Test For Exponential Better (Worse) than Used EBU (EWU) Life Distributions Based on the U- Test

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Abstract

The problem of testing exponentiality versus exponential better (worse) than used EBU(EWU) class of life distributions is considered through U- test. The percentiles of this test are tabulated for sample sizes $n = 5(1)50$. . The power estimates of the test are simulated for some commonly used distributions in reliability. Pitman's asymptotic efficiency of the test are calculated and compared. An applications of the proposed test statistic is given.

Key Words: EBU (EWU), exponentiality, efficiency, asymptotic normality.

1- Introduction

In reliability theory, various classes of life distribution have been introduced to describe several criteria of aging. Among the most well known families of life distributions are the classes of increasing failure rate (IFR), increasing faliare rate average (IFRA), new better than used (NBU), decreasing mean residual life (DMRL), harmonic new better than used in expectation (HNBUE). For some properties and interrelationships of these criteria we refer to Barlow and Proschan (1981) and Bryson and Siddiqui (1969).

The problem of testing exponentiality versus the classes (like IFR, IFRA, NBU, DMRL and HNBUE) of life distributions has seen a good deal of literature for example: Proschan and Pyke (1967), Ahmad (1994), Hollander and Proschan (1972 and 1975), Kanjo (1993) and Abu-Youssef (2002).

Definition (1.1): A life distribution F, with F(0) = 0, survival function $\overline{F}$ and finite mean μ is said to be EBU if

$$\overline{F}(x+t) \le \overline{F}(t)e^{-\frac{x}{\mu}}, \qquad x,t > 0 \qquad (1.1)$$

The dual class of life distributions that is EWU is reversing the inquality sign of relation (1.1).

Note that, the above definition is motivated by comparing the life length X_t of a component of age t with another new component of life length Y which is exponential with the same mean as X, this leads to X is EBU if and only if $X_t \leq_{st} Y$ *for all* $t \geq 0$. El-Batal (2002) introduced the above class of life distribution. He investigated their relationship to other classes of life distribution, closure properties under reliability operations, moment inequality and heritage property under shock model. The implication among EBU, NBUE and HNBUE classes of life distribution are

$$EBU \rightarrow NBUE \rightarrow HNBUE$$

The thread that connects most work mentioned here is that a measure of departure from H_0, which is often some weighted functional of F, is developed which is strictly positive under H_1 and is zero under H_0. Then, a sample version of this measure is used as test statistics and its properties are studied. In section 2, we propose a test statistic, based on the U- statistic for testing H_0 : F is exponential (μ) against H_1 : F is EBU (EWU) and not exponential.
We then present Monte Carlo null distribution critical points for sample sizes $n = 5(1)50$. In section 3 we calculate the efficiency of the test statistic for some common alternatives and compared them to other procedures. In section 4 we give simulated values of the power estimates of the test. Finally, in section 5 we consider an application in medical science, based on a set of data from Susarla and Vanryzin(1978).

2- Testing EBU(EWU) class of life distribution.

Let $X_1, X_2, ..., X_n$ represent a random sample from a population with distribution F. We wish to test the null hypothesis
H_0 : $\overline{F}$ is exponential with mean μ against
H_1 : $\overline{F}$ is $EBU(EWU)$ and not exponential, that is

$$\overline{F}(x+t) \leq \overline{F}(t) e^{-\frac{x}{\mu}}, \qquad x, t > 0 \qquad (1.1)$$

In order to test H_0 against H_1 we define the following measure of departure from H_0 as

$$\Delta_E = E[\overline{F}(t) e^{\frac{-x}{\mu}} - \overline{F}(x+t)] \qquad (2.1)$$

i.e

$$\Delta_E = \int_0^\infty [\frac{1}{2} e^{\frac{-x}{\mu}} - \int_0^\infty \overline{F}(x+t) dt] dx \qquad (2.2)$$

Note that under $H_0 : \Delta_E = 0$, while under $H_1 : \Delta_E > (<) 0$. Thus to estimate Δ_E by $\hat{\Delta}_{E_n}$, let $X_1, X_2, ..., X_n$ be a random sample from F and μ is estimated by $\overline{X}$, where $\overline{X} = \frac{1}{n}\sum X_i$ is the usual sample mean. Then $\hat{\Delta}_{E_n}$ is given by using (2.2) as

$$\hat{\Delta}_{E_n} = \frac{1}{n^3}\sum_i\sum_j\sum_k\left\{\frac{1}{2}e^{\frac{-X_i}{\overline{X}}} - I(X_j > X_i + X_k)\right\}, \qquad (2.3)$$

where

$$I(y > t) = \begin{cases} 1, & y > t \\ 0, & o.w \end{cases}$$

If we define $\phi(X_1, X_2, X_3) = \frac{1}{2}e^{\frac{-X_1}{\overline{X}}} - I(X_2 > X_1 + X_3)$, then $\hat{\Delta}_{E_n}$ in (2.3) is equivalent to the U- statistic given by

$$U_n = \frac{1}{\binom{n}{3}}\sum_{i<j<k}\varphi(X_i, X_j, X_k) \qquad (2.4)$$

The following Theorem summarizes the large sample properties of $\hat{\Delta}_{E_n}$.

Theorem 3.1. (i) As $n \to \infty$, $\sqrt{n}(U_n - \Delta_{E_n})$ is asymptotically normal with mean 0 and variance

$$\sigma^2 = Var[\frac{1}{2}e^{\frac{-x}{\mu}} + \int_0^\infty e^{\frac{-u}{\mu}}dF(u) - 2\int_0^\infty \overline{F}(X+u)dF(u) - \int_0^X \overline{F}(X-u)dF(u)] \cdot \qquad (2.5)$$

(ii) Under $H_0, \Delta_E = 0$ and $\sigma_0^2 = Var[\frac{-1}{2} + \frac{1}{2}e^{-X} + Xe^{-X}] = 0.0185$.

(iii) If F is continuous EBU, then the test is consistent.

Proof: (i) and (ii) follow from the standard theory of U- statistic cf. Lee(1990) by direct calculation. To prove Part (iii), let us write (2.1) in the following form

$$\Delta_E = \int_0^\infty\int_0^\infty [\overline{F}(t)e^{\frac{-x}{\mu}} - \overline{F}(x+t)]dF(t)dF(x) \qquad (2.6)$$

Let $D(x,t) = \overline{F}(t)e^{\frac{-x}{\mu}} - \overline{F}(x+t)$. Since F is EBU and continuous, then $D(x,t) > 0$ for at least one value of (x,t) call it (x_0, t_0).

Set $(x_1, t_1) = \inf\{(x,t) : x \le x_0, t \le t_0, \overline{F}(x) = \overline{F}(x_0)\ and\ \overline{F}(t) = \overline{F}(t_0)\}$.

Thus

$D(x_1, t_1) = \overline{F}(t_1)e^{\frac{-x_1}{\mu}}\ \overline{F}(x_1 + t_1) \ge \overline{F}(t_1)e^{\frac{-x_1}{\mu}} - \overline{F}(x_0 + t_0) = D(x_0, t_0) > 0$

and $F(x_1 + \delta) - F(x_1) > 0$ and since x_1 and t_1 are point of increase of F, thus $\Delta_E > 0$.

3- Monte carlo null distribution critical points for $\hat{\Delta}_{E_n}$ test.

In practices, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst. We have simulated the upper percentile points for 95%, 98%, 99%. Table (2.1) gives these percentile points of statistic $\hat{\Delta}_{E_n}$ in (2.3) and the calculations are based on 5000 simulated samples of sizes $n = 5(1)50$. The percentiles values change slowly as n increase.

Table (3.1) Critical Values of $\hat{\Delta}_{E_n}$

n	95%	98%	99%	n	95%	98%	99%
5	0.2869	0.3060	0.3180	23	0.2730	0.2804	0.2855
6	0.2872	0.3009	0.3111	24	0.2724	0.2789	0.2849
7	0.2857	0.2991	0.3097	25	0.2722	0.2792	0.2840
8	0.2838	0.2959	0.3055	26	0.2714	0.2770	0.2816
9	0.2819	0.2937	0.3004	27	0.2719	0.2788	0.2829
10	0.2814	0.2936	0.3003	28	0.2713	0.2772	0.2809
11	0.2806	0.2913	0.2990	29	0.2699	0.2761	0.2800
12	0.2789	0.2888	0.2979	30	0.2703	0.2760	0.2802
13	0.2781	0.2886	0.2957	32	0.2701	0.2768	0.2813
14	0.2774	0.2872	0.2952	34	0.2696	0.2756	0.2805
15	0.2772	0.2860	0.2934	36	0.2689	0.2743	0.2780
16	0.2769	0.2851	0.2913	38	0.2671	0.2724	0.2765
17	0.2764	0.2846	0.2902	40	0.2677	0.2741	0.2769
18	0.2754	0.2836	0.2892	42	0.2677	0.2725	0.2760
19	0.2753	0.2832	0.2888	44	0.2667	0.2711	0.2746
20	0.2745	0.2826	0.2879	46	0.2665	0.2716	0.2754
21	0.2748	0.2815	0.2862	48	0.2659	0.2711	0.2749
22	0.2733	0.2814	0.2860	50	0.2658	0.2711	0.2747

4- Asymptotic relative efficiency (Are)

Since the above test statistic $\hat{\Delta}_{E_n}$ in (2.3) is new and no other tests are known for these classes EBU. We compare this to some other classes. Here we choose the tests K^* and $\hat{\Delta}_n$ presented by Hollander and Proschan(1975) and Kango (1993) respectively for new better than used in expectation (NBUE) class. The comparisons are achieved by using Pitman asymptotic relative efficiency (PARE), which is defined as follows:

Let T_{1_n} and T_{2_n} be two statistics for testing $H_0 : F_\theta \in \{F_{\theta_n}\}, \theta_n = \theta + \frac{c}{\sqrt{n}}$

With c an arbitrary constant, then PARE of T_{1_n} relative to T_{2_n} is defined by

$$e(T_{1_n}, T_{2_n}) = \frac{\mu_1'(\theta_0)}{\sigma_1(\theta_0)} \Big/ \frac{\mu_2'(\theta_0)}{\sigma_2(\theta_0)}$$

where $\mu_i^{\cdot} = \lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_m)_{\to\theta_0}$ and $\sigma_\theta^2(\theta_0) = \lim_{n\to\infty} VarE(T_m)$, $i = 1,2$. Three of the most commonly used alternatives(cf. Hollander and Proschan (1972)) are:

(i) Linear failure rate family : $\overline{F}_{1\theta} = e^{-x-\frac{\theta x^2}{2}}$ $x > 0, \theta > 0$

(ii) Makeham family : $\overline{F}_{2\theta} = e^{-x-\theta(x+e^{-x}-1)}$, $x > 0, \theta > 0$

(iii) Weibull family : $\overline{F}_{3\theta} = e^{-x^\theta}$, $x \geq 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makeham families and $\theta = 1$ for Weibull family. Direct calculations of PAE of K^*, $\hat{\Delta}_n$ and $\hat{\Delta}_{E_n}$ are summarized in Table (2.2).

Table (2.2)

Distribution	K^*	$\hat{\Delta}_n$	$\hat{\Delta}_{E_n}$
F_1 linear failure rate	0.871	0.433	1.837
F_2 Makeham	0.289	0.088	0.966
F_3 Weibull	0.120	0144	1.2753

The efficiencies in Table (2.2) show clearly our U- statistic $\hat{\Delta}_{E_n}$ perform well for F_1, F_2 and F_3.

In Table (2.3) we give PARE's of $\hat{\Delta}_{E_n}$ with respect to V^* and $\hat{\Delta}_n$ whose PAE are mentioned in Table (2.2)

Table (2.3) PARE of $\hat{\Delta}_{E_n}$ with respect to V^* and $\hat{\Delta}_n$

Distribution	$e_{F_i}(\hat{\Delta}_{E_n}, \hat{\Delta}_n)$	$e_{F_i}(\hat{\Delta}_{E_n}, V^*)$
F_1 linear failure rate	2.109	4.24
F_2 Makeham	3.34	10.977
F_3 Weibull	10.62	8.85

It is clear from Table (2.3) that the statistic $\hat{\Delta}_{E_n}$ perform well for $\overline{F_1}, \overline{F_2}, \overline{F_3}$ and it is more efficient than both $\hat{\Delta}_n$ and V^* for all cases.

5- Numerical Example

Consider the data in Susarla and Van Ryzin (1978). These data represent 81 patients of melanoma, 44 of them represent whole life time (non-censored data) and the ordered values are: 13, 14, 19, 19, 20, 21, 23, 23, 25, 26, 26, 27, 27, 31, 32, 34, 34, 37, 38, 40, 46, 50, 53, 54, 57, 58, 59, 60, 65, 65, 66, 70, 85, 90, 98, 102, 103, 110, 118, 124, 130, 136, 138.

Using equation (2.3), the value of test statistics, based on the above data is $\hat{\Delta}_{E_n} = 0.2170$. This value leads to H_0 is not rejected [see Table (2.1)]. Therefore the data has not EBU Property.

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