Nonparametric tests for shifts in nonlinear regression

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Abstract

We propose nonparametric test statistics for the at most one change point (AMOC) problem in the regression function of a nonlinear regression model. The asymptotic distributions of these test statistics are investigated through strong approximation properties. We also give approximations for the test statistics limiting distributions.

Keywords : Hybrid process, Change point problem, Nonlinear regression, Strong approximation, Wiener process, Brownian bridge process, Two-parameter Gaussian process.

1. Introduction

The problem of statistical change point has been an attractive topic in statistical analysis for decades. It has originally arisen in the context of quality control. But in many practical and experimental situations some statistical properties of an observed phenomenon may change abruptly (or gradually) at some unknown time point(s). The detection and characterization of such a change are problems of interest in many scientific fields. Examples can be found in Economics (structural change), Engineering (speech signals recognition), Epidemiology (incidence of a disease), Geology (seismic signal processing), History (lindisfarne scribes), Archaeology (sits studding) and Quality control (product specifications). In statistical literature such problems are called "change point" problems. In the past four decades an extensive amount of research has been done in this area using different approaches (parametric and non-parametric) to treat the problem in both classical and non-classical (Bayesian) contexts. For review we refer to Shaban (1980), Basseville and Benveniste (1986), Lombard (1989), Csörgő and Horváth (1993), Csörgő and Horváth

(1997) and Antoch et al. (2002) as classical treatment, to Broemeling and Tsurumi (1987) and Jandhyala et al. (1999) as Bayesian and to Zacks (1983) for both.

Although, the linear regression model change point problems are extensively studied, very little work can be found in the case of nonlinear models. Dominique (1985), proposed a Kolmogorov-Smirnov type test statistic for detecting changes in the covariance structure of time series. He also investigated the behavior of the LR statistic in detecting failure occurring in the mean and the covariance of autoregressive process of bounded order. Davis et al. (1995), derived an LR test statistic for testing a change of either the coefficients, the white noise variance or the order in the AR model. They showed that the asymptotic distribution of the test is the Gumbel extreme value distribution. Jandhyala et al. (1999), introduced Bayes-type test statistics to detect parameter shifts in regression models with serially correlated random errors. The test statistics limiting distributions are derived and their critical values are approximated through simulation. Jandhyala and Al-Saleh (1999) proposed two-sided Bayes-type statistics for tests of parameter changes in case of exponential type nonlinear regression models and illustrated their methodology through data on pre-school boys' weight / height ratio. Lurie and Neerchal (1999), derived Bayes-type tests for the problem of parameter changes in general autoregressive process. They studied the statistics asymptotic distributions and showed, through simulation, that Bayes-type tests have better power than the LR tests in case of small changes. Horváth et al. (2001), used sequential empirical process of the squared residuals of an ARCH(p) sequence to detect a change in the distribution of unobservable innovations. Gómez and Drouiche (2002), proposed new homogeneity tests in autoregressive processes and deduced a test for the autoregressive coefficient nullity or randomness.

Let $X_1, X_2, \ldots, X_n$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be two independent sequences of independent random variables. Then the nonlinear regression model takes the form;

$$Y_i = \theta(X_i) + U(X_i)\varepsilon_i, \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where $\theta(.)$ is the regression function and $U(.)$ is unknown bounded variation function. Now, suppose that we are interested in testing the following hypothesis of no change in the regression function;

$$H_o : \theta(X_1) = \theta(X_2) = \ldots = \theta(X_n),$$

against the alternative

$$H_1 : \theta(X_1) = \theta(X_2) = \ldots = \theta(X_k) \neq \theta^*(X_{k+1}) = \theta^*(X_{k+2}) = \ldots = \theta^*(X_n), \tag{1.2}$$

where the change point $k, 1 \le k \le n$ and the regression functions $\theta(.)$ and $\theta^*(.)$ are all assumed unknown.

To test the above hypotheses, we extend the idea of Dieblot (1995) to the change point setup. He used the so-called Hybrid process to test the one sample hypothesis;

$\theta(.)=\theta_o(.)$ against the alternative hypothesis $\theta(.)\neq\theta_o(.)$, for a known $\theta_o(.)$. The Hybrid process of X and ε is given by

$$A(t,n)=\sum_{i=1}^{n}U(X_i)\ \ I\{X_i \le t\}\ \ \varepsilon_i\ , \qquad -\infty<t<\infty, \tag{1.3}$$

where $U(.)$ is a bounded variation function on the real line, $I\{e\}$ is the indicator function of the event e and the random variables ε_i's are called the process weights.

Dieblot (1990, 1995) and Diebolt and Laib (1991) introduced, studied and used the Hybrid process in constructing nonparametric tests for a class of nonlinear models. They showed that $A(t,n)/\sqrt{n}$ converges weakly to a time-transformed Wiener process and obtained upper bounds for the rate of convergence. Lo (1987) used exponential weights , Parzen et al. (1994) and Lin et al. (1996) used normal weights for the bootstrapped empirical process which is a special case of $A(.,.)$ of (1.3). Burke (1998), obtained an approximation for the Hybrid process when $U(t)=t\ ,-\infty<t<\infty$ and ε is a normal random variable. When $I\{X_i \le t\}=1$ and $U(t)=t\ ,-\infty<t<\infty$, Rosalsky and Sreehari (1998) studied the limiting behavior of (1.3) irrespective of the joint distributions of X and the weights ε. Horváth et al. (2000) obtained approximations for the weighted bootstrap processes as a special case of the process in (1.3), when $U(t)=1,-\infty<t<\infty$. Horváth (2000) derived almost sure rates and probability inequalities for the approximations of the Hybrid process in (1.3).

In this paper we study the limiting behavior of what we shall define and call the empirical Hybrid change point process. Then using this process, we propose several nonparametric test statistics for the hypotheses in (1.2). The limiting distributions of these test statistics are also derived and some approximations are suggested. In Section 2 we define the empirical Hybrid change point process and discuss its asymptotic distribution. Several nonparametric test statistics for the hypotheses in (1.2) are introduced in section 3 and their limiting distributions are investigated. In Section 4 we conducted a simulation study for the change point tests to estimate their critical values and powers in finite samples. Section 5 contains the proof of the main Theorem.

2. The empirical change point process

Let $\hat{\theta}(.)$, be an estimator for the regression function $\theta(.)$ of the nonlinear regression model in (1.1). Then the regression residuals can be written as;

$$Y_i-\hat{\theta}(X_i)=U(X_i)Z_i, \qquad i=1,2,...,n, \tag{2.1}$$

where $U(.)$ is the unknown bounded variation function of (1.1) and $Z_1,Z_2,...,Z_n$ is a sequence of iid random variables. Following Horváth (2000), we assume that all random variables and processes introduced so far and later on can be defined on the same probability space and the following assumptions are also satisfied:

1). The sequences X_1, $X_2,\dots,$ X_n and Z_1, $Z_2,\dots,$ Z_n are independent.

2). The variables Z_1, $Z_2,\dots,$ Z_n are iid random variables with zero mean, unit variance and the moment generating function of Z is finite in the neighborhood of zero.

3). The variables $X_1, X_2,\dots, X_n,\dots$ are independent random variables with common distribution function $F(.)$.

4). The time-transformation process and its empirical counterpart are given by:

$$G(t) = \int_{-\infty}^{t} U^2(s)\ dF(s)\ , \qquad -\infty < t < \infty \tag{2.2}$$

and

$$G_n(t) = \int_{-\infty}^{t} U^2(s)\ d\hat{F}_n(s), \qquad -\infty < t < \infty \tag{2.3}$$

where $\hat{F}_n(.)$ is the empirical distribution function of the random variable X based on n observations, i.e.

$$\hat{F}_n(s) = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \le s\}\ , \quad -\infty < s < \infty. \tag{2.4}$$

Now let us define the empirical change point Hybrid process $\{\Delta_n(t,s); -\infty < t < \infty, 0 \le s \le 1, n \ge 1\}$ by;

$$\Delta_n(t,s) = \frac{1}{\sqrt{n}}\left(\hat{A}(t,[ns]\ ) - \frac{[ns]}{n}\ \hat{A}(t,n)\right), \tag{2.5}$$

where $\hat{A}(.,\ .)$ is the empirical Hybrid process defined by;

$$\hat{A}(t,k) = \sum_{i=1}^{k} U(X_i) I\{X_i \le t\} Z_i, \quad -\infty < t < \infty,\ k = 1,2,\dots,n. \tag{2.6}$$

Let $\{W(t,s); -\infty < t, s < \infty\ \}$ be the two parameter Gaussian process, (see Csörgõ and Révész (1981), section (1.11)), with mean zero and covariance structure

$$E\ \{\ W(t_1\ ,\ s_1\ )\ W(t_2\ ,\ s_2)\ \} = (t_1\ \wedge\ t_2\ )\ (s_1\ \wedge\ s_2\ )\ , \tag{2.7}$$

where $-\infty < t_1, t_2, s_1, s_2 < \infty$.

Theorem (2.1)

Let $\Delta_n(.,.)$ and $W(.,.)$ be the two processes of (2.5) and (2.7) respectively, then we have

$$\sup_{-\infty<t<\infty}\ \sup_{0\le s\le 1} |\ \Delta_n(t,s) - \Delta(t,s)\ | \overset{Prob.}{=} o(1),$$

where

$$\Delta(t,s) = W(G(t),s) - sW(G(t),1)\ , \quad -\infty<t<\infty,\ 0\le s\le 1, \qquad (2.8)$$

and $G(.)$ is the time-transformation process of (2.2).

By the above Theorem (2.1), we have as $n\to\infty$

$$\Delta_n(t,s) \overset{D}{\to} \Delta(t,s), \quad -\infty<t<\infty, 0\le s\le 1, \qquad (2.9)$$

where $\Delta_n(.,.)$ and $\Delta(.,.)$ are as in Theorem (2.1).

It is clear that Theorem (2.1) of Horváth et al. (2000) is a special case of the above Theorem when the function $U(t)$ of the Hybrid process of (1.3) equal to one. Even though they pointed out that the distribution of the limiting random variable in this special case is not known.

3. Test statistics

In this section we suggest the use of the Kolmogrov-Smirnov goodness of fit and the Cramér-Von Mises type test statistics. So for a suspected change in the regression function of (1.1), we can use the test statistics

$$T_{n,1} = \sup_{-\infty<t<\infty}\ \sup_{0\le s\le 1} |\Delta_n(t,s)| = \max_{1\le j\le n}\max_{1\le i\le n} |\Delta_n(X_{(j)},\frac{i}{n})|, \qquad (3.1)$$

and

$$T_{n,2} = \left(\int_{-\infty}^{\infty}\int_0^1 \Delta_n^2(t,s)\,ds\,dG_n(t)\right)^{\frac{1}{2}} = \frac{1}{n}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} \Delta_n^2(X_{(j)},\frac{i}{n})\right)^{\frac{1}{2}}, \qquad (3.2)$$

where $\Delta_n(.,.)$ is the process of (2.5) and $X_{(j)}$, is the j^{th} order statistics in the X sample. Since the statistics $T_{n,1}$ and $T_{n,2}$ are continuous functionals of the process $\Delta_n(.,.)$ of (2.5) and the empirical distribution function $G_n(.)$ converges almost surely and uniformly to $G(.)$ then using (2.9), we have as $n\to\infty$;

$$T_{n,1} \overset{D}{\to} \sup_t\ \sup_s |\ \Delta(t,s)\ | = T_1 \qquad (3.3)$$

and

$$T_{n,2} \xrightarrow{D} \Big(\iint_{t\ s} \Delta^2 \ (t\ ,\ s\ )\ ds\,dG(t) \Big)^{\frac{1}{2}} = T_2\ , \tag{3.4}$$

where $\Delta(.,.)$ is the process of Theorem (2.1).

As mentioned in Horváth et al. (2000) the resulted distributions in this context such as (3.3) and (3.4) even in the special case when $U(t)=1, -\infty<t<\infty$, are not known. The random variables T_1 and T_2 in (3.3) and (3.4) are continuous functionals of a tied-down two-parameters Guassian process that depends on the unknown distribution function $G(.)$. These random variables are well defined, but unknown in the literature even if we replace $G(.)$ by its empirical counterpart from the sample. It is also easy to see that T_2 of (3.4) is a normal random variable, with unknown covariance because of $G(.)$ and its complicated form. Therefore in approximating these limiting distributions, we may follow the steps and arguments of Dieblot (1995) to suggest the following approximations. Using the covariance structure of the Gaussian processes we have

$$W(t\ ,\ s\ )-s\ W(t\ ,\ 1\ ) \overset{D}{=} W(t)\ B(s),\ -\infty<t<\infty, 0\le s\le 1 \tag{3.5}$$

Where $W(.)$ and $B(.)$ are independent Wiener and Bridge processes. Hence we suggest the approximations:

$$T_1 \approx \Gamma_n^{\frac{1}{2}} \sup_{-\infty<t<\infty} |\ W(t)\ |\ \sup_{0\le s\le 1} |\ B(s)\ |, \tag{3.6}$$

and

$$T_2 \approx \Gamma_n^{\frac{1}{2}} \Big(\int_{-\infty}^{\infty} W^2(t)\ dt \Big)^{\frac{1}{2}} \Big(\int_0^1 B^2(s) ds \Big)^{\frac{1}{2}} \tag{3.7}$$

where,

$$\Gamma_n = \lim_{t\to\infty} G_n\ (t) = \frac{1}{n} \sum_{i=1}^{n} U^2(x_i). \tag{3.8}$$

Even with the above Dieblot (1995) approximations, the distributions of the random variables T_1 and T_2 in (3.6) and (3.7) are not known and considered to be an open problem. One way out of this problem is to suggest test statistics for a specified time parameter value $t=t_\circ\in(-\infty,\ \infty)$, for the change point problem and a specified change parameter value $s=s_\circ\in[0,1]$, for the two-sample problem. To do so, we may suggest the following change point test statistics, for sufficiently large values of $t=t_\circ$;

$$T_{n3} = \frac{1}{\sqrt{d_n}} \sup_{0\le s\le 1} |\Delta_n\ (t_\circ, s)| = \frac{1}{\sqrt{d_n}} \max_{1\le k\le n} |\Delta_n\ (t_\circ, \frac{k}{n})|,$$

$$T_{n4}=\left(\frac{12}{d_n}\right)^{\frac{1}{2}}\int_0^1\Delta_n(t_*,s)\,ds=\left(\frac{12}{d_n}\right)^{\frac{1}{2}}\sum_{k=1}^{n}\frac{\Delta_n(t_*,\frac{k}{n})}{n},$$

and

$$T_{n5}=\left(\frac{1}{d_n}\int_0^1\Delta_n^2(t_*,s)\,ds\right)^{\frac{1}{2}}=\left(\frac{1}{d_n}\sum_{k=1}^{n}\frac{\Delta_n^2(t_*,\frac{k}{n})}{n}\right)^{\frac{1}{2}}, \qquad (3.9)$$

where $d_n=\frac{1}{n}\sum_{i=1}^{n}(Y_i-\hat{\theta}(X_i))^2=\frac{1}{n}\sum_{i=1}^{n}(U(X_i)Z_i)^2$, is a consistent estimator for Γ_n of (3.8) when n is large (see Proposition 2 of Dieblot (1995)).

The limiting random variables of the statistics T_{n3} , T_{n4} and T_{n5} of (3.9) are the well-known tabulated random variables $\sup_{0\le t\le 1}|B(t)|$, (where $B(.)$ is a standered Brownian bridge), $N(0,1)$, (the standered Normal random variable), and $(\int_0^1 B^2(s)ds)^{\frac{1}{2}}$, respec -tively.

For the Two-sample problem, if we put $s=s_*, 0\le s_*\le 1$, we have the following test statistics;

$$T_{n6}=(s_*(1-s_*))^{-\frac{1}{2}}\sup_{-\infty<t<\infty}|\Delta_n(t,s_*)|=(s_*(1-s_*))^{-\frac{1}{2}}\max_{1\le j\le n}|\Delta_n(X_{(j)},s_*)|,$$

$$T_{n7}=(s_*(1-s_*))^{-\frac{1}{2}}\int_{-\infty}^{\infty}\Delta_n(t,s_*)\,dt=(s_*(1-s_*))^{-\frac{1}{2}}\sum_{j=1}^{n}\frac{\Delta_n(X_{(j)},s_*)}{n},$$

and

$$T_{n8}=\{(s_*(1-s_*))^{-1}\int_{-\infty}^{\infty}\Delta_n^2(t,s_*)\,dt\}^{\frac{1}{2}}=\{(s_*(1-s_*))^{-1}\sum_{j=1}^{n}\frac{\Delta_n^2(X_{(j)},s_*)}{n}\}^{\frac{1}{2}}, \qquad (3.10)$$

where $\Delta_n(.,.)$ is defined by (2.5).

The limiting random variables of the statistics in (3.10) are $\sup_{-\infty<t<\infty}|W(G(t))|$, $\int_{-\infty}^{\infty}W(G(t))\,dt$ and $\{\int_{-\infty}^{\infty}W^2(G(t))\,dt\}^{\frac{1}{2}}$, respectively. These limiting random variables can also be approximated as in Dieblot (1995) by $d_n^{\frac{1}{2}}\sup_{-\infty<t<\infty}|W(t)|$, $d_n^{\frac{1}{2}}\int_{-\infty}^{\infty}W(t)\,dt$ and $d_n^{\frac{1}{2}}(\int_{-\infty}^{\infty}W^2(t)\,dt)^{\frac{1}{2}}$, respectively, where d_n is defined in (3.9). We note that the approximated random variables are easier to apply by using their tabulated critical values in calculating the required critical values (see for example Adler (1990) and Shorack and Wellner (1986)).

4. Simulation Study

In this Section we estimate through a Monte Carlo study the critical values and powers of the change point tests in finite samples, (the two-sample tests are just special cases of the change point tests). Similar to Antoch et al. (2002), we calculated the test statistics T_{n3},T_{n4} and T_{n5} of (3.9) using Matlab programming version 5.1. To estimate the critical values of the proposed change point tests in finite samples we conducted the following simulations. Assuming that Z of (2.6) has a standard normal we estimated the tests critical values when X has a normal distribution and again when it has exponential distribution. For simplicity we took U(X)=X of (2.6) and calculated the tests under the assumption of no change 5000 times. Then ordered the 5000 values of each test and obtained the $(1-\alpha)^{th}$ percentiles for $\alpha = 0.1, 0.05$ and 0.01. The entries of Table 1 and Table 2 below contain the results of the estimated critical values for different finite samples. These estimated critical values show that there is no that much difference between the two cases of normal and exponential simulations. We also see that there is no outliers within the estimated critical values which show that they are converging. For example, in both normal and exponential cases, and at $\alpha = 0.05$, the finite sample estimated critical values of the three tests converge to their theoretical limiting values, namely 1.36, 1.65 and 0.69 respectively.

To measure the performance of the proposed test statistics we conducted a small Monte Carlo study and reported the results in Table 3. The estimated powers were obtained at $\alpha = 0.05$, for sample size n=60 and n=100 from normal and exponential distributions (because of similarities of results we report here the normal case only). Three shift locations are considered m=n/4, n/2 and 3n/4 and the shift sizes were computed as the solution of the equation $P(X_{m+1} > X_m) = p$, where p=70%, 80% and 90%. We only considered a possible change in the mean of the random variable X as a change in the regression function. To calculate the powers, we simulated 5000 realizations of samples of size n=60 (and n=100) under the alternative hypothesis and computed the test statistics of (3.9) in each realization. Then for each simulation of 5000 realizations, we obtained the fraction of the number of times, when each test statistic exceed its critical value at $\alpha = 0.05$. Examining the results of Table 3, we can make two general remarks. The estimated powers of T_{n5} are the largest, then of T_{n3} and the smallest are of T_{n4}. This because T_{n5} counts for every deviation between the possible sub-samples, while T_{n3} considers the largest deviation only and both T_{n5} and T_{n3} are one-sided tests while T_{n4} is a two-sided test. We also notice that the estimated powers of the three tests increase naturally with the increase of the sample size and shift size . They are also decreasing as the shift location moves to the end of the sample. Thus these test statistics perform better when the shift occurs early in the sample.

Table 1

Estimated critical values for the change point tests

(Normal case)

Test	T_{n3}			T_{n4}			T_{n5}		
α	1%	5%	10%	1%	5%	10%	1%	5%	10%
n									
10	1.59	1.25	1.06	2.49	1.60	1.22	0.98	0.72	0.60
20	1.47	1.22	1.08	2.24	1.63	1.24	0.86	0.68	0.60
30	1.55	1.27	1.12	2.45	1.68	1.29	0.89	0.70	0.60
40	1.49	1.25	1.12	2.31	1.58	1.22	0.85	0.68	0.58
50	1.58	1.27	1.13	2.32	1.63	1.28	0.92	0.69	0.59
60	1.55	1.26	1.12	2.34	1.60	1.22	0.86	0.68	0.58
70	1.55	1.29	1.14	2.29	1.61	1.26	0.88	0.69	0.59
80	1.57	1.26	1.15	2.16	1.61	1.28	0.85	0.68	0.60
90	1.53	1.27	1.16	2.38	1.70	1.31	0.86	0.69	0.59
100	1.58	1.31	1.16	2.28	1.62	1.27	0.89	0.69	0.59
200	1.57	1.31	1.18	2.29	1.65	1.32	0.85	0.69	0.59
500	1.59	1.34	1.21	2.41	1.70	1.29	0.89	0.70	0.60

Table 2

Estimated critical values for the change point tests

(Exponential case)

Test	T_{n3}			T_{n4}			T_{n5}		
α	1%	5%	10%	1%	5%	10%	1%	5%	10%
n									
10	1.69	1.26	1.07	2.57	1.66	1.24	0.99	0.74	0.62
20	1.58	1.24	1.08	2.26	1.61	1.24	0.92	0.69	0.60
30	1.48	1.22	1.09	2.41	1.58	1.23	0.88	0.68	0.58
40	1.57	1.27	1.13	2.37	1.67	1.28	0.94	0.68	0.60
50	1.55	1.27	1.13	2.33	1.62	1.30	0.87	0.69	0.60
60	1.54	1.26	1.12	2.27	1.60	1.24	0.87	0.68	0.59
70	1.54	1.28	1.14	2.32	1.64	1.29	0.87	0.69	0.60
80	1.59	1.29	1.15	2.48	1.66	1.30	0.92	0.69	0.59
90	1.55	1.29	1.16	2.56	1.68	1.32	0.90	0.69	0.60
100	1.57	1.28	1.15	2.34	1.67	1.34	0.86	0.68	0.60
200	1.55	1.30	1.17	2.25	1.66	1.25	0.87	0.68	0.59
500	1.61	1.34	1.19	2.43	1.72	1.30	0.89	0.69	0.60

Table 3

Estimated power percentages for the change point tests

(Normal case and $\alpha = 5\%$)

m	n = 60			n = 100		
	15	30	45	25	50	75
Prob. of change =70%						
T_{n3}	40.8	35.3	28.8	43.8	37.9	33.1
T_{n4}	12.7	12.5	12.2	12.7	12.8	12.5
T_{n5}	58.4	53.5	49.2	59.4	54.1	50.1
Prob. of change =80%						
T_{n3}	64.2	54.4	40.8	66.8	58.2	43.7
T_{n4}	17.1	15.0	15.4	17.8	16.6	16.6
T_{n5}	75.4	68.5	59.5	75.7	69.3	60.0
Prob. of change =90%						
T_{n3}	86.7	76.3	60.0	90.2	80.9	62.6
T_{n4}	21.7	20.7	21.4	21.8	21.7	21.5
T_{n5}	91.8	83.7	71.7	92.3	84.9	72.2

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4. Proofs:

Proof of theorem (2.1): By the definition of the processes $\Delta_n(.,.)$ and $\Delta(.,.)$ in (2.5) and (2.8), we have almost surely

$$
\begin{aligned}
L_n &= \sup_{-\infty<t<\infty} \sup_{0\le s\le 1} | \Delta_n(t, s) - \Delta(t, s) |, \\
&\le \sup_{-\infty<t<\infty} \sup_{0\le s\le 1} | \frac{1}{\sqrt{n}} \hat{A}_n(t, [ns]) - W(G(t), s) | \\
&\quad + \sup_{-\infty<t<\infty} \sup_{0\le s\le 1} | \frac{1}{\sqrt{n}} \frac{[ns]}{n} \hat{A}_n(t, n) - s\, W(G(t), 1) |, \\
&\le \sup_{-\infty<t<\infty} \sup_{0\le s\le 1} | \frac{1}{\sqrt{n}} \hat{A}_n(t, [ns]) - W(G(t), s) | \\
&\quad + \sup_{-\infty<t<\infty} \sup_{0\le s\le 1} | \frac{1}{\sqrt{n}} \frac{[ns]}{n} \hat{A}_n(t, n) - \frac{[ns]}{n} W(G(t), 1) | \\
&\quad + \sup_{0\le s\le 1} | \frac{[ns]}{n} - s | . \sup_{-\infty<t<\infty} |W(G(t), 1)| . \qquad (4.1)
\end{aligned}
$$

Using Theorem (2.2) of Horváth (2000) and the finiteness of the two-parameter Gaussian process $W(.,.)$, we get from (4.1) and as $n \to \infty$;

$$L_n = O_{a.s.}(n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}}) + O_{a.s.}(n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}}) + o(1). O_p(1) \overset{Prob.}{=} o(1).$$

References:

Adler, R. J. (1990): *An Introduction to Continuity, Extrema, and Related Topics ,.. General Gaussian Processes.* IMS Lecture Notes, Vol. 12, Hayward. USA.

Antoch, J., Hušková, M. and Jauršková, D. (2002): *Change point detection.* Springer Verlag (preprint).

Basseville, M. and Benveniste, A. (1986): *Detection of abrupt changes in signals and dynamical systems*. New York. Basel.

Broemeling, L. and Tsurumi, H. (1987): *Econometrics and Structural change*. Marcel Dekker, Inc. New York. Basel.

Burke, M. D. , (1998): A Gaussian bootstrap approach to estimation and tests. In : Szyszkowicz, B. (Ed.), *Asymptotic Methods in Probability and Statistics.* Elsevier Science B.V., Amsterdam, PP. 697-706.

Csörgõ, M. and Horváth, L. (1988): Invariance principles for change point problems. *J. Multivariate. Anal.* **27**, 151-168.

Csörgõ, M. and Horváth, L. (1993): *Weighted Approximations in Probability and Statistics.* Wiley, Chichester.

Csörgõ, M. and Horváth, L. (1997): *Limit Theorems in Change-Point Analysis.* Wiley-Chichester.

Csörgõ, M. and Révész , P. (1981): *Strong Approximations in Probability and Statistics*. Academic Press, New York.

Davis, R. A., Huang, D. and Yao, Y.-C. (1995): Testing for a change in parameter values and order of an autoregressive model. *Ann. Statist.*, **23**, 282-304.

Diebolt, J. , (1990): Testing functions defining a nonlinear autoregressive time series. *Stochastic Proc. Appl.* **36,** 85-106.

Diebolt, J. , (1995): A nonparametric test for regression function. Asymptotic Theory. *J. Statist. Plan. Inf.* **44,** 1-17.

Diebolt, J. and Laib, N., (1991): Un Principe d'invariance faible pour l'etude d'un test nonparametrique reltife a la fonction de regression. *Comptes Rendus de l'Academie des Sciences de Paris.* **312, 887- 891**.

Dominique, P. (1985): Testing and estimating change-points in time series. *Adv. Appl. Prob*., **17, 841-867**.

Gómez, P. M. and Drouiche, K. (2002): A test of homogeneity for autoregressive processes. *Inter. J. Adap. Contr. Signa. Process.* **16**, 231-242.

Horváth, L. (2000) : Approximations for hybrids of empirical and partial sums processes. *J. Statist. Plan. Inf.* **88**, 1-18.

Horváth, L., Kokoszka, P. and Steinbach, J., (2000): Approximations for weighted bootstrap process. *Statist. Probab. Lett.*. **48**, 59-70.

Horváth, L., Kokoszka, P. and Teyssiére, G. (2001): Empirical process of the squared residuals of an ARCH sequence. *Ann. Statist.* **29**, 445-469.

Jandhyala, V. , Zacks, S. and El-Shaarawi, A. (1999): Change-point methods and their applications: contribution to Ian MacNeill. *Environmetrics* **10**, 657-676.

Jandhyala, V. and Al-Saleh, J. (1999): Parameter changes at unknown times in non-linear regression. *Environmetrics* **10**, 711-724.

Lin, D. Y., Robins, J. M. and Wei, L. J. (1996): Comparing two-failure time distributions in the presence of dependent censoring. *Biometrika* **83**, 381-393.

Lo, A. Y., (1987): A large sample study of the Bayesian bootstrap. *Ann. Statist.* **15**, 360-375.

Lombard, F. (1989): Some recent developments in the analysis of change-point data. *South African Statist. J.*, **23**, 1-21.

Lurie, A. and Neerchal, N. (1999): Bays-type tests for constancy of autoregressive parameters. *Environmetrics* **10**, 737-752.

Parzen, M. I., Wei, L. J., and Ying, Z., (1994): A resampling method based on pivotal estimating functions. *Biometrika*, **81**, 341-350.

Rosalsky, A., and Sreehari, M. (1998) On the limiting behavior of randomly weighted partial sums. *Statist. Prob. Lett.* **40**, 403-410.

Sen, P. K. (1988): Robust tests for change point models. In: S. Kotz and N.L. Johnson, (Eds. in chief), *Encyclopedia of Statistical Sciences*, **8**, John Wiley and Sons, New York.

Shaban, S. A. (1980): Change point problem and two-phase regression, an annotated bibliography. *Inter. Statist. Rev.* **48**, 83-93.

Shorack, G. and Wellner, J. (1986): *Empirical Processes with Applications to Statistics*. Wiley, New York.

Zacks, S. (1983): Survey of classical and Bayesian approaches to the change-point problem: fixed and sequential approaches of testing and estimation. In: M.H. Rizvi, J.S. Rustagi and D. Siegmund, (Eds.), *Recent Advances in Statistics*. Academic press, New York.