Characterization Of Some Continuous Distributions Via The Conditional Variance

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We present new characterization using the first and second partial derivatives of the logarithm of the survival function in connection with the conditional variance to characterize several continuous probability distributions. An application of interest is also given.

1.Introduction

Several directions have been used to characterize probability distributions in terms of the connection between the conditional moments and the failure rates of the respective directions. See. e.g. Osaki and Li (1988) and Ahmed (1991) among others.

Ideas drawn from statistical inference have also been used to characterize probability distributions. In particular, a regression of a higher order statistic on a lower one in the same sample has been used by many authors. See, for example Dallas (1987) and Ahmed and Yehia (1993), among others. El-Arishy (1995) used the partial derivative of the logarithm of the survival function in connection with truncated moments to characterize several probability distributions. Based on the moments of upper record and K-th upper records and the recurrence relation between the single moments, characterizations of the linear exponential distribution are studied by EL-Sayed(2003). In (1997), Ahmad, Fakhry and Jaheen introduced a characterization of Burr distribution using the conditional variance.

Let Æ be the exponential class of continuous distribution having densities with respect to the lebesgue measure in the form

$$(1.1)\quad f(y,\theta) = \exp[y\, g(\theta) + s(y) + p(\theta)],\quad a < y < b$$

where a, and b don't involve θ, and s(y) is a continuous function of y. Let $\bar{F}(y,\theta) = 1 - F(y,\theta)$ be the survival function of the random variable y having a density in the form (1.1) . and $r(y) = \frac{f(y)}{\bar{F}(y,\theta)}$ be its failure rate function

In this paper, we use the relation between the conditional variance and the first and second partial derivatives of the logarithm of the survival function, which is extensively used in inference to characterize several probability distributions. Section 2. contains the main result. A characterization of Gamma distribution is given in section 3.

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2.Main results

In this section, it is worth noting that although the results of this section hold in general for all members of the exponential family of distributions, we shall consider only continuous distributions

Lemma 2.1

Let f(y) be as given in (1.1). Then

$$E(y) = -\frac{p'(\theta)}{g'(\theta)} = m(\theta)$$

and

$$\mathrm{Var}(y) = \sigma^2 = \frac{m'(\theta)}{g'(\theta)}$$

Proof: See El-Arishy (1995).

Theorem 2.2

A density f belongs to Æ if and only if

$$(2.3)\quad \mathrm{Var}(Y : Y \geq y) = \sigma^2 - \frac{g''(\theta)}{g'^3(\theta)}\frac{\partial \ln \bar{F}(y,\theta)}{\partial\theta} - \frac{1}{g'^2(\theta)}\frac{\partial^2 \ln \bar{F}(y,\theta)}{\partial\theta^2}$$

provided that $\bar{F}'(0,\theta) = \bar{F}'(\infty,\theta) = 0$.

Proof:

Let us first assume that (2.3) holds. Then by Lemma 2.1, it can be written as follows:

$$(2.4)\quad \bar{F}(y,\theta)\int_y^\infty u^2 f(u)d(u) - \left[\int_y^\infty uf(u)du\right]^2 = \frac{g''(\theta)p'(\theta) - g'(\theta)p''(\theta)}{g'^3(\theta)}\bar{F}^2(y,\theta) -$$

$$\frac{g''(\theta)}{g'^3(\theta)}\bar{F}(y,\theta)\frac{\partial\bar{F}(y,\theta)}{\partial\theta} + \frac{\bar{F}(y,\theta)}{g'^2(\theta)}\frac{\partial^2\bar{F}(y,\theta)}{\partial\theta^2} - \frac{1}{g'^2(\theta)}\left[\frac{\partial\bar{F}(y,\theta)}{\partial\theta}\right]^2$$

Differentiate both sides of (2.4) three times with respect to y, and use f, g, p instead of f (y, θ), g (θ), and p(θ) for simplicity, one gets

(2.5) $A\ \bar{F}(y,\theta)+B\ \bar{F}'(y,\theta)+C=0$

where

(2.6) $$A=2f-\frac{2}{g'^2f}\frac{\partial^2 f}{\partial\theta^2}\left[\frac{\partial f}{\partial y}\right]^2+\frac{2g''}{g'^3f^2}\frac{\partial f}{\partial\theta}\left[\frac{\partial f}{\partial y}\right]^2+\frac{1}{g'^2f}\frac{\partial^2 f}{\partial\theta^2}\frac{\partial^2 f}{\partial y^2}+\frac{g''}{g'^3}\frac{\partial^3 f}{\partial\theta\partial y^2}$$
$$-\frac{g''}{g'^3f}\frac{\partial f}{\partial\theta}\frac{\partial^2 f}{\partial y^2}+\frac{2}{g'^2f}\frac{\partial f}{\partial y}\frac{\partial^3 f}{\partial\theta^2\partial y}-\frac{2g''}{g'^3f}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial\theta\partial y}-\frac{1}{g'^2}\frac{\partial^4 f}{\partial\theta^2\partial y^2}.$$

(2.7) $$B=\frac{4}{g'^2f^2}\frac{\partial f}{\partial\theta}\left[\frac{\partial f}{\partial y}\right]^2-\frac{2}{g'^2f}\frac{\partial f}{\partial\theta}\frac{\partial^2 f}{\partial y^2}-\frac{4}{g'^2f}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial\theta\partial y}+\frac{2}{g'^2}\frac{\partial^3 f}{\partial\theta\partial y^2}$$

and

(2.8) $$C=\frac{4}{g'^2f}\frac{\partial f}{\partial y}\left[\frac{\partial f}{\partial y}\right]^2+\frac{g''}{g'^3}\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial y}-\frac{1}{g'^2}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial\theta^2}+\frac{3f}{g'^2}\frac{\partial^3 f}{\partial\theta^2\partial y}-\frac{3g''f}{g'^3}\frac{\partial^2 f}{\partial\theta\partial y}$$
$$-\frac{6}{g'^2}\frac{\partial f}{\partial\theta}\frac{\partial^2 f}{\partial\theta\partial y}+\frac{2f}{g'^3}\frac{\partial f}{\partial y}(g''p'-g'p'').$$

Without any loss of generality, assume that A, B, and C are finite for all choices of y, and θ. Now, let $y\to\infty$ in equation (2.5) this gives $\bar{F}(\infty,\theta)=0$ and $\bar{F}'(\infty,\theta)=0$; implaying that

(2.9) $C=0.$

Note, also, that if $y\to 0$, then we must have $\bar{F}(0,\theta)=1$, and $\bar{F}'(0,\theta)=0$; implying that

(2.10) $A=0.$

In this case, equation (2.5) reduces to

$$B\ \bar{F}'(y,\theta)=0 \qquad \forall\ y\ ,\theta$$

which is true only if

(2.11) $B=0.$

Differentiating equation (2.11) with respect to θ, we get

(2.12) $$2\frac{\partial^2 f}{\partial\theta^2}\left[\frac{\partial f}{\partial y}\right]^2+2\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y\partial\theta}+f\frac{\partial f}{\partial\theta}\frac{\partial^3 f}{\partial y^2\partial\theta}-\left[\frac{\partial f}{\partial\theta}\right]^2\frac{\partial^2 f}{\partial y^2}-f\frac{\partial^2 f}{\partial\theta^2}\frac{\partial^2 f}{\partial y^2}$$
$$-2f\left[\frac{\partial^2 f}{\partial y\partial\theta}\right]^2-2f\frac{\partial f}{\partial y}\frac{\partial^3 f}{\partial y\partial\theta^2}+f^2\frac{\partial^4 f}{\partial y^2\partial\theta^2}=0.$$

Solving equations (2.9), (2.10) and (2.12), one gets:

$$g'^2=\left[\frac{1}{f^2}\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial y}-\frac{1}{f}\frac{\partial^2 f}{\partial y\partial\theta}\right]^2$$

or

$$(2.13)\ g'(\theta)=\frac{1}{f(y,\theta)}\frac{\partial^2 f(y,\theta)}{\partial y\partial\theta}-\frac{1}{f^2(y,\theta)}\frac{\partial f(y,\theta)}{\partial\theta}\frac{\partial f(y,\theta)}{\partial y}=\frac{\partial^2 \ln f(y,\theta)}{\partial y\partial\theta}$$

and

$$(2.14)\ g''(\theta)=\frac{\partial^3 \ell n f(y,0)}{\partial y\partial\theta^3}$$

using (2.10) and (2.11) and substituting in equation (2.8) we have

$$g''(\theta)\left[p'(\theta)-\frac{\partial \ell n f(y,\theta)}{\partial\theta}\right]=g'(\theta)\left[p''(\theta)-\frac{\partial^2 \ell n f(y,\theta)}{\partial\theta^2}\right]$$

or

$$(2.15)\qquad \frac{\partial^2}{\partial\theta^2}[\ell n f(y,\theta)-p(\theta)]-\frac{g''(\theta)}{g'(\theta)}\frac{\partial}{\partial\theta}[\ell n f(y,\theta)-p(\theta)]=0$$

which is a second order partial differential equation whose solution is

$$(2.16)\qquad \ell n f(y,\theta)-p(\theta)=g(\theta)y+s(y)$$

where s(y) is some arbitrary function of y. But equation (2.16) is equivalent to

$$f(y,\theta)=\exp[g(\theta)y+p(\theta)+s(y)]$$

yielding that f belongs to Æ.

To prove the converse, we note that

$$(2.17)\qquad \bar{F}(y,\theta)=\int_y^\infty f(u,\theta)du=\int_y^\infty \exp[g(\theta)u+p(\theta)+s(y)]du$$

Differentiating both sides of (2.17) partially with respect to θ, gives:

$$(2.18)\qquad \frac{\partial\bar{F}}{\partial\theta}(y,\theta)=\int_y^\infty [ug'(\theta)+p'(\theta)]f(u,\theta)du=g'(\theta)\int_y^\infty uf(u,\theta)du+p'(\theta)\bar{F}(y,\theta)$$

Hence

$$(2.19)\qquad \int_y^\infty uf(u,\theta)du=\frac{1}{g'\theta}\frac{\partial\bar{F}(y,\theta)}{\partial\theta}-\frac{p'(\theta)}{g'(\theta)}\bar{F}(y,\theta)$$

Differentiating both sides of (2.18) again partially with respect to θ, gives:

$$\frac{\partial^2\bar{F}(y,\theta)}{\partial\theta^2}=\int_y^\infty((ug'(\theta)+p'(\theta))^2+ug''(\theta)+p''(\theta))f(u,\theta)du$$

$$=g'^2(\theta)\int_y^\infty u^2 f(u,\theta)du+\frac{g''(\theta)+2g'(\theta)p'(\theta)}{g'(\theta)}\frac{\partial\bar{F}(y,\theta)}{\partial\theta}$$

$$+\frac{\left[p''(\theta)\,g'(\theta)-g''(\theta)\,p'(\theta)-g'(\theta)p'^{2}(\theta)\right]\bar{F}(y,\theta)}{g'(\theta)}$$

or

$$(2.20)\qquad \int_{y}^{\infty}u^{2}f(u,\theta)\,du=\frac{1}{g'^{2}(\theta)}\frac{\partial^{2}\bar{F}(u,\theta)}{\partial\theta^{2}}-\frac{g''(\theta)+2g'(\theta)\;p'(\theta)}{g'^{3}(\theta)}\frac{\partial\bar{F}(y,\theta)}{\partial\theta}$$

$$+\frac{g''(\theta)\,p'(\theta)-p''(\theta)g'(\theta)+g'(\theta)p'^{2}(\theta)}{g'^{3}(\theta)}\bar{F}(y,\theta)$$

Using equation (2.19) and (2.20), implying that

$$Var(Y\mid Y\geq y)=\sigma^{2}-\frac{g''(\theta)}{g'^{3}(\theta)}\frac{\partial\ell n\bar{F}(y,\theta)}{\partial\theta}+\frac{1}{g'^{2}(\theta)}\frac{\partial^{2}\ell n\bar{F}(y,\theta)}{\partial\theta^{2}}\,.$$

Remark

$$Var(y)=\sigma^{2}$$

3-Application

3.1 Characterization for the gamma distribution

Lemma 3.1

(i) $\bar{F}(y,\alpha+1,\beta)=\bar{F}(y,\alpha,\beta)+\frac{y}{\alpha}f(y,\alpha,\beta)$

(ii) $f(y,\alpha+1,\beta)=\frac{\beta y}{\alpha}f(y,\alpha,\beta)$

Proof: see Osaki and Li (1988).

Theorem Let y be a non-negative continuous random variable with c.d.f. F(t), pdf f (y,θ) and mean μ, then y has a gamma distribution with c.d.f F(y,α,β) iff

$$(3.1)\qquad V(Y\mid Y\geq y)=\frac{\alpha}{\beta^{2}}+\frac{(\alpha-2\beta+1)y+\beta y^{2}}{\beta^{2}}r(y)-\frac{y^{2}}{\alpha^{2}}r^{2}(y)$$

for all y>0 .

proof

A. Necessity (see osaki and Li (1988)).

B.sufficiency:

Consider (3.1) holds and it can be written with unknown function f(y)

$$Var(Y \mid Y \geq y) = \frac{\int_y^\infty x^2 f(x)dx}{\bar{F}(y,\theta)} - \left[\frac{\int_y^\infty x f(x)dx}{\bar{F}(y,\theta)}\right]^2$$

$$= \frac{\alpha}{\beta^2} + \left[\frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2}\right] r(y) - \frac{y^2}{\alpha^2} r^2(y)$$

Or

$$(3.2)\ \bar{F}(y,\theta)\int_y^\infty x^2 f(x)d - \left[\int_y^\infty x f(x)dx\right]^2 = \frac{\alpha}{\beta^2}\bar{F}^2(y,\theta) + \frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2} f(y)\bar{F}(y,\theta) - \frac{y^2}{\alpha^2} f^2$$

Differentiate (3.2) three times with respect to y and after simplification one can get

$$(3.3)\qquad A\bar{F}(y,\theta) + B = 0$$

where

$$A = 2f^3 - 2\left(\frac{\alpha+1-2\beta+2\beta y}{\beta^2}\right) f f'^2 - 3\left[\frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2}\right] f f' f''$$

$$+\frac{2}{\beta} f^2 f' + 2\left(\frac{\alpha+1-2\beta+2\beta y}{\beta^2}\right) f^2 f'' - 2\left[\frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2}\right] f'^3$$
$$+\left[\frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2}\right] f^2 f'''$$

and

$$B = -\frac{6}{\beta} f^4 + \left(\frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2}\right) f^2 f'^2 - 5\left(\frac{1-2\beta+2\beta y}{\beta^2}\right) f^3 f'$$
$$-3\left(\frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2} - \frac{2y}{\alpha^2}\right) f^3 f'' + \frac{2y^2}{\alpha^2} f^3 f'''$$

Let $y \to \infty$ in Eq (3.3) then $\bar{F}(\infty,\theta) = 0$, implying B=0 and let $y \to 0$ then $\bar{F}(0,\theta) = 1$,implying A= 0.

Now, the equation A=0 can be written as following

$$\frac{(\alpha-2\beta+1)y+\beta y^2}{\beta^2}\frac{\partial^3 \ell nf}{\partial y^3} + 2\left(\frac{(\alpha-2\beta+1)+2\beta y}{\beta^2}\right)\frac{\partial^2 \ell nf}{\partial y^2} + \frac{2}{\beta}\frac{\partial \ell nf}{\partial y} + 2 = 0$$

Which is a partial differential equation of order 3 whose solution is

$$f(y) = C y^{\alpha-1} e^{-\beta y}$$

where C ,is a constant, can be determined by

$$\int_0^\infty f(y)dy=1$$

Thus , we get

$$f(y)=\frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{\beta y}$$

which is the gamma probability density function of y.

Remark when $\alpha=1$ then f is exponential iff -

$$var(Y/Y\geq y)=\frac{1}{\beta^2}+\left(\frac{2(1-\beta)y+\beta y^2}{\beta^2}\right)r(y)-y^2r^2(y)$$

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