MOMENTS AND RECURRENCE RELATIONS OF RECORD VALUES FROM TRUNCATED WEIBULL DISTRIBUTION

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Abstract

In this paper, some exact forms and recurrence relations for single and double moments of upper record values arising from left truncated Wiebull distribution are established. Our results are the generalization of the results given by Balakrishnan and Chan (1993).

Keywords: *Upper record values; recurrence relations; single moments; double moments; truncated distributions; Wiebull distribution.*

1. Introduction

Record values are of great importance in several real-life problems involving weather, economic and sports data. In statistical studies, record values are introduced by Chandler (1952) as a model for successive extremes then has now spread in different directions. Interested readers may refer to Nagaraja (1988), Ahsanallah (1988) and Arnold, Balakrishnan and Nagaraja (1992, 1998).

Some work has been done in this area for the Rayliegh and Weibull distributions by Balakrishnan and Chan (1993) and for the logistic by Balakrishnan, Ahsanullah and Chan (1995) for the generalized Pareto by Sultan and Moshref(2000).

Balakrishnan, Ahsanullah and Chan (1992) have established some recurrence relations between the moments of lower record values from Gumble distribution in a very simple recursive process. Balakrishnan and Ahsanullah (1994, 1995) established some recurrence relations between the moments of upper record values from the generalized Pareto and the exponential distributions, respectively, in a simple recursive manner. Pawlas and Szynal (1999) have established some recurrence relations for single and double moments of k^{th} record values from Pareto, generalized

Pareto and Burr distributions. Sultan and Balakrishnan (1999) have obtained the best linear unbiased estimates (BLUE's) of the location and scale parameters of Rayligh and Weibull distributions. Raqab (2000) has established some general relations for expectations of functions of record values, which may be used to obtain recurrence relations for moments of record values. Sultan (2000) has discussed the moments of record values from uniform distribution and associated inference.

Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the first n upper record values arising from a sequence of independent and identically distribution (i.i.d) random variables (r.v.'s) $X_1, X_2, \ldots, X_n$ with continuous distribution function (cdf) F(x). Then the probability density function (pdf) of the n^{th} upper record value $X_{U(n)}$, n=1,2,3, ... is given by

$$f_n(x) = \frac{1}{\Gamma(n)} f(x)\{-\log(1-F(x))\}^{n-1}, \quad (1)$$

while the joint pdf of $X_{U(n)}$ and $X_{U(m)}$, $m < n$ is given by [see Arnold, Balakrishnan and Nagaraja (1998)].

$$f_{m,n}(x,y) = \frac{1}{\Gamma(m)\Gamma(n-m)}[-\log(1-F(x))]^{m-1}\frac{f(x)}{1-F(x)} \quad (2)$$
$$\times[-\log(1-F(y))+\log(1-F(x))]^{n-m-1} f(y), \qquad -\infty < x < y < \infty$$

In this paper, the pdf of the upper record values from left truncated Weibull distribution (LTW) are introduced in Section 2. Exact moments of record values arising from LTW and some recurrence relations satisfied by the single and double moments of upper record values from LTW are derived in Section 3. In Section 4, a numerical study and conclusion are pointed out.

2. The pdf of the upper record values from LTW

Weibull distribution is quite popular as a life-testing model and for many other applications. It is quite flexible and has the advantage of having a closed form of cdf.

A random variable X has a Weibull distribution if there is a positive value of the parameter c (shape) such that $v = \frac{x^c}{c}$ has the standard exponential distribution with pdf $f(v) = e^{-v}$, $v \geq 0$. Then the pdf of the Weibull distribution is given by

$$f(x) = x^{c-1} e^{-\frac{x^c}{c}}, \qquad x > 0, \quad c > 0, \quad (3)$$

with the corresponding cumulative distribution function

$$F(x) = 1 - e^{-\frac{x^c}{c}} \quad , \qquad x > 0 \, , \; c > 0. \qquad (4)$$

In (3), if we put (c=1) the distribution became exponential distribution and if (c=2) it is Rayleigh distribution.[For more details for Weibull and related distributions, see Johnson, Kotz and Balakrishnan (1994)].

Let us consider that x be a random variable follows LTW distribution denoted by $x \sim LTW\,(c,a)$ so the pdf of x has the form

$$f_t(x) = g(x) = x^{c-1} e^{-\frac{1}{c}(x^c - a^c)} \quad , \qquad x \geq a \, , \; c > 0, \qquad (5)$$

and the cdf is given by

$$F_t(x) = G(x) = 1 - e^{-\frac{1}{c}(x^c - a^c)} \quad , \qquad x \geq a \, , \; c > 0. \qquad (6)$$

Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the upper record values arising from LTW, (5), then the pdf of the n^{th} upper record value $X_{U(n)}$ is obtained from (1) to be

$$f_n(x) = \frac{1}{c^{n-1}\Gamma(n)} x^{c-1} (x^c - a^c)^{n-1} e^{-\frac{1}{c}(x^c - a^c)}, \quad x \geq a \, , \; c > 0. \quad (7)$$

Similarly the joint pdf of $X_{U(m)}$ and $X_{U(n)}$ is obtained to be

$$f_{m,n}(x,y) = \frac{1}{c^{n-2}\Gamma(m)\Gamma(n-m)} x^{c-1} y^{c-1} (x^c - a^c)^{m-1}$$
$$\times (y^c - x^c)^{n-m-1} e^{-\frac{1}{c}(y^c - a^c)}, \quad a \leq x < y < \infty, m < n. \qquad (8)$$

3. Recurrence relations for single and double moments

It is difficult to obtain the exact moments of record values from LTW in a closed form, so in this section, we use the relation between the pdf and cdf of LTW distribution to establish some recurrence relations for the single and double moments of upper record values. From (5) and (6), we have

$$x\, g(x) = a^c (1 - G(x)) + c(1 - G(x))[-\log(1 - G(x))]. \qquad (9)$$

In the following results, we use this relation to establish some recurrence relations between the single and double moments of record values from LTW distribution.

Result 1

For $n \geq 1$ and $k = 0,1,2,\ldots$, the single moments of upper record value from LTW satisfy the recurrence relations

$$\mu_{n+1}^{(k)} = \frac{nc + k - a^c}{nc}\mu_n^{(k)} + \frac{a^c}{nc}\mu_{n-1}^{(k)}. \qquad (10)$$

Proof

The single moment of upper record values from LTW distribution can be written through (9) as

$$\begin{aligned}\mu_n^{(k)} &= \frac{1}{\Gamma(n)}\int_a^\infty x^k[-\log(1-G(x))]^{n-1} g(x)\,dx \\ &= \frac{a^c}{\Gamma(n)}\int_a^\infty x^{k-1}[-\log(1-G(x))]^{n-1}[1-G(x)]\,dx \\ &+ \frac{c}{\Gamma(n)}\int_a^\infty x^{k-1}[-\log(1-G(x))]^{n}[1-G(x)]\,dx. \\ &= I_1(x) + I_2(x), \end{aligned} \qquad (11)$$

where for the first part $I_1(x)$, and integrating by parts treating x^{k-1} for integration and the rest of the integrand for differentiation, we get

$$\begin{aligned} I_1(x) &= \frac{a^c}{k\Gamma(n)}\int_a^\infty x^k[-\log(1-G(x))]^{n-1} g(x)\,dx \\ &- \frac{a^c(n-1)}{k\Gamma(n)}\int_a^\infty x^k[-\log(1-G(x))]^{n-2} g(x)\,dx \end{aligned}$$

$$I_1(x) = \frac{a^c}{k}\mu_n^{(k)} - \frac{a^c}{k}\mu_{n-1}^{(k)},$$

while for the second part $I_2(x)$, we get

$$\begin{aligned} I_2(x) &= \frac{c}{k(n-1)!}\int_a^\infty x^k[-\log(1-G(x))]^{n} g(x)\,dx \\ &- \frac{cn}{k(n-1)!}\int_a^\infty x^k[-\log(1-G(x))]^{n-1} g(x)\,dx \end{aligned}$$

$$I_2(x) = \frac{nc}{k}\mu_{n+1}^{(k)} - \frac{nc}{k}\mu_n^{(k)}.$$

Substituting the above expressions, $I_1(x)$ and $I_2(x)$, into the expression of $\mu_n^{(k)}$ in (11), we get

$$\mu_n^{(k)} = \frac{a^c}{k}\mu_n^{(k)} - \frac{a^c}{k}\mu_{n-1}^{(k)} + \frac{nc}{k}\mu_{n+1}^{(k)} - \frac{nc}{k}\mu_n^{(k)},$$

hence

$$\mu_{n+1}^{(k)} = \frac{nc + k - a^c}{nc}\,\mu_n^{(k)} + \frac{a^c}{nc}\mu_{n-1}^{(k)}.$$

Remarks:

1. We can use LTW distribution as a father distribution and simply obtain the results of left truncated exponential (LTE) and left truncated Rayleigh (LTR) distributions as follows:
   a. By setting c=1,2 in the *pdf* of the n^th^ upper record values and joint of $X_{u(n)}$ and $X_{u(m)}$ from LTW distribution in (7) and (8), we obtain the *pdf* of the n^th^ upper record values and joint of $X_{u(n)}$ and $X_{u(m)}$ for each LTE and LTR distributions, respectively.
   b. By setting c = 1 in (10), we get the single moments of upper record values from LTE distribution as follows:

$$\mu_{n+1}^{(k)} = \frac{n + k - a}{n}\mu_n^{(k)} + \frac{a}{n}\,\mu_{n-1}^{(k)}.$$

   c. By setting c = 2 in (10), we get the single moments of upper record values from LTR distribution as follows :

$$\mu_{n+1}^{(k)} = \frac{2n + k - a^2}{2n}\mu_n^{(k)} + \frac{a^2}{2n}\,\mu_{n-1}^{(k)}.$$

2. By setting a = 0 in (10), we get

$$\mu_{n+1}^{(k)} = \frac{nc + k}{nc}\,\mu_n^{(k)},$$

which is given by Balakrishnan and Chan (1993). It can be, also obtained from Remark 2.1, which is given by Raqab(2000).

3. By setting a = 0 and k=1 in (10), we get

$$\mu_{n+1} = \frac{nc + 1}{nc}\,\mu_n,$$

which is given by Raqab (2000) through the equation (2.8), page 1635.

Once again, upon using the relation in (9), we derive a simple recurrence relation for the double moments of n upper record values from LTW distribution.

Result 2

For $m \geq 1$ and k, s =1, 2, ... the double moments of upper record values from the LTW distribution is given by

$$\mu_{m,m+1}^{(k,s)} = \frac{a^c}{k+cm}\mu_m^{(k+s)} - \frac{a^c}{k+cm}\mu_{m-1,m}^{(k,s)} + \frac{cm}{cm+k}\mu_{m+1}^{(k+s)}, \qquad (12)$$

and for $1 \leq m \leq n-2$ and k, s =0,1,2, ...

$$\mu_{m+1,n}^{(k,s)} = \frac{k+cm}{cm}\mu_{m,n}^{(k,s)} + \frac{a^c}{cm}\mu_{m-1,n-1}^{(k,s)} - \frac{a^c}{cm}\mu_{m,n-1}^{(k,s)}. \qquad (13)$$

Proof

The double moments of record values from LTW distribution may be written as:

$$\mu_{m,n}^{(k,s)} = \frac{1}{\Gamma(m)\Gamma(n-m)}\int_a^{\infty} y^s g(y)\, I(y)\, dy, \qquad (14)$$

where

$$I(y) = \int_a^y x^k[-\log(1-G(x))]^{m-1}[-\log(1-G(y)) + \log(1-G(x))]^{n-m-1}\frac{g(x)}{1-G(x)}dx.$$

(i) For n = m+1

$$\mu_{m,m+1}^{(k,s)} = \frac{1}{\Gamma(m)}\int_a^{\infty} y^s g(y)\, I_0(y)\, dy,$$

where

$$I_0(y) = \int_a^y x^k[-\log(1-G(x))]^{m-1}\frac{g(x)}{1-G(x)}\, dx.$$

Upon using the relation in (9) into the above expression, yields

$$I_0(y) = a^c\int_a^y x^{k-1}[-\log(1-G(x))]^{m-1}dx + c\int_a^y x^{k-1}[-\log(1-G(x))]^m\, dx.$$

Upon integrating by parts treating x^{k-1} for integration and the rest of the integrand for differentiation in each part of the right hand side, the integral, $I_0(y)$, can be written as follows:

$$I_0(y)=\frac{a^c y^k}{k}[-\log(1-G(y))]^{m-1}-\frac{a^c(m-1)}{k}\int_a^y x^k\frac{g(x)}{1-G(x)}$$

$$\times[-\log(1-G(x))]^{m-2}\,dx+\frac{cy^k}{k}[-\log(1-G(y))]^m$$

$$-\frac{cm}{k}\int_a^y x^k[-\log(1-G(x))]^{m-1}\frac{g(x)}{1-G(x)}\,dx.$$

Upon substituting the above expression of $I_0(y)$ instead of $I(y)$ in (14), we obtain

$$\mu_{m,m+1}^{(k,s)}=\frac{a^c}{k\Gamma(m)}\int_a^\infty y^{k+s}[-\log(1-G(y))]^{m-1}g(y)\,dy-\frac{a^c(m-1)}{k\Gamma(m)}\int_a^\infty\int_a^y y^s x^k\frac{g(x)}{1-G(x)}$$

$$\times[-\log(1-G(x))]^{m-2}g(y)\,dx\,dy+\frac{cm}{k\Gamma(m)}\int_a^\infty y^{k+s}[-\log(1-G(y))]^m g(y)\,dy$$

$$-\frac{cm}{k\Gamma(m)}\int_a^\infty\int_a^y y^s x^k[-\log(1-G(x))]^{m-1}\frac{g(x)}{1-G(x)}g(y)\,dx\,dy.$$

The relation (12) is proved simply upon using the definition of the single and double moment of the upper record values in the above equation and simplifying the resulting expression to get.

$$\mu_{m,m+1}^{(k,s)}=\frac{a^c}{k}\mu_m^{(k+s)}-\frac{a^c}{k}\mu_{m-1,m}^{(k,s)}+\frac{cm}{k}\mu_{m+1}^{(k+s)}-\frac{cm}{k}\mu_{m,m+1}^{(k,s)}.$$

Hence

$$\mu_{m,m+1}^{(k,s)}=\frac{a^c}{k+cm}\mu_m^{(k+s)}-\frac{a^c}{k+cm}\mu_{m-1,m}^{(k,s)}+\frac{cm}{cm+k}\mu_{m+1}^{(k+s)}.$$

(ii) For $1\le m\le n-2$

$$I(y)=\int_a^y x^k[-\log(1-G(x))]^{m-1}[-\log(1-G(y))+\log(1-G(x))]^{n-m-1}$$

$$\times\frac{g(x)}{1-G(x)}dx.$$

Upon using the relation in (9) in the above expression, yields

$$I(y)=a^c\int_a^y x^{k-1}[-\log(1-G(x))]^{m-1}[-\log(1-G(y))+\log(1-G(x))]^{n-m-1}\,dx$$

$$+c\int_a^y x^{k-1}[-\log(1-G(x))]^m[-\log(1-G(y))+\log(1-G(x))]^{n-m-1}\,dx$$

$$=I_1(y)+I_2(y). \tag{15}$$

For the first part $I_1(y)$, upon integrating by parts treating x^{k-1} for integration and the rest of the integrand for differentiation, we obtain

$$I_1(y)=\frac{a^c(n-m-1)}{k}\int_a^y x^k[-\log(1-G(x))]^{m-1}[-\log(1-G(y))$$
$$+\log(1-G(x))]^{n-m-2}\frac{g(x)}{1-G(x)}dx-\frac{a^c(m-1)}{k}\int_a^y x^k$$
$$\times[-\log(1-G(x))]^{m-2}[-\log(1-G(y))+\log(1-G(y))]^{n-m-1}\frac{g(x)}{1-G(x)}dx.$$

For the second part $I_2(y)$, upon integrating by parts treating x^{k-1} for integration and the rest of the integrand for differentiation, we obtain

$$I_2(y)=\frac{c(n-m-1)}{k}\int_a^y x^k[-\log(1-G(x))]^{m}[-\log(1-G(y))$$
$$+\log(1-G(x))]^{n-m-2}\frac{g(x)}{1-G(x)}dx-\frac{c(m-1)}{k}\int_a^y x^k$$
$$\times[-\log(1-G(x))]^{m-1}[-\log(1-G(y))+\log(1-G(y))]^{n-m-1}\frac{g(x)}{1-G(x)}dx.$$

Upon substituting the above expression of $I_1(y)$ and $I_2(y)$ into (15) and (14), we obtain

$$\mu_{m,n}^{(k,s)}=\frac{a^c(n-m-1)}{k\Gamma(m)\Gamma(n-m)}\int_a^\infty\int_a^y y^s x^k[-\log(1-G(x))]^{m-1}\frac{g(x)}{1-G(x)}$$
$$\times[-\log(1-G(y))+\log(1-G(x))]^{n-m-2}g(y)\,dx\,dy$$
$$-\frac{a^c(m-1)}{k\Gamma(m)\Gamma(n-m)}\int_a^\infty\int_a^y y^s x^k[-\log(1-G(x))]^{m-2}\frac{g(x)}{1-G(x)}$$
$$\times[-\log(1-G(y))+\log(1-G(x))]^{n-m-1}g(y)\,dx\,dy$$
$$+\frac{c(n-m-1)}{k\Gamma(m)\Gamma(n-m)}\int_a^\infty\int_a^y y^s x^k[-\log(1-G(x))]^{m}\frac{g(x)}{1-G(x)}$$
$$\times[-\log(1-G(y))+\log(1-G(x))]^{n-m-2}g(y)\,dx\,dy$$
$$-\frac{cm}{k\Gamma(m)\Gamma(n-m)}\int_a^\infty\int_a^y y^s x^k[-\log(1-G(x))]^{m-1}\frac{g(x)}{1-G(x)}$$
$$\times[-\log(1-G(y))+\log(1-G(x))]^{n-m-1}g(y)\,dx\,dy.$$

The relation in (13) is proved simply upon using the definition of the single and double moments of upper record values in the above equation and simplifying the resulting expression to get

$$\mu_{m,n}^{(k,s)}=\frac{a^c}{k}\mu_{m,n-1}^{(k,s)}-\frac{a^c}{k}\mu_{m-1,n-1}^{(k,s)}+\frac{cm}{k}\mu_{m+1,n}^{(k,s)}-\frac{cm}{k}\mu_{m,n}^{(k,s)}.$$

Hence

$$\mu_{m+1,n}^{(k,s)}=\frac{k+cm}{cm}\mu_{m,n}^{(k,s)}+\frac{a^c}{cm}\mu_{m-1\,n-1}^{(k,s)}-\frac{a^c}{cm}\mu_{m,n-1}^{(k,s)}.$$

Remarks:

1. By setting c = 1 in (12) and (13), we get the double moments of upper record values from LTE distribution as follows

$$\mu_{m,m+1}^{(k,s)} = \frac{a}{k+m}\mu_{m}^{(k+s)} - \frac{a}{k+m}\mu_{m-1,m}^{(k,s)} + \frac{m}{m+k}\mu_{m+1}^{(k+s)},$$

and for $1 \le m \le n-2$ and $k,\ s = 0,1,2,\ldots$

$$\mu_{m+1,n}^{(k,s)} = \frac{k+m}{m}\mu_{m,n}^{(k,s)} + \frac{a}{m}\mu_{m-1,n-1}^{(k,s)} - \frac{a}{m}\mu_{m,n-1}^{(k,s)}.$$

2. By setting c = 2 in (12) and (13), we get the double moments of upper record values from LTR distribution as follows

$$\mu_{m,m+1}^{(k,s)} = \frac{a^2}{k+2m}\mu_{m}^{(k+s)} - \frac{a^2}{k+2m}\mu_{m-1,m}^{(k,s)} + \frac{2m}{2m+k}\mu_{m+1}^{(k+s)},$$

and for $1 \le m \le n-2$ and $k,\ s = 0,1,2,\ldots$

$$\mu_{m+1,n}^{(k,s)} = \frac{k+2m}{2m}\mu_{m,n}^{(k,s)} + \frac{a^2}{2m}\mu_{m-1,n-1}^{(k,s)} - \frac{a^2}{2m}\mu_{m,n-1}^{(k,s)}.$$

3. It is noted that, the results in (12) and (13) for double moments of record value from LTW distribution are generalization of the results given by Balakrishnan and Chan (1993). By setting a = 0 in (12) and (13), we get

$$\mu_{m,m+1}^{(k,s)} = \frac{cm}{cm+k}\mu_{m+1}^{(k+s)}, \qquad (16)$$

and

$$\mu_{m+1,n}^{(k,s)} = \frac{cm+k}{cm}\mu_{m,n}^{(k,s)}, \qquad (17)$$

respectively, which is given by Balakrishnan and Chan (1993). Furthermore, the two resulting expressions (16) and (17) can be obtained from Theorem 2.3 which is given by Raqab (2000).

4. Numerical study and conclusion

Let us assume that X_1, X_2, … are the upper record values observed from LTW distribution. The single moments at different values of the shape parameter c and truncated point a, in (10), are given in Table 1.

Table 1 Single moments of record values from LTW

c, a \ n	1	2	3	4	5	6	7
3.0,0.5	1.343	1.734	2.015	2.235	2.419	2.579	2.721
3.0,1.0	1.797	1.797	2.097	2.297	2.471	2.625	2.762
5.0,0.5	1.275	1.522	1.673	1.784	1.873	1.948	2.013
5.0,1.0	1.547	1.547	1.702	1.805	1.890	1.962	2.026

It is noted that, from Table 1, as c increases, the single moments decrease. While, as a increases, the single moments increase. And, as n increases, the single moments increase.

The double moments at different values of the shape parameter c and truncated point a are given in Table 2.

Table 2 Double moments of record values from LTW

c, a	n \ m	1	2	3	4	5	6
3.0,0.5	2	2.426					
	3	2.775	3.627				
	4	3.044	3.986	4.632			
	5	3.278	4.290	4.985	5.530		
	6	3.482	4.556	5.294	5.873	6.357	
	7	3.664	4.794	5.571	6.180	6.689	7.131
3.0,1.0	2	2.865					
	3	2.775	3.960				
	4	3.524	4.320	4.918			
	5	3.779	4.630	5.269	5.788		
	6	4.004	4.903	5.579	6.127	6.595	
	7	4.207	5.149	5.858	6.434	6.924	7.354
5.0,0.5	2	1.980					
	3	2.775	2.583				
	4	2.294	2.745	3.018			
	5	2.403	2.875	3.161	3.371		
	6	2.496	2.985	3.283	3.501	3.675	
	7	2.576	3.082	3.388	3.614	3.794	3.945
5.0,1.0	2	2.165					
	3	2.775	2.692				
	4	2.481	2.849	3.099			
	5	2.594	2.978	3.239	3.438		
	6	2.691	3.088	3.359	3.565	3.734	
	7	2.775	3.185	3.464	3.676	3.850	3.997

It is noted that, from Table 2, as c increases, the double moments increase. While, as a increases, the double moments increase. And, as n increases, the double moments increase. Finally, we can say that our results for LTW are generations of the results of Balakrishnan and Chan (1993) and Raqab(2000). Furthermore, the recurrence relations for single and double moments of LTW can be use to calculate the coefficients of the best linear unbiased estimate by following the manner of Sultan and Balakrishnan (1999).

When a=0, Sultan and Balakrishnan (1999) have derived exact moments of upper record values from Weibull distribution and use them to obtain the BLUE's for the location and scale parameters. As well as, they have constructed confidence intervals for the parameters.

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