{G}(\iota,\varsigma)\mathcal{L}(\varsigma,g(\varsigma),\mathfrak{r}_1(\varsigma)))\nabla\varsigma \right| \\
&+ \left| \int_0^{\mathbb{T}} \varphi_A(\iota,g(\varsigma))\big[\mathcal{B}(\varsigma)u(\varsigma)+\mathcal{L}(\varsigma,g(\varsigma),\mathfrak{r}_2(\varsigma))\mathcal{N}g(\varsigma)\nabla\varsigma\big] \right|\Bigg] \\
&- \Bigg[\left| \int_0^{\mathbb{T}} \mathfrak{G}(\iota,\varsigma)\mathcal{L}(\varsigma,h(\varsigma),\mathfrak{r}_1(\varsigma)))\nabla\varsigma \right| \\
&+ \left| \int_0^{\mathbb{T}} \varphi_A(\iota,h(\varsigma))\big[\mathcal{B}(\varsigma)u(\varsigma)+\mathcal{L}(\varsigma,g(\varsigma),\mathfrak{r}_2(\varsigma))\mathcal{N}g(\varsigma)\nabla\varsigma\big] \right|\Bigg].
\end{aligned}
$$

Employing the assumption (H2) yields,

$$
\begin{aligned}
|(\pi g)(\iota) - (\pi h)(\iota)| &\leq \mathscr{E}|g(\varsigma) - h(\varsigma)| + \mathscr{F}|\mathfrak{r}_1(\varsigma) - \mathfrak{r}_2(\varsigma)|, \\
|(\pi g)(\iota) - (\pi h)(\iota)| &\leq \frac{\mathscr{E}}{1-\mathscr{F}}|g(\varsigma) - h(\varsigma)|.
\end{aligned}
$$

Hence,

$$
|(\pi g)(\iota) - (\pi h)(\iota)| \leq \mathcal{L}||g - h||,
$$

which implies

$$
\mathscr{E}_{\mathscr{F}} = \mathcal{L} + \epsilon\mathbb{T}^{\gamma} < 1.
$$

Thus, Equation (4.1) exhibits controllability over $\mathbb{I}$. □

## 5. Stability Results

**Definition 5.9.** *If PBVP (1.1) holds Hyers-Ulam stability (HUS), then $\mathcal{N}_{\mathcal{L}} > 0$ such that for all $\epsilon > 0$ and for $g \in \mathfrak{M}_\alpha$ that meets the requirements*

$$
|{}^{C}D^{\gamma}g(\iota) - \mathcal{L}(\varsigma, g(\varsigma), {}^{C}D^{\gamma}g(\varsigma))| \leq \epsilon \quad for\ \ all \quad \iota \in \mathbb{T}_k \tag{5.1}
$$

*there exists a solution of $h \in \mathfrak{M}_\alpha$ for PBVP (1.1) such that*

$$
|g(\iota) - h(\iota)| \leq \mathcal{N}_{\mathcal{L}}\epsilon \quad for\ \ all \quad \iota \in \mathbb{T}.
$$

*Any positive value of $\mathcal{N}_{\mathcal{L}} > 0$ signifies constant for HUS.*

**Definition 5.10.** *Suppose PBVP (1.1) has '+ 've continuous function $\mathcal{H}_{\mathcal{L}}$ which satisfies the condition $\mathcal{H}_{\mathcal{L}}(0) = 0$ , it consequently demonstrates generalized Hyers-Ulam stability (GHUS), such that for every $g \in \mathfrak{M}_\alpha$ meeting the condition (4.1), there exists solution $h \in \mathfrak{M}_\alpha$ of (1.1) such that*

$$
|g(\iota) - h(\iota)| \leq \mathcal{H}_{\mathcal{L}}(\epsilon) \quad for\ \ all \quad \iota \in \mathbb{T}.
$$

**Definition 5.11.** *Consider $K$ as a set of positive, non-decreasing ld-continuous real-valued function defined on $\mathbb{T}$. The Hyers-Ulam-Rassias stability (HURS) of PBVP (1.1) is classified as type $K$ if, for every instance $\varphi \in K$ and $\epsilon > 0$, there exists $\mathcal{N}_{\mathcal{L},\varphi} > 0$ such that for each $g \in \mathfrak{M}_\alpha$ which satisfies*

$$|{}^C D^\gamma g(\iota) - \mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma))| \leq \epsilon\varphi(\iota) \quad \text{for all} \quad \iota \in \mathbb{T}_k \tag{5.2}$$

*there exists a solution of $h \in \mathfrak{M}_\alpha$ of (1.1) such that*

$$|g(\iota) - h(\iota)| \leq \epsilon \mathcal{N}_{\mathcal{L},\varphi}\varphi(\iota) \quad \text{for all} \quad \iota \in \mathbb{T}.$$

*Here $\mathcal{N}_{\mathcal{L},\varphi} > 0$ is known as HURS constant.*

**Remark 4.** *A function $g \in \mathfrak{C}^1_{rd}(\mathbb{T}, \mathcal{R})$ are a solution of (5.2) if there exists a function $\mathcal{H} \in \mathfrak{C}^1_{rd}(\mathbb{T}, \mathcal{R})$ possessing the following characteristics:*

- $|\mathcal{H}(\iota)| \leq \epsilon\varphi(\iota) \quad for \;\; all \quad \iota \in \mathbb{T}$ .
- ${}^C D^\gamma g(\iota) = \mathcal{L}(\varsigma, g(\varsigma), {}^C D^\gamma g(\varsigma)) + \mathcal{H}(\iota) \quad for \;\; all \quad \mathbb{T}_k.$

**Theorem 5.7.** *Assume that (H1)-(H5) hold true for $\dfrac{A\mathscr{E}}{1-\mathscr{F}} < 1$. Then, (1.1) contains HURS type of $K$.*

*Proof.* Let $g \in \mathfrak{C}^1_{rd}(\mathbb{T}, \mathcal{R})$ meets the requirements outlined in (5.2). Thus, as noted in Remark 5.4 there exists for $\mathcal{H} \in \mathfrak{C}^1_{rd}(\mathbb{T}, \mathcal{R})$ satisfying $|\mathcal{H}(\iota)| \leq \epsilon\varphi(\iota)$ such that

$${}^C D^\gamma[g(\iota) - h(\iota, h)] = \mathcal{L}[\iota, g(\iota, \mathcal{N}(g(\iota))), {}^C D^\gamma g(\iota)] + \mathcal{H}(\iota). \quad for \;\; all \quad \iota \in \mathbb{T}_k$$

For ${}^C D^\gamma g(\iota) = q(\iota), \iota \in \mathbb{T}_k$ along $q \in \mathfrak{M}_\alpha$, then according to Lemma 3.1, it is asserted that

$$g(\iota) = \int_0^{\mathbb{T}} \mathfrak{G}(\iota, \varsigma)\mathcal{L}\big(\varsigma, g(\varsigma), \mathcal{N}g(\varsigma), q(\varsigma)) + \mathcal{H}(\iota)\big)\nabla\varsigma. \tag{5.3}$$

For $\varphi \in K$, then according to Remark 5.4, it follows that

$$\Big|g(\iota) - \int_0^{\mathbb{T}} \mathfrak{G}(\iota, \varsigma)\mathcal{L}\big(\varsigma, g(\varsigma), \mathcal{N}g(\varsigma)), q(\varsigma)\big)\nabla\varsigma\Big| \leq A\epsilon\varphi(\iota) \quad \iota \in \mathbb{T}_k. \tag{5.4}$$

Let $h \in \mathfrak{M}_\alpha$ is a solution of (1.1). Then for $\iota \in \mathbb{T}$, we have

$${}^C D^\gamma[h(\iota) - g(\iota, h)] = \mathcal{L}[\iota, h(\iota, \mathcal{N}h(\iota)), {}^C D^\gamma h(\iota)]. \quad for \;\; all \quad \iota \in \mathbb{T}_k \tag{5.5}$$

For ${}^C D^\gamma h(\iota) = \mathfrak{r}(\iota), \iota \in \mathbb{T}_k$ with $\mathfrak{r} \in \mathfrak{M}_\alpha$, applying Lemma 3.1, one can obtain

$$h(\iota) = \int_0^{\mathbb{T}} g(\iota, \varsigma)\mathcal{L}\big(\varsigma, h(\varsigma), \mathcal{N}(h(\varsigma), \mathfrak{r}(\varsigma)).\big)\nabla\varsigma. \tag{5.6}$$

From (5.5) & (5.6), one can obtain

$$\begin{aligned}|g(\iota)-\hbar(\iota)| &= \Big| g(\iota)-\int_0^{\mathbb{T}}\mathfrak{G}(\iota,\varsigma)\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),q(\varsigma)\big)\nabla\varsigma \\ &\quad+\int_0^{\mathbb{T}}\mathfrak{G}(\iota,\varsigma)\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),q(\varsigma)\big)\nabla\varsigma \\ &\quad-\int_0^{\mathbb{T}}\mathfrak{G}(\iota,\varsigma)\mathcal{L}\big(\varsigma,\hbar(\varsigma),\mathcal{N}(\hbar(\varsigma),\mathfrak{r}(\varsigma))\big)\nabla\varsigma\Big|, \\ &\le \Big| g(\iota)-\int_0^{\mathbb{T}}\mathfrak{G}(\iota,\varsigma)\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),q(\varsigma)\big)\nabla\varsigma\Big| \\ &\quad+\Big|\int_0^{\mathbb{T}}\mathfrak{G}(\iota,\varsigma)\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),q(\varsigma)\big)\nabla\varsigma \\ &\quad-\int_0^{\mathbb{T}}\mathfrak{G}(\iota,\varsigma)\mathcal{L}\big(\varsigma,\hbar(\varsigma),\mathcal{N}\hbar(\varsigma),\mathfrak{r}(\varsigma))\big)\nabla\varsigma\Big|.\end{aligned}$$

Using Equation (5.4), results in

$$\begin{aligned}|g(\iota)-\hbar(\iota)| \le A\epsilon\varphi(\iota)+\Big|\int_0^{\mathbb{T}}\mathfrak{G}(\iota,\varsigma)\Big| \\ \Big|\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),q(\varsigma)\big)-\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),\mathfrak{r}(\varsigma)\big)\Big|.\end{aligned} \tag{5.7}$$

According to (1.1), for $\iota\in\mathbb{T}$,

$$|\mathfrak{r}(\varsigma)-q(\varsigma)| = \big|\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),q(\varsigma)\big)-\mathcal{L}\big(\varsigma,g(\varsigma),\mathcal{N}g(\varsigma),\mathfrak{r}(\varsigma)\big)\big|.$$

Applying the assumption (H2), one can obtain

$$|\mathfrak{r}(\varsigma)-q(\varsigma)| = \big[\mathscr{E}|\hbar(\varsigma)-g(\varsigma)|+\mathscr{F}|\mathfrak{r}(\varsigma)-q(\varsigma)\big]|\mathcal{N}g(\varsigma)|.$$

i.e.,

$$|\mathfrak{r}(\varsigma)-q(\varsigma)| = \frac{\mathscr{E}}{1-\mathscr{F}}|\hbar(\varsigma)-g(\varsigma)||\mathcal{N}g(\varsigma)|.$$

From (5.7), we get

$$\begin{aligned}|g(\iota)-\hbar(\iota)| &\le \frac{A\mathscr{E}}{1-\mathscr{F}}+A\epsilon\varphi(\iota), \\ &\le \frac{A}{1-\dfrac{A\mathscr{E}}{1-\mathscr{F}}}\epsilon\varphi(\iota)|\mathcal{N}g(\varsigma)|, \\ &\le \mathcal{N}\epsilon\varphi(\iota)|\mathcal{N}g(\varsigma)|.\end{aligned}$$

Thus, equation (1.1) encompasses HURS characterized by type $K$ alongside a constant HURS $\dfrac{A(1-\mathscr{F})}{1-A\mathscr{E}-\mathscr{F}}>0$. □

## 6. Application I

Suppose $\mathbb{T}=[1,2]\cup[3,4]$ and $\mathbb{T}=3$. Hence, $\mathbb{I}=[1,3]\cap\mathbb{T}=[1,2]\cup\{3\}$. Assume the PBVP

$$\begin{cases} D^{1.5}[\hbar(\iota)-g(\iota,\hbar)] = \dfrac{e^{-2\iota}}{4}+\dfrac{sin|\hbar(\iota)|+sin|^{C}D^{1/2}\hbar(\iota)|\mathcal{N}_g(\iota)}{10+e^{-5\iota}}, for\ \ all\ \ \iota\in\mathbb{T}_k, \\ \hbar(0)=\hbar(2)=0. \end{cases} \tag{6.1}$$

Here $\mathcal{L}(\iota, \hbar(\iota), \mathcal{N}\hbar(\iota), D^{1.5}\hbar(\iota)) = \dfrac{e^{-2\iota}}{4} + \dfrac{sin|\hbar(\iota)| + sin|^C D^{1/2}\hbar(\iota)|\mathcal{N}_g(\iota)}{10 + e^{-5\iota}}$ satisfying the assumption (H1). For $q_i \in \mathcal{L}, i = 2, 3$. Let $D^{1.5}q_i(\iota) = \mathfrak{r}_i(\iota)$ & for $\iota \in \mathbb{T}$, one can obtain

$$
\begin{aligned}
&|\mathcal{L}(\iota, q_1(\iota), \mathfrak{r}_1(\iota)) - \mathcal{L}(\iota, q_2(\iota), \mathfrak{r}_2(\iota))| \\
&\quad = \left| \frac{e^{-2\iota}}{4} + \frac{[sin|q_1(\iota)| + sin|\mathfrak{r}_1(\iota)|]\mathcal{N}_g(\iota)}{10 + e^{-5\iota}} - \frac{e^{-2\iota}}{4} - \frac{[sin|q_2(\iota)| + sin|\mathfrak{r}_2(\iota)|]\mathcal{N}g(\iota)}{10 + e^{-5\iota}} \right|, \\
&\quad = \left| \frac{[sin|q_1(\iota)| + sin|\mathfrak{r}_1(\iota)|]\mathcal{N}g(\iota)}{10 + e^{-5\iota}} - \frac{[sin|q_2(\iota)| + sin|\mathfrak{r}_2(\iota)|]\mathcal{N}g(\iota)}{10 + e^{-5\iota}} \right|, \\
&\quad \leq \left[ \frac{1}{10}|q_1(\iota) - q_2(\iota)| + \frac{1}{10}|\mathfrak{r}_1(\iota) - \mathfrak{r}_2(\iota)| \right] \big|\mathcal{N}g(\iota)\big|.
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&|\mathcal{L}(\iota, q_1(\iota), \mathfrak{r}_1(\iota)) - \mathcal{L}(\iota, q_2(\iota), \mathfrak{r}_2(\iota))| \\
&\quad \leq \left[ \frac{1}{10}|q_1 - q_2| + \frac{1}{10}|\mathfrak{r}_1 - \mathfrak{r}_2| \right] \big|\mathcal{N}g(\iota)\big|.
\end{aligned}
$$

Thus,the assumption (H2) meets the requirements $\mathscr{E} = \mathscr{F} = \dfrac{1}{10}$. Also, for $q \in \mathcal{L}$, consider $D^{1.5}q(\iota) = \mathfrak{r}(\iota)$.
For $\iota \in \mathbb{T}$,

$$
|\mathcal{L}(\iota, q(\iota), \mathfrak{r}(\iota, \mathcal{N}(\iota)))| \leq \frac{1}{4} + \frac{1}{10}|q(\iota)| + \frac{1}{10}|\mathfrak{r}(\iota)| + 1.
$$

Thus, the assumption (H3) meets the requirements alongside $\mathcal{P} = \dfrac{1}{4}, \mathfrak{r} = \dfrac{1}{10}, \mathcal{Q} = \dfrac{1}{10}$. With the data provided, the inequality now demonstrates $\dfrac{\mathscr{E}K}{1 - \mathscr{F}} < 1$ which gives $K < 19$. Furthermore, utilizing this principle once again in

$$
\frac{(k + m)(||\mathcal{P}|| + \mathfrak{r}\alpha)(\mathcal{N}g(\varsigma))}{1 - \mathcal{Q}} \leq \alpha, \quad \alpha > 0.
$$

yields $m < \dfrac{380}{4\alpha + 10}, \quad \alpha > 0$. Furthermore, employing the boundary condition $\hbar(0) = \hbar(4) = 0$, and as prop 2.13 indicates,

$$
\begin{aligned}
\left| \int_0^2 \mathfrak{G}(\iota, \varsigma)\nabla\varsigma \right| &\leq \left| \int_0^2 \mathfrak{G}(\iota, \varsigma)d\varsigma \right|, \\
&\leq \left| \int_0^\iota \frac{(\iota - \varsigma)^{0.5}}{\Gamma(\frac{2}{1})}d\varsigma - \frac{\iota}{2}\int_0^4 \frac{(\iota - \varsigma)^{0.5}}{\Gamma(\frac{2}{1})}d\varsigma - \frac{\iota}{2}\int_4^\iota \frac{(\iota - \varsigma)^{0.5}}{\Gamma(\frac{2}{1})}d\varsigma \right|, \\
&\leq 1.
\end{aligned}
$$

As a result, the assumption (H4) holds with $A = 1$ and satisfies the assumption (H5). Consequently, all requirements of Theorems 3.5 and 3.6 are met. Thus, the PBVP (6.1) possesses a unique solution $\hbar$ & applying Lemma 3.2 confirms its solution,

$$
\hbar(\iota) = \int_0^4 \mathfrak{G}(\iota, \varsigma)\left( \frac{e^{-2\iota}}{4} + \frac{sin|\hbar(\varsigma) + sin|^C D^{\frac{1}{2}}\hbar(\varsigma)|}{10 + e^{-5\varsigma}} \right)\nabla\varsigma. \tag{6.2}
$$

Further, if $g \in C^1_{ld}(\mathbb{T}, \mathcal{R})$ meets the requirements

$$\left| {}^C D^{1/2} g(\iota) - \frac{e^{-2\iota}}{4} + \frac{sin|g(\iota) + sin|{}^C D^{\frac{1}{2}} g(\iota)|}{10 + e^{-5\iota}} \right| \leq \epsilon,$$

Applying Definition (5.1), results in

$$|g(\iota) - h(\iota)| \leq \frac{17}{16}\epsilon.$$

Thus PBVP (6.1) demonstrates robust Hyers-Ulam stability supported by a fixed HUS constant $\frac{17}{16}$. Fig 1 demonstrates a remarkable alignment between numerical solution and exact solution throughout entire interval.

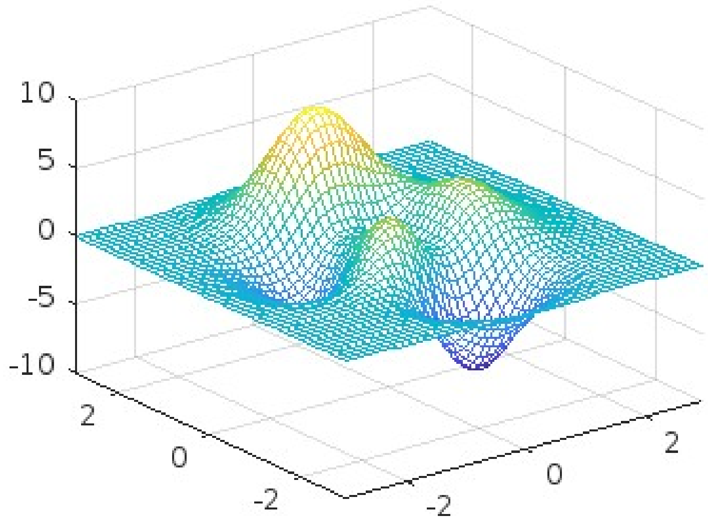


Fig 1: Graph depicting the estimated solution for $h(\iota)$.

## 7. Conclusion

This paper delves into critical findings regarding a fractional dynamic system characterized by partial neutral integro-differential equations in Caputo fractional nabla derivative and is governed by periodic boundary conditions across time scales. Our analysis employs traditional fixed-point methods for system evaluation. Additionally, we present an illustrative application, accompanied by a MATLAB-generated graph. Future work will focus on advancing numerical methods for fractional systems with delays or nonlocal conditions, exploring new control strategies, and applying these systems to communication networks and biomedical fields. Key areas include enhancing stability analysis and leveraging machine learning for optimized control.

## 8. Conflict of Interest

None.

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