ORDERING RESIDUAL LIVES USING THE MOMENT GENERATING TRANSFORM

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ABSTRACT: This paper presents new notions of stochastic comparisons and ageing classes based on the moment generating function order. The relationships to other stochastic orders and ageing classes are given. Some preservation properties are presented and some applications to coherent systems and shock models are also studied.

1. INTRODUCTION.

In the context of reliability theory, several orderings of random variables have been considered to give definitions and characterizations of orderings and ageing classes of life time distributions. In fact, notions of positive ageing play an important role in reliability theory, survival analysis and other fields. Therefore an abundance of classes of distributions describing notions of ageing have been considered in literature; see e.g. Barlow and Proschan (1981), Deshpande et al (1986) or Kijima (1997) for an overview.

One of these classes frequently used is the so called L-class. Within this class, Klar (2002) introduced an example of a distribution with an infinite third moment and having the property that the hazard rate tends to zero as time goes to infinity. This example leads to serious doubts if L-class should be considered as a reasonable notion of positive ageing. Recently, Muller and Klar (2002) introduced a new ageing class of life distribution (called M-class) based on the comparison of two non-negative random variables on the moment generating function order (written $\leq_{mgf}$). The M-class is related to the L-class only by replacing the laplace transform order by the moment generating function (M.g.f) order.

Actuarial statisticians usually say that, the risk X is smaller than the risk Y in the exponential order, denoted $X \leq_{exp} Y$, where the inequality

$$E[\exp(tx)] \leq E[\exp(ty)]$$

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Holds for all $t \geq 0$; provided that the expectations involved exist. This order has been examined in Goovaerts et al. (1990, Sectionn 4.3 p. 45) and in Kaas et al. (1994, Section 3, p. 53). It has been used by Kaas and Gerber (1994) in order to compare different approximations for the total claims of a risk portfolio.

In this paper, we further investigate new properties of this order and introduce a new ageing class based on the moment generating function of residual lives order.

Section 2. includes some definitions and properties. Sections 3 and 4 present new order and new ageing class based on the m.g.f. of residual lives. Finally, Section 5 includes some preservation results.

2- DEFINITIONS AND PROPERTIES

In this section, we state definitions of some ageing orders that have been found useful for modeling, or the design of better systems. (see Shaked and Shanthikumar (1994) for a general reference).

Definition 2.1: A random variable X is said to be smaller than a random variable Y

(i) in the stochastic order (denoted by $X \leq_{st} Y$) if and only if,

$$\bar{F}_X(x) \leq \bar{F}_Y(x) \qquad \text{for all } x.$$

(ii) in the increasing convex order ($X \leq_{icx} Y$) if and only if,

$$\int_x^\infty \bar{F}_X(u)du \leq \int_x^\infty \bar{F}_y(u)du, \qquad \text{for all } x \geq 0.$$

(iii) in moment generating function order ($X \leq_{mgf} Y$) if and only if,

$$\int_0^\infty e^{su}\, \bar{F}_X(u)du \leq \int_0^\infty e^{su} \bar{F}_y(u)du, \qquad \text{for all } s \geq 0.$$

Definition 2.2: Let X and Y be two non-negative random variables and let $X_t = X\text{-}t \mid X > t$, and $Y_t = Y\text{-}t \mid Y > t$ be their residual lives, $t > 0$ then the random variable X is said to be smaller than Y

(i) in the hazard rate order (denoted by $X \leq_{hr} Y$) if

$$X_t \leq_{st} Y_t \qquad \text{for all } t > 0.$$

(ii) in the mean residual life order (denoted by $X \leq_{m\,r\ell} Y$) if

$$E(X_t) \leq E(Y_t) \qquad \text{for all } t \geq 0$$

Provided $E(X_t)$ and $E(Y_t)$ are finite.

(iii) in the Laplace transform order of residual life ($X \leq_{Lt\text{-}r\ell} Y$) if and only if

$$X_t \leq_{Lt} Y_t \qquad \text{for all } t \geq 0$$

3- M.G.F. ORDER of RESIDUAL LIVES

In this section, we introduce a new partial order based on the moment generating function order of residual lives. (Mg-rl)

Definition 3.1. Let X and Y be two non-negative random variables. X is said to be smaller than Y in the moment generating function order of residual lives (denoted by $X \leq_{Mg\text{-}r\ell} Y$) if

$$X_t \leq_{mgf} Y_t \qquad \text{for all } t \geq 0$$

The following proposition introduces the necessary and sufficient condition for the $\leq_{Mg\text{-}r\ell}$ order.

Proposition 3.2. Let X and Y be two continuous non-negative random variables, then

$$X \leq_{Mg\text{-}r\ell} Y \ \text{ iff } \ \int_t^\infty e^{su}\, \bar{F}_X(u)du \Big/ \int_t^\infty e^{su}\bar{F}_y(u)du \downarrow \text{in } t \geq 0$$

for all $s > 0$

Proof: Notice that

$$M_{X_t}(s) = \int_0^{\infty} e^{sx} \frac{\overline{F}_X(x+t)}{\overline{F}_X(t)} dx$$

$$= \int_t^{\infty} e^{s(u-t)} \left[\overline{F}_X(u) / \overline{F}_X(t)\right] du$$

$$= \left[1 \big/ \left[e^{st} \overline{F}_X(t)\right]\right] \int_t^{\infty} e^{su} \overline{F}_X(u) du$$

$$= -\int_t^{\infty} e^{su} \overline{F}_X(u) du \Bigg/ \frac{\partial}{\partial t}\left(\int_t^{\infty} e^{su} \overline{F}_X(u) du\right);$$

Now, by Definition 3.1 and given s > 0

$$X_t \le_{mgf} Y_t \qquad \text{for all } t \ge 0$$

iff $\quad M_{X_t}(s) \le M_{Y_t}(s) \qquad \text{for all } t \ge 0$

iff $$-\frac{\int_t^{\infty} e^{su} \overline{F}_X(u) du}{\frac{\partial}{\partial t}\left(\int_t^{\infty} e^{su} \overline{F}_X(u) du\right)} \le -\frac{\int_t^{\infty} e^{su} \overline{F}_Y(u) du}{\frac{\partial}{\partial t}\left(\int_t^{\infty} e^{su} \overline{F}_Y(u) du\right)}$$

iff $$\frac{\int_t^{\infty} e^{su} \overline{F}_X(u) du}{\int_t^{\infty} e^{su} \overline{F}_Y(u) du} \downarrow \text{ in } t \ge 0.$$

Remark: the implications among some of the previous order are shown as:

$$\begin{array}{ccc} X \leq_{st} Y \Rightarrow X \leq_{icv} Y \Rightarrow & X \leq_{Lt} Y \Leftrightarrow & \\ \Downarrow & X \leq_{mgf} Y & \\ X \leq_{icx} Y & \Uparrow & \\ & X \leq_{Mg\text{-}r\ell} Y & \end{array}$$

4. NEW AGEING CLASSES

If A denotes some ageing property, a general procedure to define or characteriz A is by means of stochastic orders of residual life times of the form

$$X \in A \Leftrightarrow X_{t'} \leq_{st.ord} X_t \qquad \forall\, t < t'$$

and $$X \in A \Leftrightarrow X_t \leq_{st.ord} X \qquad \forall\, t \geq 0,$$

where $\leq_{st.ord}$ denotes some stochastic order and $X \in A$ means that X has the ageing property A.

Another characterization is of the form

$$X \in A \Leftrightarrow X \leq_{st.ord} Y,$$

where Y is an exponential random variable with mean E(X) (for details, see Pellerey and Shaked (1997), Deshpand et al (1986), Belzunce et al. (1996), Klefsjo (1982,1983) Shaked and Shanthikumar (1994) and Belzunce et al (1999).

In this section, we propose new ageing classes following the previous procedures for the m.g.f order and the m.g.f order residual lives.

Definition 4.1. X is DRL_{mgf} (decreasing residual life in the moment generating function order)

If

$$X_{t'} \leq_{mgf} X_t \qquad \forall\, t < t',\ t,t' \geq 0$$

Note: X is $DRL_{mgf} \Leftrightarrow X_{t'} \leq_{Mg\text{-}r\ell} X_t \quad \forall\, t < t'$ and $t,t' \geq 0$

$$\Leftrightarrow X_t \leq_{Mg\text{-}r\ell} X \qquad \forall\, t \geq 0$$

Definition 4.2.

X is NBU_{mgf} (new better than used in the moment generating function order) if

$X_t \leq_{mgf} X \qquad \forall\, t \geq 0$

Definition 4.3.

X is EBU_{mgf} (exponentially better than used in the moment generating function order) if for an exponential random variable Y, with mean E(X), then

$$X \leq_{Mg\text{-}r\ell} Y.$$

Note: The following implications are easy to prove,

$$\begin{array}{ccccccc} DRL_{mgf} & \Rightarrow & DMRL & \Rightarrow & L\text{-class} & & \\ \Downarrow & & & & \Uparrow & & \\ NBU_{mgf} & \Rightarrow & NBUE & \Rightarrow & HNBUE & \Rightarrow & GHNBUE \\ & & \Uparrow & & & & \\ & & EBU_{mgf} & \Rightarrow & M\text{-class} & & \end{array}$$

Now, let us consider a renewal process with independent and identically distributed non-negative inter arrival times X_i with common distribution F where F(0) = 0. Let $S_o = 0$ and $S_k = \sum_{i=1}^{k} X_i$ and consider the renewal counting process $N(t) = \sup\{n : S_n \leq t\}$.

Several papers have investigated some characteristics of the renwal process related to ageing properties of F. See for example, Barlow and Proschan (1981). In (1994), Chen investigated the relationship between the behaviour of the renewal function $M(t) \equiv E(N(t))$ and the ageing property of F. Some other results are given for the excess lifetime at time $t \geq 0$, that is, $\gamma(t) = S_{N(t)+1} - t$, which is the time of the next event at time t. For example,

(i) Chen (1994) showed that:
   a) if $\gamma(t)$ is stochastically $\downarrow$ in $t \geq 0$, then $F \in NBU$, and
   b) if $E[\gamma(t)] \downarrow$ in $t \geq 0$, then $F \in NBUE$.

(ii) Li et al. (2000) showed that:
   if $\gamma(t) \downarrow$ in $t \geq 0$, in the increasing concave order, then $F \in NBUC$

(iii) Li and Kochar (2001) showed that:

if $\gamma(t)\downarrow$ in $t \geq 0$, in the increasing concave order, then $F \in NBU(2)$.

(iv) Balzunce et al. (2001) showed that,
if $\gamma(t)\downarrow$ in $t \geq 0$, in the laplace order, then $F \in NBU_{Lt}$.

The following theorem includes a similar result using the moment generating function order and the NBU_{Mg} class.

Theorem 4.1

$\gamma(t)\downarrow$ in $t \geq 0$, in moment generating order, iff $F \in NBU_{Mg}$.

Proof: a) **First,** let $\gamma(t)\downarrow$ in $t \geq 0$, and we observe that

$$P(\gamma(t) \geq x) = \overline{F}(t+x) + \int_0^t P(\gamma(t-y) \geq x)\, dF(y) \qquad \ldots(1)$$

(see Karlin and Taylor (1975), p. 193).

The transform $M_{\gamma(t)}(s)$ can, then be written as

$$M_{\gamma(t)}(s) = \frac{\int_t^\infty e^{sx}\overline{F}(x)dx}{e^{st}} + \int_0^t \int_0^\infty P(\gamma(t-y) \geq x)dx\, dF(y)$$

Since $\gamma(t)\downarrow$ in m.g.f. order, then

$$M_{\gamma(t)}(s) \geq \frac{\int_t^\infty e^{sx}\overline{F}(x)dx}{e^{st}} + F(t)M_{\gamma(t)}(s)$$

or $$M_{\gamma(t)}(s) \geq \frac{\int_t^\infty e^{sx}\overline{F}(x)dx}{e^{st}\overline{F}(t)}, \qquad \ldots(2)$$

but $\gamma(t) \leq_{mgf} \gamma(0), \quad t \geq 0. \qquad \ldots(3)$

using equations (1),(2), and (3) we get

$$\int_t^{\infty} e^{sx}\overline{F}(x)dx = M_{\gamma(0)}(s) \geq M_{\gamma(t)}(s) \geq \frac{\int_t^{\infty} e^{sx}\overline{F}(x)dx}{e^{st}\overline{F}(t)}$$

and then $F \in NBU_{Mg}$.

Second Let $F \in NBU_{Mg}$ and consider the equality

$$P(\gamma(t) \geq u) = \overline{F}(t+u) + \int_0^t \overline{F}(t-x+u)\, dM(x)$$

(see Barlow and Proschan (1981)).

Now,

$$M_{\gamma(t)}(s) = \frac{\int_t^{\infty} e^{su}\overline{F}(u)du}{e^{st}} + \frac{\int_0^t \int_{t-x}^{\infty} e^{su}\overline{F}(u)du\, dM(x)}{e^{s(t-x)}}.$$

Since $F \in NBU_{Mg}$ then

$$M_{\gamma(t)}(s) \leq \overline{F}(t)\int_0^{\infty} e^{su}\overline{F}(u)du + \int_0^t \overline{F}(t-x)\int_0^{\infty} e^{su}\overline{F}(u)du dM(x)$$

$$= \int_0^{\infty} e^{su}\overline{F}(u)du\left[\overline{F}(t) + \int_0^t \overline{F}(t-x)dM(x)\right]$$

$$= \int_0^{\infty} e^{su}\overline{F}(u)du \quad P(\gamma(t) \geq 0) =$$

$$= \int_0^{\infty} e^{su}\overline{F}(u)du = M_{\gamma(0)}(s)$$

Thus $\gamma(t) \leq_{mgf} \gamma(0)$.

Definition 4.4. A function $g:[0,\infty) \to [0,\infty)$ is said to be star-shaped if $g(0) = 0$ and $\frac{g(x)}{x}\uparrow$ in $x \geq 0$

(see Marshall and Olkin (1979)).

Proposition 4.2.

Let X be a random variable and Y = h(x) where h is ↑ and let g be the inverse of h. If X is NBU_{Mg} and g is a star-shaped function then

Y is NBU_{Mg}.

Proof:

X is NBU_{Mg} means that

$$\int_{t}^{\infty} e^{sx}\overline{F}_X(x)dx \le e^{st}\,\overline{F}_X(t)\int_{0}^{\infty} e^{sx}\overline{F}_X(x)dx.$$

Now, let

$$I = e^{st}\,\overline{F}_y(t)\int_{0}^{\infty} e^{sx}\,\overline{F}_y(x)dx - \int_{t}^{\infty} e^{sx}\,\overline{F}_y(x)dx$$

$$= e^{st}\,\overline{F}_x(g(t))\int_{0}^{t} e^{sx}\,\overline{F}_x(g(x))dx - [1 - e^{st}\overline{F}_x(g(t))]\int_{t}^{\infty} e^{sx}\,\overline{F}_x(g(x))dx.$$

Since g is star-shaped then

$$I > e^{st}\,\overline{F}_x(g(t))\int_{0}^{t} e^{sx}\,\overline{F}_x\left(\frac{g(t)}{t}x\right)dx - [1 - e^{st}\overline{F}_x(g(t))]\int_{t}^{\infty} e^{sx}\,\overline{F}_x\left(\frac{g(t)}{t}x\right)dx.$$

Let $y = \frac{g(t)}{t}x$ and $s' = \frac{st}{g(t)}$,

$$I > e^{s'g(t)}\,\overline{F}_x(g(t)\int_{0}^{\infty} e^{s'y}\,\overline{F}_x(y)dy - \int_{g(t)}^{\infty} e^{s'y}\,\overline{F}_x(y)dy \ge 0.$$

Since X is NBU_{Mg},, then Y is NBU_{Mg}.

Definition 4.5. Let X and Y be two random variables with joint density f(x,y). Then f(x,y) is said to be totally positive of order 2 (TP_2) if $f(x_1,y_1)\,f(x_2,y_2) \ge f(x_2,y_1)\,f(x_1,y_2)$ for all $x_1 < x_2$, $y_1 < y_2$ in the domain of X and Y.

5- PRESERVATION RESULTS

The following theorems present some preservation results for compositions of the survival function of one of the new ageing class.

Theorem 5.1.

Let X_1 and X_2 have distribution function F_1 and F_2 respectively and $X_1 \leq_{Mg-r\ell} X_2$. Let Y be independent of X_1 and X_2 and has distribution function G. If Y has log concave density, then

$$X_1 + Y \leq_{Mg-r\ell} X_2 + Y$$

Proof: For fixed $s \geq 0$ and $i = 1,2$, we notice that

$$\Phi(i,t) = \int_0^\infty e^{sv} \bar{F}_{X_i+Y}(v+t)dv$$

$$= \int_0^\infty e^{sv} \int_0^\infty \bar{F}_i(v+t-u)dF_y(u)dv$$

$$= \int_0^\infty e^{sv} \int_{-\infty}^t \bar{F}_i(v+z)f_y(t-z)dz\,dv$$

$$= \int_{-\infty}^t \bar{\bar{F}}_y(t-z)\Psi(i,z)dz \ .$$

Since $X_1 \leq_{Mg-r\ell} X_2$, then

by proposition 3.2, we can say that $\psi(i,z)$ is TP_2 in (i,z).

Moreover, since Y has log concave density, then $f_Y(t-z)$ is TP_2 in (t,z), and by the composition formula (see Barlow and Proschan (1981) page 100), then

$\phi(i,t)$ is TP_2.

Corollary 5.2. Let $X_i \leq_{Mg-r\ell} Y_i$ for all $i = 1,\ldots,n$ and X_i and Y_i (all independent) have log concave densities. Then

$$\sum_{i=1}^n X_i \leq_{Mg-r\ell} \sum_{i=1}^n Y_i$$

Proof: This is done by repeated applications of Theorem 5.1.

Definition 5.3. Let X and Y be random variables with densities f and g respectively, such that $\frac{f(t)}{g(t)}$ decreases over the union of the supports of X and Y are equivalently $f(u)\, g(v) \geq f(v)g(u)$ for all $u \leq v$, then X is said to be smaller than Y in the likelihood ratio order and is denoted by $X \leq_{\ell r} Y$.

The closure of Mg-rℓ order under mixture is given in the following theorem.

Theorem 5.4

Let $\{X(\theta),\ \theta \in R^{+}\}$ be a family of random variables having distribution function F_θ, and let θ_i be random variable having distribution function G_i, i=1,2 and are independent of X.

If $\theta_1 \leq_{\ell r} \theta_2$ and if

$$X(\theta_1) \leq_{Mg\text{-}r\ell} X(\theta_2) \quad \text{whenever } \theta_1 \leq \theta_2,$$

then $X(\theta_1) \leq_{Mg\text{-}r\ell} X(\theta_2)$.

Proof: Let F_i be the distribution function of $X(\theta_i)$, i =1,2. We know that,

$$\overline{F}_i(x)=\int_0^\infty \overline{F}_\theta(x)dG_i(\theta).$$

We need to prove that

$$\Phi(i,t)=\int_0^\infty e^{sx}\overline{F}_i(x+t)dx \quad \text{is } TP_2 \text{ in } (i,t).$$

Actually
$$\Phi(i,t)=\int_0^\infty e^{sx}\overline{F}_i(x+t)dx=\int_0^\infty e^{sx}\int_0^\infty \overline{F}_\theta(x+t)dG_i(\theta)dx$$
$$=\int_0^\infty g_i(\theta)\int_0^\infty e^{sx}\overline{F}_\theta(x+t)dxd\theta$$
$$=\int_0^\infty g_i(\theta)\Psi(\theta,t)d\theta$$

Since $X(\theta_1) \leq_{Mg\text{-}r\ell} X(\theta_2)$ for $\theta_1 \leq \theta_2$, then $\psi(\theta,t)$ is TP_2 in (i, θ). But $\theta_1 \leq_{\ell r} \theta_2$, it follows that $g_i(\theta)$ is TP_2 in (i, θ). Thus the result follows.

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