BAYESIAN PREDICTION OF THE MEDIAN OF FUTURE OBSERVATIONS BASED ON FINITE MIXTURE MODELS

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ABSTRACT

Bayesian predictive density functions of the median of a set of odd and even number of future order statistics are obtained when the observations (informative and future) are assumed to follow a finite mixture of components of general form and type 1 censoring is imposed on the informative sample. The prior belief of the experimenter is measured by a general class of distributions which includes most priors used in literature. A mixture of two Weibull components model is given as an application. A numerical example presents Bayesian prediction bounds of the future median of observations based on a finite mixture of two exponential components.

Key Words: Type 1 censoring; Finite mixture; Two-sample prediction; Prediction intervals.

1. INTRODUCTION

Bayesian prediction of future order ststistics based on homogeneous populations, that can be represented by single-component distributions, have been investigeted by several authors. Among others, are Dunsmore [(1974), (1976)], Geisser [(1975), (1985), (1986), (1990), (1993)], Lingappaiah [(1978), (1979), (1980), (1986), (1989)], Howlader and Hossain (1995), Dunsmore and Amin (1998), AL-Hussaini and Jaheen [(1995), (1996), (1999)], AL-Hussaini [(1999)[(b)], (2001)[(a)]], Lee and Lio (1999), Corcuera and Giummolè (1999) and Johnson, Evans and Green (1999). Prediction of future records has been considered by AL-Hussaini and Ahmad (2003). Bayesian prediction of future median based on a homogeneous population has been studied, in the nonparametric setting, by Guilbaud (1983) and by AL-Hussaini and Jaheen (1999), AL-Hussaini (2001)[(b)] in the parametric case. For heterogeneous populations, that can be represented by finite mixtures of distributions, certain problems in Bayesian prediction have been studied in AL-Hussaini [(1999)[(a)], (2003)] and AL-Hussaini, Nigm and Jaheen (2001).

Applications of finite mixtures in different disciplines are numerous. Examples of such applications may be found in Everitt and Hand (1981), Titterington, Smith and Makov (1985), McLachlan and Basford (1988) and Lindsay (1995). Applications to reliability and hazard were presented by AL-Hussaini and Sultan (2001).

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Suppose that a heterogeneous population can be described by a finite mixture of k components whose density function is given by

$$f(x) = \sum_{i=1}^{k} p_i f_i(x) , \tag{1.1}$$

where the mixing proportions p_i are such that $0 \le p_i \le 1$, $\sum_{i=1}^{k} p_i = 1$ and the i^{th} component $f_i(x)$ represents the density function of the i^{th} subpopulation. Suppose that, for $i = 1, ..., k$, $x > 0$,

$$f_i(x) = \lambda_i'(x) exp[-\lambda_i(x)] , \tag{1.2}$$

where $\lambda_i'(x)$ is the derivative of $\lambda_i(x)$ with respect to x, which is assumed to exist, and

$$\lambda_i(x) \equiv \lambda_i(x; \theta) , \ \theta \in \Theta . \tag{1.3}$$

The function $\lambda_i(x)$ is chosen so that $\lambda_i(x) \to 0$ as $x \to 0^+$ and $\lambda_i(x) \to \infty$ as $x \to \infty$. It may be noted that the i^{th} component $f_i(x)$, given by (1.2), is composed of the product of the hazard rate function (HRF) $\lambda_i'(x)$ and survival function (SF) $exp[-\lambda_i(x)]$. This general probability density function (PDF) includes, among others, the Weibull, compound Weibull (or three-parameter Burr type XII) , Pareto, beta, Gompertz and compound Gompertz distributions, [see AL-Hussaini and Osman (1997)].

It was pointed out, in AL-Hussaini (2001)[(b)], that the class of all finite mixtures, given by (1.1) and (1.2), is identifiable, provided that $\lambda_1(x), ..., \lambda_k(x)$ are distinct. For the concept of identifiability and details, see Maritz and Lwin (1989) or any of the above references on mixtures.

A general iteration scheme was developed in AL-Hussaini and Osman (1997) to compute the median of a finite mixture of k components whose PDF is given by (1.1) and (1.2), with a slight parameter modification. From now on, we shall restrict our study to the special case of only $k = 2$ components, so that

$$f(x) = p_1 f_1(x) + p_2 f_2(x) , \tag{1.4}$$

where $f_i(x)$ is given by (1.2).

It is assumed that n items are subjected to a life testing experiment and that r units have failed during the interval $(0,x_0)$ (type 1 censoring): r_1 units from the first subpopulation and r_2 units from the second subpopulation such that $r=r_1+r_2$ and $n - r$ units, which cannot be identified as to subpopulation are still functioning. Suppose that the two sub-populations have density functions $f_1(x)$ and $f_2(x)$, given by (1.2), mixed with proportions p_1 and p_2=(1-p_1), so that the population is described by a mixture whose PDF is given by (1.4). The corresponding cumulative distribution function (CDF) and SF are given by

$$F(x) = p_1 F_1(x) + p_2 F_2(x) , \tag{1.5}$$

$$R(x) = p_1 R_1(x) + p_2 R_2(x) , \tag{1.6}$$

where, for $i = 1, 2$, $F_i(x)$ and $R_i(x)$, corresponding to $f_i(x)$, are given, respectively, by

$$F_i(x) = 1 - exp[-\lambda_i(x)] , \ R_i(x) = exp[-\lambda_i(x)] . \tag{1.7}$$

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Let x_{ij} denote the failure time of the j^{th} unit belonging to the i^{th} subpopulation and that $x_{ij} \le x_0$, $j = 1, ..., r_i$. The likelihood function is given, in Mendenhall and Hader (1958), by

$$L(\theta; \underline{x}) \propto [\prod_{j=1}^{r_1} p_1 f_1(x_{1j})][\prod_{j=2}^{r_2} p_2 f_2(x_{2j})][R(x_0)]^{n-r} .$$

Suppose that the prior belief of the experimenter is measured by a general density function, denoted by $\pi(\theta; \gamma)$ of the form

$$\pi(\theta; \gamma) \propto C(\theta; \gamma) exp[-D(\theta; \gamma)] , \theta \in \Theta , \gamma \in \Omega . \tag{1.8}$$

It was shown in AL-Hussaini (2003), by using (1.2), (1.6), (1.7), L and π, that the posterior density function, denoted by $\pi^*(\theta \mid \underline{x})$ is given by

$$\pi^*(\theta \mid \underline{x}) \propto \sum_{j_1=0}^{n-r} \eta_{j_1}(\theta; \underline{x}) exp[-\zeta_{j_1}(\theta; \underline{x})] , \tag{1.9}$$

where

$$\eta_{j_1}(\theta; \underline{x}) = Q_{j_1}(\theta; \underline{x}) C(\theta; \gamma) \quad , \quad \zeta_{j_1}(\theta; \underline{x}) = S_{j_1} + D(\theta; \gamma) , \tag{1.10}$$

$$Q_{j_1}(\theta; \underline{x}) = \binom{n-r}{j_1} p_1^{r_1+j_1} p_2^{n-r_1-j_1} \phi_1(\theta)\phi_2(\theta) , \tag{1.11}$$

$$S_{j_1}(\theta; \underline{x}) = j_1 \lambda_1(x_0) + (n - r - j_1)\lambda_2(x_0) + \psi_1(\theta) + \psi_2(\theta) , \tag{1.12}$$

$\underline{x} = (x_{11}, ..., x_{1r_1}, x_{21}, ..., x_{2r_2})$ and, for $i = 1, 2$,

$$\phi_i(\theta) = \prod_{j=1}^{r_i} \lambda_i'(x_{ij}) , \quad \psi_i(\theta) = \sum_{j=1}^{r_i} \lambda_i(x_{ij}) . \tag{1.13}$$

It is assumed, for $r = r_1 + r_2$, that $x_r = max\{x_{ij}\} \le x_0$, where x_r and x_{ij} are realizations of the random variables X_r and X_{ij}, $i = 1, 2$ and $j = 1, ..., r_i$ and x_0 is the censoring time.

Let Y_s denote the ordered lifetime of the s^{th} unit to fail in a future sample of size m, $1 \le s \le m$, drawn from a population whose density function is given by (1.4) and (1.2). It was shown in AL-Hussaini (2003) that the density function of Y_s is given by

$$f_{Y_s}(y \mid \theta) = A_1 \Sigma_1 K_1 p_1^{j_3} p_2^{\omega} exp[-\{j_3 \lambda_1(y) + \omega \lambda_2(y)\}][p_1 \lambda_1'(y)$$

$$exp\{-\lambda_1(y)\} + p_2 \lambda_2'(y) exp\{-\lambda_2(y)\}] , y > 0 , \tag{1.14}$$

where A_1 is a normalizing constant, $\omega = m - s + j_2 - j_3$,

$$\Sigma_1 = \sum_{j_2=0}^{s-1} \sum_{j_3=0}^{m-s+j_2} , \quad K_1 = (-1)^{j_2} \binom{s-1}{j_2} \binom{m-s+j_2}{j_3} . \tag{1.15}$$

So that the predictive density function, denoted by $f_{Y_s}(y \mid \underline{x})$, was shown to be

$$f_{Y_s}^*(y \mid \underline{x}) = A_1^* \Sigma_1^* K_1 [I_1 + I_2] , y > 0 , \tag{1.16}$$

where A_1^* is a normalizing constant, K_1 is given by (1.15),

$$\Sigma_1^* = \sum_{j_1=0}^{n-r} \Sigma_1 = \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{s-1} \sum_{j_3=0}^{m-s+j_2} , \tag{1.17}$$

and, for $\ell = 1, 2$

$$I_\ell(y) = \int_\Theta C_\ell^*(\theta; y) exp[-D_\ell^*(\theta; y)] dy , \tag{1.18}$$

$$\begin{aligned} C_1^*(\theta; y) &= p_1^{j_3+1} p_2^{\omega} \lambda_1'(y) \eta_{j_1}(\theta; \underline{x}), D_1^*(\theta; y) = \zeta_{j_1}(\theta; \underline{x}) + (j_3+1)\lambda_1(y) + \omega\lambda_2(y), \\ C_2^*(\theta; y) &= p_1^{j_3} p_2^{\omega+1} \lambda_2'(y) \eta_{j_1}(\theta; \underline{x}), D_2^*(\theta; y) = \zeta_{j_1}(\theta; \underline{x}) + j_3\lambda_1(y) + (\omega+1)\lambda_2(y). \end{aligned} \tag{1.19}$$

The median of m observations, denoted by $\tilde{Y}_m$, is defined by

$$\tilde{Y}_m = \begin{cases} Y_{k:m} & , \quad m = 2k-1, \\ \frac{1}{2}[Y_{k:m} + Y_{k+1:m}] & , \quad m = 2k, \end{cases} \tag{1.20}$$

where k is a positive integer ≥ 1.

2. PREDICTIVE DENSITY FUNCTIONS OF FUTURE MEDIAN

2.1. The Case of Odd m

It is easy to obtain the density function of $\tilde{Y}_m$, for a given θ, when m is odd, by noticing, from (1.20), that in this case the median of $m = 2k - 1$ observations is simply the k^{th} order statistic. So, by substituting $s = k$ and $m = 2k - 1$ in (1.14), we obtain the density function $f_{\tilde{Y}_{2k-1}}(y \mid \theta)$ of the median of $m = 2k - 1$ observations, where A_1 is a normalizing constant, $\Sigma_1 = \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{k-1+j_3}$, $K_1 = (-1)^{j_2} \binom{k-1}{j_2} \binom{k-1+j_2}{j_3}$ and $\omega = k - 1 + j_2 - j_3$.

Similarly, the predictive density function of the future median, $f^*_{\tilde{Y}_{2k-1}}(y \mid \underline{x})$ takes the form (1.16) with $s = k$ and $m = 2k - 1$. In such case, A_1^* is the normalizing constant, K_1 and ω are given above and $\Sigma_1^* = \sum_{j_1=0}^{n-r} \Sigma_1$. The integrals $I_\ell(y)$, $\ell = 1, 2$ are given by (1.18) with the constituents $C_\ell^*(\theta; y)$, $D_\ell^*(\theta; y)$ as in (1.19).

2.2. The Case of Even m

2.2.1. Density function of the median

If $m = 2k$, then the density function of the median of m observations is given, for $y > 0$, by

$$f_{\tilde{Y}_{2k}}(y \mid \theta) = D_2(k) \Sigma_2 K_2 I_\ell(y \mid \theta) , \tag{2.1}$$

where

$$D_2(k) = \frac{2(2k)!}{[(k-1)!]^2} , \quad \Sigma_2 = \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{j_2} \sum_{j_4=0}^{k-1} \sum_{\ell=1}^{4} , \tag{2.2}$$

$$K_2 = (-1)^{j_2} \binom{k-1}{j_2} \binom{j_2}{j_3} \binom{k-1}{j_4} , \tag{2.3}$$

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and, for $\ell = 1, 2, 3, 4,$

$$I_\ell(y \mid \theta) = G_\ell(p_1) \int_0^y u_\ell(t, 2y-t) exp[-v_\ell(t, 2y-t)] dt \ , \tag{2.4}$$

$$G_1(p_1) = p_1^{\omega_1+2} p_1^{\omega_2}, G_2(p_1) = G_3(p_1) = p_1^{\omega_1+1} p_2^{\omega_2+1}, G_4(p_1) = p_1^{\omega_1} p_2^{\omega_2+2}, \tag{2.5}$$

$$\begin{aligned} u_1(t, 2y-t) &= \lambda_1'(t)\lambda_1'(2y-t) \ , \ u_2(t, 2y-t) = \lambda_1'(t)\lambda_2'(2y-t), \\ u_3(t, 2y-t) &= \lambda_2'(t)\lambda_1'(2y-t) \ , \ u_4(t, 2y-t) = \lambda_2'(t)\lambda_2'(2y-t), \end{aligned} \tag{2.6}$$

$$\begin{aligned} v_1(t, 2y-t) &= (j_3+1)\lambda_1(t) + (j_2-j_3)\lambda_2(t) + (j_4+1)\lambda_1(2y-t) + \\ &\quad (k-1-j_4)\lambda_2(2y-t), \\ v_2(t, 2y-t) &= (j_3+1)\lambda_1(t) + (j_2-j_3)\lambda_2(t) + j_4\lambda_1(2y-t) + \\ &\quad (k-j_4)\lambda_2(2y-t), \end{aligned}$$

$$\begin{aligned} v_3(t, 2y-t) &= j_3\lambda_1(t) + (j_2-j_3+1)\lambda_2(t) + (j_4+1)\lambda_1(2y-t) + \\ &\quad (k-1-j_4)\lambda_2(2y-t), \\ v_4(t, 2y-t) &= j_3\lambda_1(t) + (j_2-j_3+1)\lambda_2(t) + j_4\lambda_1(2y-t) + \\ &\quad (k-j_4)\lambda_2(2y-t), \end{aligned} \tag{2.7}$$

$$\omega_1 = j_3 + j_4 \qquad , \qquad \omega_2 = k - 1 + j_2 - j_3 - j_4 \ . \tag{2.8}$$

For proof, see APPENDIX A.

2.2.2. Predictive density function of future median

The predictive density function of the future median of $m = 2k$ observations is given by

$$f^*_{\tilde{Y}_{2k}}(y \mid \underline{x}) = \int_\Theta f_{\tilde{Y}_{2k}}(y \mid \theta)\pi^*(\theta \mid \underline{x}) d\theta \ ,$$

where $f_{\tilde{Y}_{2k}}(y \mid \theta)$ is given by (2.1) and the posterior density function $\pi^*(\theta \mid \underline{x})$ by (1.9). It then follows that the predictive density function $f^*_{\tilde{Y}_{2k}}(y \mid \underline{x})$ is given, for $y > 0$, by

$$f^*_{\tilde{Y}_{2k}}(y \mid \underline{x}) = A_2^* \Sigma_2^* K_2 I_\ell^*(y \mid \underline{x}) \ , \tag{2.9}$$

where A_2^* is a normalizing constant, $\Sigma_2^* = \sum_{j_1=0}^{n-r} \Sigma_2$, Σ_2 is given by (2.2), K_2 by (2.3) and, for $\ell = 1, 2, 3, 4,$

$$I_\ell^*(y \mid \underline{x}) = \int_\Theta \int_0^y u_\ell^*(t, y; \theta) exp[-v_\ell^*(t, y, ; \theta)] dt d\theta \ , \tag{2.10}$$

$$u_\ell^*(t, y; \theta) = G_\ell(p_1) u_\ell(t, 2y-t) \eta_{j_1}(\theta; \underline{x}), v_\ell^*(t, y; \theta) = v_\ell(t, 2y-t) + \zeta_{j_1}(\theta; \underline{x}), \tag{2.11}$$

$G_\ell(p_1)$, $u_\ell(t, 2y-t)$ and $v_\ell(t, 2y-t)$ are given by (2.5), (2.6), (2.7) and $\eta_{j_1}(\theta; \underline{x})$, $\zeta_{j_1}(\theta; \underline{x})$ by (1.10).

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3. A SPECIALIZATION

3.1. Weibull Components

Suppose that, for $i = 1,2$ and $x > 0$, $\lambda_i(x) = \theta_i x^{\beta_i}$, so that $\lambda_i'(x) = \theta_i \beta_i x^{\beta_i - 1}$. In this case, the i^{th} subpopulation is Weibull(θ_i, β_i), $\theta_i, \beta_i > 0$. It is assumed that β_1 and β_2 are known and that a prior density is of the form (1.8) where $C(\theta,\gamma) = p_1^{a_1-1} p_2^{a_2-1} \theta_1^{b_1-1} \theta_2^{b_2-1}$, $D(\theta,\gamma) = \gamma_1\theta_1 + \gamma_2\theta_2$, $p_2 = 1 - p_1$, $\theta = (\theta_1, \theta_2, p_1)$ and $\gamma = (a_1, a_2, b_1, b_2, \gamma_1, \gamma_2)$. This prior assumes the independence of θ_1, θ_2 and p_1, where $\theta_i \sim gamma(b_i, \gamma_i)$ and $p_1 \sim beta(a_1, a_2)$.
It follows, from (1.11), (1.12) and (1.13), that

$$Q_{j_1}(\theta, \underline{x}) \propto \binom{n-r}{j_1} p_1^{r_1+j_1} p_2^{n-r_1-j_1} \theta_1^{r_1} \theta_2^{r_2} \ , \quad S_{j_1}(\theta, \underline{x}) = c_1\theta_1 + c_2\theta_2 \ ,$$

where

$$c_1 = j_1 x_0 + \sum_{j=0}^{r_1} x_{1j} \ , \quad c_2 = (n - r - j_1)x_0 + \sum_{j=0}^{r_2} x_{2j} \ . \tag{3.1}$$

So that, from (1.10),

$$\eta_{j_1}(\theta, \underline{x}) \propto \binom{n-r}{j_1} p_1^{\delta_1-1} p_2^{\delta_2-1} \theta_1^{\eta_1-1} \theta_2^{\eta_2-1} \ , \quad \zeta_{j_1}(\theta, \underline{x}) = \xi_1\theta_1 + \xi_2\theta_2 \ ,$$

$$\delta_1 = r_1 + j_1 + a_1, \delta_2 = n - r_1 - j_1 + a_2, \eta_i = r_i + b_i, \xi_i = c_i + \gamma_i, (i = 1,2) \ . \tag{3.2}$$

3.1.1. When $m = 2k - 1$

It follows, from (1.14) that the density function of $\tilde{Y}_{2k-1}$, is given by

$$f_{\tilde{Y}_{2k-1}}(y \mid \theta) = A_1 \Sigma_1 K_1 p_1^{j_3} p_2^{\omega} exp[-\{j_3\theta_1 y^{\beta_1} + \omega\theta_2 y^{\beta_2}\}][p_1\theta_1\beta_1 y^{\beta_1-1}$$

$$exp\{-\theta_1 y^{\beta_1}\} + p_2\theta_2\beta_2 y^{\beta_2-1}) exp\{-\theta_2 y^{\beta_2}\}] \ , \ y > 0 \ , \tag{3.3}$$

where A_1 is a normalizing constant, Σ_1 and K_1 are given by (1.15) with $s = k$ and $m = 2k - 1$.
From (1.16), the predictive density function is given, for $y > 0$, by

$$f^*_{\tilde{Y}_{2k-1}}(y \mid \underline{x}) = A_1^* \Sigma_1^* K_1^* [I_1^*(y) + I_2^*(y)] \ , \tag{3.4}$$

where

$$\Sigma_1^* = \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{k-1+j_2} \ , \tag{3.5}$$

$$K_1^* = K_1 \binom{n-r}{j_1} \frac{B(j_3 + \delta_1, \omega + \delta_2)\Gamma(\eta_1)\Gamma(\eta_2)}{j_3 + \omega + \delta_1 + \delta_2} \ , \tag{3.6}$$

K_1 is given by (1.15) and $B(.,.)$ is the standard beta function.

$$I_1^*(y) = \frac{\eta_1(j_3 + \delta_1)\beta_1 y^{\beta_1-1}}{B_{11}^{\eta_1+1} B_{12}^{\eta_2}} \ , \ I_2^*(y) = \frac{\eta_2(\omega + \delta_2)\beta_2 y^{\beta_2-1}}{B_{21}^{\eta_1} B_{22}^{\eta_2+1}} \ , \tag{3.7}$$

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where

$$B_{11}=\xi_1+(j_3+1)y^{\beta_1}\quad,\quad B_{12}=\xi_2+\omega y^{\beta_2}\quad,$$
$$B_{21}=\xi_1+j_3y^{\beta_1}\quad,\quad B_{22}=\xi_2+(\omega+1)y^{\beta_2}\ , \tag{3.8}$$

3.1.2. When m=2k

In this case, the density function of future median is given, for $y>0$, by (2.1) where $D_2(k)$ and Σ_2 are given by (2.2), K_2 by (2.3) and for $\ell=1,2,3,4$, $I_\ell(y\mid\theta)$ is given by (2.4), $G_\ell(p_1)$ by (2.5),
$u_1(t,2y-t)=\theta_1^2\beta_1^2t^{\beta_1-1}(2y-t)^{\beta_1-1}$,
$u_2(t,2y-t)=\theta_1\theta_2\beta_1\beta_2t^{\beta_1-1}(2y-t)^{\beta_2-1}$,
$u_3(t,2y-t)=\theta_1\theta_2\beta_1\beta_2t^{\beta_2-1}(2y-t)^{\beta_1-1}$,
$u_4(t,2y-t)=\theta_2^2\beta_2^2t^{\beta_2-1}(2y-t)^{\beta_2-1}$,
and $v_\ell(t,2y-t)=D_{\ell1}\theta_1+D_{\ell2}\theta_2$,
where
$D_{11}=(j_3+1)t^{\beta_1}+(j_4+1)(2y-t)^{\beta_1}$, $D_{12}=(j_2-j_3)t^{\beta_2}+(k-2-j_4)(2y-t)^{\beta_2}$,
$D_{21}=(j_3+1)t^{\beta_1}+j_4(2y-t)^{\beta_1}$, $D_{22}=(j_2-j_3)t^{\beta_2}+(k-1-j_4)(2y-t)^{\beta_2}$,
$D_{31}=j_3t^{\beta_1}+(j_4+1)(2y-t)^{\beta_1}$, $D_{32}=(j_2-j_3+1)t^{\beta_2}+(k-2-j_4)(2y-t)^{\beta_2}$,
$D_{41}=j_3t^{\beta_1}+j_4(2y-t)^{\beta_1}$, $D_{42}=(j_2-j_3+1)t^{\beta_2}+(k-1-j_4)(2y-t)^{\beta_2}$.

The predictive density function of future median is given, for $y>0$, by (2.9), where for $\ell=1,2,3,4$, $G_\ell(p_1)$ is as in (2.5),
$u_1^*(t,y;\theta)=\binom{n-r}{j_1}p_1^{\delta_1+\omega_1+1}p_2^{\delta_2+\omega_2-1}\theta_1^{\eta_1+1}\theta_2^{\eta_2-1}\beta_1^2t^{\beta_1-1}(2y-t)^{\beta_1-1}$,
$u_2^*(t,y;\theta)=\binom{n-r}{j_1}p_1^{\delta_1+\omega_1}p_2^{\delta_2+\omega_2}\theta_1^{\eta_1}\theta_2^{\eta_2}\beta_1\beta_2t^{\beta_1-1}(2y-t)^{\beta_2-1}$,
$u_3^*(t,y;\theta)=\binom{n-r}{j_1}p_1^{\delta_1+\omega_1}p_2^{\delta_2+\omega_2}\theta_1^{\eta_1}\theta_2^{\eta_2}\beta_1\beta_2t^{\beta_2-1}(2y-t)^{\beta_1-1}$,
$u_4^*(t,y;\theta)=\binom{n-r}{j_1}p_1^{\delta_1+\omega_1-1}p_2^{\delta_2+\omega_2+1}\theta_1^{\eta_1-1}\theta_2^{\eta_2+1}\beta_2^2t^{\beta_2-1}(2y-t)^{\beta_2-1}$,
and, for $\ell=1,2,3,4$ and $i=1,2$,
$v_\ell^*(t,y;\theta)=H_{\ell1}\theta_1+H_{\ell2}\theta_2$, $H_{\ell i}=\xi_i+D_{\ell i}$, ξ_i is given by (3.2)
By substituting in (2.9) and simplifying, we finally obtain the predictive density function in the form

$$f^*_{Y_{2k}}(y\mid\underline{x})=A_2^*\Sigma_2^*K_2^*I_\ell^{**}(y)\ , \tag{3.9}$$

where $\Sigma_2^*=\sum_{j_1=0}^{n-r}\Sigma_2$, Σ_2 is as in (2.2),

$$K_2^*=K_2\binom{n-r}{j_1}B(\delta_1+\omega_1+1,\delta_2+\omega_2+1)\Gamma(\eta_1+1)\Gamma(\eta_2+1)\ , \tag{3.10}$$

K_2 is given by (2.3) and $I_\ell^{**}(y)\equiv I_\ell^{**}(y\mid\underline{x})$,

$$I_1^{**}(y)=\frac{(\eta_1+1)(\delta_1+\omega_1+1)}{\eta_2(\delta_2+\omega_2)}\int_0^y\frac{\beta_1^2t^{\beta_1-1}(2y-t)^{\beta_1-1}}{H_{11}^{\eta_1+2}H_{12}^{\eta_2}}dt\ ,$$
$$I_2^{**}(y)=\int_0^y\frac{\beta_1\beta_2t^{\beta_1-1}(2y-t)^{\beta_2-1}}{H_{21}^{\eta_1+1}H_{22}^{\eta_2+1}}dt\ ,$$
$$I_3^{**}(y)=\int_0^y\frac{\beta_1\beta_2t^{\beta_2-1}(2y-t)^{\beta_1-1}}{H_{31}^{\eta_1+2}H_{32}^{\eta_2}}dt\ ,$$
$$I_4^{**}(y)=\frac{(\eta_2+1)(\delta_2+\omega_2+1)}{\eta_1(\delta_1+\omega_1)}\int_0^y\frac{\beta_2^2t^{\beta_2-1}(2y-t)^{\beta_2-1}}{H_{41}^{\eta_1}H_{42}^{\eta_2+2}}dt\ . \tag{3.11}$$

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3.2. Predictive Survival Functions and Predictive Intervals

The predictive survival function (PSF) of the future median $\tilde{Y}_m$ is given by

$$P[\tilde{Y}_m > \nu \mid \underline{x}] = \int_{\nu}^{\infty} f_{\tilde{Y}_m}*(y \mid \underline{x})dy .$$

By substituting $f_{\tilde{Y}_m}*(y \mid \underline{x})$, when m is odd, given by (1.16), with $s = k$ and $m = 2k-1$, we obtain

$$P[\tilde{Y}_{2k-1} > \nu \mid \underline{x}] = \frac{\Sigma_1^* K_1^*[J_1^*(\nu) + J_2^*(\nu)]}{\Sigma_1^* K_1^*[J_1^*(0) + J_2^*(0)]}, \tag{3.12}$$

where, for $\ell = 1,2$, $J_\ell^*(\nu) = \int_\nu^\infty I_\ell^*(y)dy$, $I_\ell^*(y)$ is given by (3.7).

By substituting $f_{\tilde{Y}_m}*(y \mid \underline{x})$, when $m = 2k$ is even, given by (2.9), we obtain

$$P[\tilde{Y}_{2k} > \nu \mid \underline{x}] = \frac{\Sigma_2^* K_2^* J_\ell^{**}(\nu)}{\Sigma_2^* K_2^* J_\ell^{**}(0)}, \tag{3.13}$$

where, for $\ell = 1,2,3,4$, $J_\ell^{**}(\nu) = \int_\nu^\infty I_\ell^{**}(y)$, $I_\ell^{**}(y)$ is given by (3.11).

Observe that $P[\tilde{Y}_m > 0 \mid \underline{x}] = 1$, so that $A_1^* = \{\Sigma_1^* K_1^*[J_1^*(0) + J_2^*(0)]\}^{-1}$ and $A_2^* = \{\Sigma_2^* K_2^* J_\ell^{**}(0)\}^{-1}$.

Lower and upper bounds, L and U, of a $100\tau\%$ predictive interval of future median $\tilde{Y}_m$ are obtained by solving the following two equations for L and U:

$$\frac{1+\tau}{2} = P[\tilde{Y}_m > L \mid \underline{x}], \quad \frac{1-\tau}{2} = P[\tilde{Y}_m > U \mid \underline{x}], \tag{3.14}$$

where, for $\nu > 0$, $P[\tilde{Y}_m > \nu \mid \underline{x}]$ is the PSF , given by (3.12) and (3.13), respectively.

3.3. Special Cases

In the above application of a mixture of two Weibull components model, the parameters β_i were assumed to be known. Two important special cases may be obtained by setting $\beta_i = 1, 2$ in the Weibull(θ_i, β_i) components.

The first case is that of Weibull$(\theta_i, \beta_i = 1)$ components, $i = 1, 2$, in which $\lambda_i(x) = \theta_i x$. This leads to a mixture of two exponential(θ_i), $i = 1, 2$, components.

The second case is that of Weibull$(\theta_i, \beta_i = 2)$ components, i=1,2, in which $\lambda_i(x) = \theta_i x^2$. This leads to a mixture of two Rayleigh(θ_i), $i = 1, 2$, components.

The applications of the exponential, Rayleigh and Weibull distributions are numerous, see for example Johnson, Kotz and Balakrishnan (1994). All of the results obtained under a mixture of Weibull(θ_i, β_i) components regarding the predictive densities, survival functions and predictive intervals can be specialized to the exponential(θ_i) and Rayleigh(θ_i) components by setting $\beta_i = 1, 2$ in the Weibull(θ_i, β_i) components, respectively.

4. NUMERICAL EXAMPLE

In this section, an example is given to illustrate how Bayesian prediction bounds for the median of m future observations are obtained when the underlying population distribution is a mixture of two exponential components. In this case, the predictive density functions are given by (3.4) and (3.9) in the odd and even cases, respectively, where β_1 and β_2 are set to be equal to 1 in (3.7), (3.8), (3.11) and the functions $H_{\ell i}$, $\ell = 1,2,3,4$ and $i = 1,2$.

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In this example, the elements of the vector of parameters $\theta = (\theta_1 = 2.23, \theta_2 = 2.35, p_1 = 0.35)$ are generated from gamma($b_1 = 1.8, \gamma_1 = 5.7$), gamma($b_2 = 5.22, \gamma_2 = 2.17$) and beta($a_1 = 1.7, a_2 = 3.5$) distributions, respectively, where the elements of the vector of hyperparameters $\gamma = (a_1, a_2, b_1, b_2, \gamma_1, \gamma_2)$ have known given values. If the hyperparameters are unknown, they could be estimated, based on 'past samples' using empirical Bayes method [see, for example, Maritz and Lwin (1989)] or by using hierarchical Bayes method [see, for example, Bernardo and Smith (1994) or Geisser (1990)].

For $n = 20$, the following sample is generated from a mixture of two exponential components with parameters $\theta_1 = 2.23$, $\theta_2 = 2.35$ and $p_1 = 0.35$, where the censoring time is $x_0 = 6.5$:
0.1573,0.2914,0.3527,0.3981,0.4213,0.5213,0.7112,0.7932,0.9225,1.2113,2.5317 ($r_1 = 11$),
3.1792,4.5170,5.5117,6.3790,6.4230 ($r_2 = 5$).

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Table(1): Odd m

m	k	L	U	Length
3	2	0.2101	0.5622	0.3521
5	3	0.2955	0.7176	0.4221
7	4	0.3501	0.8418	0.4917
9	5	0.3922	0.9155	0.5233
11	6	0.5320	1.2632	0.6212
13	7	0.6119	1.5677	0.9558
15	8	0.7312	2.1039	1.3727
17	9	0.8503	2.3515	1.5012
19	10	1.1331	3.0432	1.9101

Table(2): Even m

m	k	L	U	Length
2	1	0.1719	0.5440	0.3721
4	2	0.2133	0.6288	0.4155
6	3	0.2944	0.7867	0.4923
8	4	0.3371	0.8472	0.5101
10	5	0.4411	1.0133	0.5722
12	6	0.5100	1.1351	0.6251
14	7	0.6831	1.4964	0.8133
16	8	0.7737	1.9892	1.2156
18	9	0.8935	2.5060	1.6125
20	10	1.1311	2.9411	1.8111

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Equations (3.14) are solved, when $\tau = 0.95$, with an initial value (equals 0.8575 in our example), chosen to be the median of the informative sample, to obtain the lower and upper bounds L and U of a prediction interval with cover $\tau = 0.95$, for the median of m future observations, when $m = 2k - 1$ and $m = 2k$.
Tables (1) and (2) show the lower and upper bounds of two-sided intervals with cover $\tau = 0.95$ for the median of m of future observations for different values of m, when $m = 2k - 1$ is odd and $m = 2k$ is even and also show the lengths of the intervals.

5. CONCLUDING REMARKS

1. In this paper, Bayesian predictive densities for the median of m future observations are obtained when m is odd or even, assuming that both of the informative and future samples are drawn from a population whose distribution is a mixture of two components each of which is a member of a general class that includes important distributions used in life testing (and other areas as well). The prior belief of the experimenter is measured by a general class of distributions (1.8), suggested by AL-Hussaini (1999)$^{(b)}$, includes most priors used in literature. Lower and upper bounds of a predictive interval with cover τ of future median can then be obtained by solving the two equations, given by (3.14).
2. Other heterogeneous populations (that can be represented by finite mixtures of two (or more) components) than those composed of the Weibull components can be similarly treated. For example, the predictive density functions, given by (1.16) with $s = k$ and $m = 2k - 1$ in the odd case, and by (2.9) in the even case hold true for any components (such as the compound Weibull (or three-parameter Burr type XII), Pareto, beta, Gompertz and compound Gompertz components) that belong to the general class of density functions (1.2). Prediction bounds for the median of m future observations, assumed to be drawn from any of such populations, can thus be obtained by using the survival function corresponding to the predictive density obtained.
3. It may be noticed, from Tables (1) and (2), that by increasing the value of m, the lengths of the intervals increase since wider intervals are expected to include larger m.

APPENDIX

To prove (2.1), we use the definition of the median of m observations $\tilde{Y}_m$, given by (1.20), when $m = 2k$. That is,

$$\tilde{Y}_m = \frac{1}{2}[Y_{k:2k} + Y_{k+1:2k}]\,. \tag{A.1}$$

This transformation is applied to the joint density function of $Y_{k:2k}$ and $Y_{k+1:2k}$ (written, for simplicity, as Y_k and Y_{k+1}), given by

$$f_{Y_k,Y_{k+1}}(x, z \mid \theta) = \frac{(2k)!}{[(k-1)!]^2}[1 - R(x)]^{k-1}[R(z)]^{k-1} f(x) f(z), 0 < x < z, \tag{A.2}$$

where $f(.)$ and $R(.)$ are given by (1.4) and (1.6), respectively.
By expanding $[1 - R(x)]^{k-1}$ and then substituting (1.6) in (A.2), we obtain

$$f_{Y_k,Y_{k+1}}(x, z \mid \theta) = \frac{D_2(k)}{2} \Sigma K_2 p_1^{\omega_1} p_2^{\omega_2} [R_1(x)]^{j_3} [R_2(x)]^{j_2 - j_3}$$
$$[R_1(z)]^{j_4} [R_2(z)]^{k-1-j_4} f(x) f(z), 0 < x < z, \tag{A.3}$$

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where $D_2(k)$ and K_2 are given by (2.2) and (2.3), respectively, ω_1 and ω_2 by (2.8) and $\Sigma = \sum_{j_2=0}^{k-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{k-1}$.
By substituting in (A.3), $R_i(.)$ and $f(.)$, given by (1.7) and (1.4), where the components $f_i(.)$, $i = 1, 2$, are given by (1.2), it follows that

$$f_{Y_k,Y_{k+1}}(x, z \mid \theta) = \frac{D_2(k)}{2}\Sigma_2 K_2 G_\ell(p_1) u_\ell(x, z) exp[-v_\ell(x, z)], 0 < x < z, \qquad (A.4)$$

where $\Sigma_2 = \Sigma\sum_\ell^4$ and, for $\ell = 1, 2, 3, 4$, $G_\ell(p_1)$ is given by (2.5),
$u_1(x, z) = \lambda_1'(x)\lambda_1'(z)$, $u_2(x, z) = \lambda_1'(x)\lambda_2'(z)$;
$u_3(x, z) = \lambda_2'(x)\lambda_1'(z)$, $u_4(x, z) = \lambda_2'(x)\lambda_2'(z)$,
$v_1(x, z) = (j_3 + 1)\lambda_1(x) + (j_2 - j_3)\lambda_2(x) + (j_4 + 1)\lambda_1(z) + (k - 1 - j_4)\lambda_2(z)$,
$v_2(x, z) = (j_3 + 1)\lambda_1(x) + (j_2 - j_3)\lambda_2(x) + j_4\lambda_1(z) + (k - j_4)\lambda_2(z)$,
$v_3(x, z) = j_3\lambda_1(x) + (j_2 - j_3)\lambda_2(x) + j_4\lambda_1(z) + (k - j_4)\lambda_2(z)$,
$v_4(x, z) = j_3\lambda_1(x) + (j_2 - j_3 + 1)\lambda_2(x) + j_4\lambda_1(z) + (k - j_4)\lambda_2(z)$.

Applying transformation (A.1) to the density function (A.4), [by writing $y = \frac{1}{2}(x + z)$ and $t = x$, to obtain the joint density function $f_{Y_k,\tilde{Y}_{2k}}(t, y \mid \theta)$ and then integrating out t], we get the density function $f_{\tilde{Y}_{2k}}(y \mid \theta)$, given by (2.1), where, for $\ell = 1, 2, 3, 4$, $u_\ell(t, 2y - t)$ and $v_\ell(t, 2y - t)$ are given, respevtively, by (2.6) and (2.7).

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