RECURRENCE RELATIONS FOR MOMENTS OF PROGRESSIVELY TYPE-II RIGHT CENSORED ORDER STATISTICS FROM WEIBULL DISTRIBUTION

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Abstract:

In this paper, we derive recurrence relations for the single and product moments of progressively Type-II right censored order statistics from Weibull distribution. In addition, some well known results are deduced as special cases.

1 Introduction

Progressive Type-II censored sampling is an important method of obtaining data in lifetime studies. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter, and a compromise can be achieved between time consumption and the observation of some extreme values [see Aggarwala and Balakrishnan (1998)]. Some early works on progressive censoring can be found in Cohen(1963) and Mann(1971). The idea of obtaining moments of usual order statistics in a recursive manner has been explored for a number of distributions, Malik, Balakrishnan and Ahmed (1988) have reviewed several recurrence relations and identities available for the single and product moment of order

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statistics from an arbitrary continuous distribution, and they have pointed out the interrelationship between many of these recurrence relations. Also, Balakrishnan, Malik and Ahmed (1988) have reviewed several recurrence relations and identities established for the single and product moments of order statistics from specific continuous distributions, Balakrishnan and Sultan (1998) have updated the reviews of Malik, Balakrishnan and Ahmed (1988) and Balakrishnan, Malik and Ahmed (1988) and discuss several recurrence relations and identities for moments and product moments of order statistics. Aggarwala and Balakrishnan (1998) have established independence result for general progressive Type-II censored samples from the standard uniform population. That result was used in order to obtain moments for general progressive Type-II censored order statistics from the standard uniform distribution. This independence result also gave rise to a second algorithm for the generation of general progressive Type-II censored order statistics from any continuous distribution. Finally, they have derived the best linear unbiased estimators for the parameters of one- and two-parameter uniform distributions, and they have discussed problem of maximum likelihood estimation. Thomas and Wilson (1972) have developed a computational method for calculating the single and product moments of order statistics from progressively censored samples by making use of the corresponding moments of the usual order statistics, the absence of an explicit representation for the marginal and joint density function of order statistics under progressive censoring makes their method extremely tedious. By deriving the required marginal and joint density function in explicit form, Ng, Chan, and Balakrishnan(2002) have obtained an alternative, highly efficient, method for computing the desired moments. Let us consider the following progressive Type-II censoring scheme: Suppose n units taken from the same population are placed on a life test. At the first failure time of one of the n units, a number R_1 of the surviving units is randomly withdrawn from the test; at the second failure time, another R_2 surviving units are selected at random and taken out of the experiment, and so on. Finally, at the time of the mth failure, the remaining $R_m = n - R_1 - R_2 - \dots - R_{m-1} - m$ objects are removed. In this scheme, $\mathbf{R} = (R_1, R_2, \dots, R_m)$ is prespecified. The resulting m ordered failure times, which we will denote by $X_{1:m:n}^{(R_1,R_2,\dots,R_m)}, X_{2:m:n}^{(R_1,R_2,\dots,R_m)}, \dots, X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$ are referred to as *progressive Type-II right censored order statistics*. The special case when $R_1 = R_2 = \dots = R_{m-1} = 0$ so that $R_m = n - m$ is the case of conventional Type-II right censored sampling. Also when $R_1 = R_2 = \dots = R_m = 0$, so that $m = n$, the progressively Type-II right censoring scheme reduces to the case of no censoring (ordinary order statistics). If the failure times are based on an absolutely continuous distribution function F with probability density function f , the joint probability density function of the progressively censored failure times $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$, is given by [see Balakrishnan and Aggarwala (2000)].

$$f_{X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n}}(x_1, x_2, \dots, x_m) = A_{n;R_1,\dots,R_{m-1}} \prod_{i=1}^{m} f(x_i)\{1 - F(x_i)\}^{R_i},$$
$$-\infty < x_1 < x_2 < \dots < x_m < \infty, \qquad (1.1)$$

where $f(.)$ and $F(.)$ are, respectively, the pdf and cdf of the random sample and

$$A_{n;R_1,\dots,R_{m-1}} = n(n - R_1 - 1)\dots(n - R_1 - R_2 - \dots - R_{m-1} - m + 1). \qquad (1.2)$$

For simplicity, we write $A_{n;R_1,\dots,R_{m-1}} = A_{n;\tilde{R}_{m-1}}$; $1 \leq m \leq n$ and $A_{n;\tilde{R}_0} = n$.
In this paper, we are concerned with progressive Type-II censored data from Weibull distribution which is widely used in reliability analysis, the probability density function of Weibull distribution has the form

$$f(x) = \delta x^{\delta-1} e^{-x^{\delta}}, \quad 0 \leq x < \infty, \quad \delta > 0. \tag{1.3}$$

and its cumulative distribution function

$$F(x) = 1 - e^{-x^{\delta}}, \quad 0 \leq x < \infty, \quad \delta > 0. \tag{1.4}$$

It is easy to note that

$$f(x) = \delta x^{\delta-1}\{1 - F(x)\}, \quad x \geq 0. \tag{1.5}$$

In this paper, we establish some recurrence relations for the single moments of progressively Type-II right censored order statistics from Weibull distribution in Section 2. Also, we establish some recurrence relations for the product moments of progressively Type-II right censored order statistics from Weibull distribution in Section 3. Finally, some well known results are deduced as special cases in Section 4.

2 Single moments

Let $X_{1:m:n}^{(R_1,R_2,\dots,R_m)}, X_{2:m:n}^{(R_1,R_2,\dots,R_m)}, \dots, X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$ be a progressive Type-II right censored order statistics with censoring scheme $(R_1, R_2, \dots, R_m)$ from Weibull distribution. The single moments of the progressive Type-II censoring can be written from (1.1) as follows

$$\begin{aligned} \mu_{i:m:n}^{(R_1,\dots,R_m)(k)} &= E[X_{i:m:n}^{(R_1,\dots,R_m)(k)}] \\ &= A_{n;\tilde{R}_{m-1}} \int\int_{0<x_1<x_2<\dots<x_m<\infty}\cdots\int x_i^k f(x_1)[1 - F(x_1)]^{R_1} \\ &\times f(x_2)[1 - F(x_2)]^{R_2} \dots f(x_m)[1 - F(x_m)]^{R_m} dx_1 \dots dx_m, \end{aligned} \tag{2.1}$$

where $A_{n;\tilde{R}_{m-1}}$ is defined in (1.2). When $k = 1$, the superscript in the notation of the mean of the progressively Type-II right censored order statistics may be omitted without any confusion. The single moment of progressive Type-II right censored order statistics given in (2.1) satisfies the following recurrence relations.

Theorem 2.1 *For $2 \leq m \leq n$, $k \geq 0$ and $\delta > 0$*

$$\begin{aligned} \mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k+\delta)} &= \frac{1}{(R_1+1)}\Big\{\frac{\delta+k}{\delta}\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)(k)} \\ &- (n - R_1 - 1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)(k+\delta)}\Big\}. \end{aligned} \tag{2.2}$$

For $m = 1$, $n = 1,2,...$, $k \geq 0$ and $\delta > 0$

$$\mu_{1:1:n}^{(n-1)^{(k+\delta)}} = \frac{\delta + k}{n\delta}\mu_{1:1:n}^{(n-1)^{(k)}}. \tag{2.3}$$

Proof

Starting from (2.1), we write

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,...,R_m)^{(k)}} &= A_{n;\tilde{R}_{m-1}} \int\int_{0<x_2<...<x_m<\infty}...\int I(x_2)f(x_2)[1-F(x_2)]^{R_2}...f(x_m) \\ &\times [1-F(x_m)]^{R_m}dx_2...dx_m,\end{aligned} \tag{2.4}$$

where $A_{n;\tilde{R}_{m-1}}$ is defined in (1.2) and

$$I(x_2) = \int_0^{x_2} x_1^k[1-F(x_1)]^{R_1}f(x_1)dx_1,$$

which upon using (1.5) and integrating by parts gives

$$\begin{aligned}I(x_2) &= \frac{\delta}{k+\delta}x_2^{k+\delta}[1-F(x_2)]^{R_1+1} \\ &+ \frac{\delta}{k+\delta}(R_1+1)\int_0^{x_2} x_1^{k+\delta}f(x_1)[1-F(x_1)]^{R_1}dx_1.\end{aligned} \tag{2.5}$$

By using the above expression of $I(x_2)$ into (2.4), and simplifying in view of the definition of the single moments given in (2.1), we obtain (2.2).

Now, for $m = 1, n = 1,2,....$, $k \geq 0$ and $\delta > 0$, and by using (2.1), we have

$$\begin{aligned}\mu_{1:1:n}^{(R_1)^{(k)}} &= A_{n;\tilde{R}_0}\int_0^{\infty} x_1^k f(x_1)[1-F(x_1)]^{R_1}dx_1 \\ &= \frac{n\delta}{(\delta+k)}(R_1+1)\int_0^{\infty} x_1^{k+\delta}[1-F(x_1)]^{R_1}f(x_1)dx_1\end{aligned}$$

so,

$$\mu_{1:1:n}^{(n-1)^{(k)}} = \frac{n\delta}{\delta+k}\mu_{1:1:n}^{(n-1)^{(k+\delta)}}.$$

and hence (2.3) is proved.

Theorem 2.2 *For $2 \leq i \leq m-1$, $k \geq 0$ and $\delta > 0$*

$$\begin{aligned}\mu_{i:m:n}^{(R_1,...,R_m)^{(\delta+k)}} &= \frac{1}{(R_i+1)}\Bigg\{\frac{\delta+k}{\delta}\mu_{i:m:n}^{(R_1,R_2,...,R_m)^{(k)}} \\ &- (n-R_1-...-R_i-i)\mu_{i:m-1:n}^{(R_1,...,R_{i-1},R_i+R_{i+1}+1,...,R_m)^{(\delta+k)}} \\ &+ (n-R_1-...-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,...,R_{i-1}+R_i+1,R_{i+1},...,R_m)^{(\delta+k)}}\Bigg\}\end{aligned} \tag{2.6}$$

Proof Again, staring from (2.1), we write

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= A_{n;\dot{R}_{m-1}} \int\limits_{0<x_1<\cdots<x_{i-1}<x_{i+1}<\cdots<\infty}\cdots\int\cdots\int \left\{\int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{R_i} dx_i\right\} \\ &\times f(x_1)[1-F(x_1)]^{R_1} \times \ldots \times f(x_{i-1})[1-F(x_{i-1})]^{R_{i-1}} \\ &\times f(x_{i+1})[1-F(x_{i+1})]^{R_{i+1}} \ldots \\ &\times f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_m. \end{aligned} \quad (2.7)$$

By using (1.5) in (2.7) and integrating the innermost integral by parts with some simplifications, we get (2.6).

Theorem 2.3 *For $2 \le m \le n$ and $k \ge 0$,*

$$\begin{aligned}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(\delta+k)}} &= \frac{1}{(R_m+1)}\left\{\frac{\delta+k}{\delta}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}\right. \\ &+ \left.(n-R_1-\ldots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-1}+R_m+1)^{(k+\delta)}}\right\} \end{aligned} \quad (2.8)$$

Proof

Form (2.1), we have

$$\begin{aligned}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= A_{n;\dot{R}_{m-1}} \int\limits_{0<x_1<\cdots<x_{m-1}<\infty}\int\cdots\int \left\{\int_{x_{m-1}}^{\infty} x_m^k [1-F(x_m)]^{R_m} f(x_m) dx_m\right\} \\ &\times f(x_1)[1-F(x_1)]^{R_1} \ldots f(x_{m-1})[1-F(x_{m-1})]^{R_{m-1}} dx_1 \ldots dx_{m-1}, \end{aligned} \quad (2.9)$$

which upon using (1.5) and integrating the innermost integral by parts, we obtain (2.8).

3 Product moments

The $(i,j)-th$ product moment of the progressively Type-II right censored order statistics from Weibull distribution can be written from (1.1) as

$$\begin{aligned}\mu_{i,j:m:n}^{(R_1,\ldots,R_m)^{(r,s)}} &= E[X_{i:m:n}^{(R_1,\ldots,R_m)^r} X_{j:m:n}^{(R_1,\ldots,R_m)^s}] \\ &= A_{n;\dot{R}_{m-1}} \int\limits_{0<x_1<x_2<\cdots<x_m<\infty}\int\cdots\int x_i^r x_j^s f(x_1)[1-F(x_1)]^{R_1} \\ &\times f(x_2)[1-F(x_2)]^{R_2} \ldots f(x_m)[1-F(x_m)]^{R_m} dx_1 \ldots dx_m, \end{aligned} \quad (3.10)$$

where $f(.)$ and $F(.)$ are given respectively, by (1.3) and (1.4) and $A_{n;\dot{R}_{m-1}}$ is defined in (1.2). The product moment defined in (3.10) satisfies the following recurrence relations.

Theorem 3.1 *For $1 \leq i < j \leq m-1$, $m \leq n$, and $k, \delta > 0$*

$$\begin{aligned}\mu_{i,j:m:n}^{(R_1,\dots,R_m)^{(k,\delta)}} &= \frac{1}{(R_j+1)}\Big\{\mu_{i:m:n}^{(R_1,\dots,R_m)^{(k)}} \\ &- (n-R_1-\dots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,\dots,R_j+R_{j+1}+1,\dots,R_m)^{(k,\delta)}} \\ &+ (n-R_1-\dots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\dots,R_{j-1}+R_j+1,\dots,R_m)^{(k,\delta)}}\Big\} \quad (3.11)\end{aligned}$$

Proof

From (3.10), we write

$$\begin{aligned}\mu_{i:m:n}^{(R_1,\dots,R_m)^{(k)}} &= E[X_{i:m:n}^{(R_1,\dots,R_m)^{(k)}}\{X_{j:m:n}^{(R_1,\dots,R_m)}\}^0] \\ &= A_{n;\tilde{R}_{m-1}} \int\dots\int\dots\int_{0<x_1<\dots<x_{j-1}<x_{j+1}<\dots<x_m<\infty} x_i^{(k)} \\ &\times \left\{\int_{x_{j-1}}^{x_{j+1}} x_j^0 f(x_j)[1-F(x_j)]^{R_j} dx_j\right\} f(x_1)[1-F(x_1)]^{R_1} \\ &\times \dots \times f(x_{j-1})[1-F(x_{j-1})]^{R_{j-1}} f(x_{j+1})[1-F(x_{j+1})]^{R_{j+1}} \\ &\times \dots \times f(x_m)[1-F(x_m)]^{R_m} dx_1\dots dx_{j-1}dx_{j+1}\dots dx_m, \quad (3.12)\end{aligned}$$

which upon using (1.5) and integrating the innermost integral by parts and simplifying, we get (3.11).

Theorem 3.2 *For $1 \leq i \leq m-1$, $m \leq n$ and $k > 0$*

$$\begin{aligned}\mu_{i,m:m:n}^{(R_1,\dots,R_m)^{(k,\delta)}} &= \frac{1}{(R_m+1)}\{\mu_{i:m:n}^{(R_1,\dots,R_m)^{(k)}} \\ &+ (n-R_1-\dots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,\dots,R_{m-1}+R_m+1)^{(k,\delta)}}\}. \quad (3.13)\end{aligned}$$

Proof

The proof can be done easily by following the same manner as in theorem 3.1.

4 Deductions

Letting $\delta \to 1$ in the recurrence relations above, we deduce the recurrence relations given by Balakrishnan and Aggarwalla (2000) for the single and product moments of progressively Type-II right censored order statistics from the exponential distribution.

- From Theorem 2.1 we have: For $2 \leq m \leq n$ and $k \geq 0$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k+1)}} &= \frac{1}{(R_1+1)}\{(1+k)\mu_{1:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} \\ &- (n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\dots,R_m)^{(k+1)}}\}, \quad (4.14)\end{aligned}$$

for $m = 1$, $n = 1, 2, \ldots$ and $k \geq 0$,

$$\mu_{1:1:n}^{(n-1)(k+1)} = \frac{k+1}{n} \mu_{1:1:n}^{(n-1)(k)}.$$

Which is the relation established by Balakrishnan and Aggarwalla (2000).

- From **Theorem 2.2** we have: For $2 \leq i \leq m-1$ and $k \geq 0$.

$$\begin{aligned} \mu_{i:m:n}^{(R_1,\ldots,R_m)(k+1)} &= \frac{1}{(R_i+1)}\{(k+1)\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)(k)} \\ &- (n-R_1-\ldots-R_i-i)\mu_{i:m-1:n}^{(R_1,\ldots,R_{i-1},R_i+R_{i+1}+1,\ldots,R_m)(k+1)} \\ &+ (n-R_1-\ldots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\ldots,R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)(k+1)} \end{aligned} \quad (4.15)$$

which is the relation established by Balakrishnan and Aggarwalla (2000).

- From **Theorem 2.3** we have: For $2 \leq m \leq n$ and $k \geq 0$,

$$\begin{aligned} \mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)(k+1)} &= \frac{(k+1)}{(R_m+1)}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)(k)} \\ &+ \mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-1}+R_m+1)(k+1)}, \end{aligned} \quad (4.16)$$

which is the relation established by Balakrishnan and Aggarwalla (2000).

- From **Theorem 3.1** we have: For $1 \leq i < j \leq m-1$, $m \leq n$, and $k = 1$

$$\begin{aligned} \mu_{i,j:m:n}^{(R_1,\ldots,R_m)} &= \frac{1}{(R_j+1)}\{\mu_{i:m:n}^{(R_1,\ldots,R_m)} \\ &- (n-R_1-\ldots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,\ldots,R_j+R_{j+1}+1,\ldots,R_m)} \\ &+ (n-R_1-\ldots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,\ldots,R_{j-1}+R_j+1,\ldots,R_m)}\}, \end{aligned} \quad (4.17)$$

which is the relation established by Balakrishnan and Aggarwalla(2000).

- From **Theorem 3.2** we have: For $1 \leq i \leq m-1$, $m \leq n$, and $k = 1$

$$\begin{aligned} \mu_{i,m:m:n}^{(R_1,\ldots,R_m)} &= \frac{1}{(R_m+1)}\{\mu_{i:m:n}^{(R_1,\ldots,R_m)} \\ &+ (n-R_1-\ldots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,\ldots,R_{m-1}+R_m+1)}\}, \end{aligned} \quad (4.18)$$

which is the relation established by Balakrishnan and Aggarwalla(2000).

1. For the special case $R_1 = \ldots = R_m = 0$ so that $m = n$ in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, the relations established above for Weibull distribution reduce to the following:

(a) From **Theorem 2.1** we have: For $n \geq 2$ and $k \geq 0$

$$\mu_{1:n}^{(k+\delta)} = \{\frac{\delta+k}{\delta}\mu_{1:n}^{(k)} - (n-1)\mu_{1:n-1:n}^{(1,0,\dots,0)(k+\delta)}\} \tag{4.19}$$

(b) From **Theorem 2.2** we have: For $2 \leq i \leq n-1$ and $k \geq 0$

$$\begin{aligned} \mu_{i:n}^{(\delta+k)} &= \{\frac{\delta+k}{\delta}\mu_{i:n}^{(k)} \\ &- (n-i)\mu_{i:n-1:n}^{(0,\dots,0,1,0,\dots,0)(\delta+k)} \\ &+ (n-i+1)\mu_{i-1:n-1:n}^{(0,\dots,0,1,0,\dots,0)(\delta+k)}\} \end{aligned} \tag{4.20}$$

(c) From **Theorem 2.3** we have: For $n \geq 2$ and $k \geq 0$,

$$\mu_{n:n}^{(\delta+k)} = \frac{\delta+k}{\delta}\mu_{n:n}^{(k)} + \mu_{n-1:n}^{(0,\dots,1)(k+\delta)} \tag{4.21}$$

(d) From **Theorem 3.1** we have: For $1 \leq i < j \leq n-1$ and $k > 0$,

$$\begin{aligned} \mu_{i,j:n}^{(k,\delta)} &= \{\mu_{i:n}^{(k)} \\ &- (n-j)\mu_{i,j:n-1:n}^{(0,\dots,1,\dots,0)(k,\delta)} \\ &+ (n-j+1)\mu_{i,j-1:n-1:n}^{(0,\dots,1,\dots,0)(k,\delta)}\} \end{aligned} \tag{4.22}$$

(e) From **Theorem 3.2** we have: For $1 \leq i \leq n-1$ and $k > 0$,

$$\mu_{i,n:n}^{(k,\delta)} = \{\mu_{i:n}^{(k)} + \mu_{i,n-1:n-1:n}^{(0,\dots,1)(k,\delta)}\}. \tag{4.23}$$

2. Now, if $R_1 = R_2 ... = R_{j-1} = 0$ so that there is no censoring before the time of the j-th failure, then the first j progressively Type-II right censored order statistics are simply the first j usual order statistics. Thus, the relations above reduce to:

(a) From (4.19)we have: For $n \geq 2$ and $k \geq 0$,

$$\mu_{1:n}^{(k+\delta)} = \frac{\delta+k}{n\delta}\mu_{1:n}^{(k)}$$

(b) From (4.20)we have: For $2 \leq i \leq n-1$ and $k \geq 0$,

$$\mu_{i:n}^{(\delta+k)} = \frac{\delta+k}{\delta(n-i+1)}\mu_{i:n}^{(k)} + \mu_{i-1:n}^{(\delta+k)}$$

(c) From (4.21)we have:For $n \geq 2$ and $k \geq 0$,

$$\mu_{n:n}^{(\delta+k)} = \frac{\delta+k}{\delta}\mu_{n:n}^{(k)} + \mu_{n-1:n}^{(k+\delta)}$$

(d) From (4.22)and (4.23) we have: For $1 \le i < j \le n$ and $k > 0$,

$$\mu_{i,j:n}^{(k,\delta)} = \frac{1}{(n-j+1)}\mu_{i:n}^{(k)} + \mu_{i,j-1:n}^{(k,\delta)}.$$

- For the special case $R_1 = ... = R_m = 0$ so that $m = n$ in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, ..., X_{n:n}$, the relations established above for exponential distribution reduce to the following:

1. From(4.14): For $k \ge 0$,

$$\mu_{1:n}^{(k+1)} = (k+1)\mu_{1:n}^{(k)} - (n-1)\mu_{1:n-1:n}^{(1,...,0)(k+1)}.$$

2. From(4.15): For $2 \le i \le n-1$, and $k \ge 0$,

$$\mu_{i:n}^{(k+1)} = (k+1)\mu_{i:n}^{(k)} - (n-i)\mu_{i:n-1:n}^{(0,...,1,...,0)(k+1)} + (n-i+1)\mu_{i-1:n-1:n}^{(0,...,1,...,0)(k+1)},$$

   where, in the superscript of the second term on the right hand side, the 1 is in the i-th position, and in the superscript of the third term on the right hand side, the 1 is in the (i-1)-th position.

3. From(4.16): For $n \ge 2$ and $k \ge 0$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n-1:n}^{(0,...,0,1)(k+1)}.$$

4. From(4.17): For $1 \le i < j \le n-1$

$$\mu_{i,j:n} = \mu_{i:n} - (n-j)\mu_{i,j:n-1:n}^{(0,...,1,...,0)} + (n-j+1)\mu_{i,j-1:n-1:n}^{(0,...,1,...,0)},$$

5. From(4.18): For m=n . $1 \le i \le n-1$

$$\mu_{i,n:n} = \mu_{i:n} + \mu_{i,n-1:n-1:n}^{(0,...,1)}.$$

- Now, if $R_1 = R_2 ... = R_{j-1} = 0$ so that there is no censoring before the time of the j-th failure, then the first j progressively Type-II right censored order statistics are simply the first j usual order statistics. Thus, the relations above reduce to:

  - From(4.14): For$2 \le n$ and $k \ge 0$

$$\mu_{1:n}^{(k+1)} = \frac{(k+1)}{n}\mu_{1:n}^{(k)}.$$

  - From(4.15): for $2 \le i \le n-1$, and $k \ge 0$,

$$(n-i+1)\mu_{i:n}^{(k+1)} = (k+1)\mu_{i:n}^{(k)} + (n-i+1)\mu_{i-1:n}^{(k+1)}.$$

– From(4.16): For $k \geq 0$,

$$\mu_{n:n}^{(k+1)} = (k+1)\mu_{n:n}^{(k)} + \mu_{n-1:n}^{(k+1)}.$$

These recurrence relations are equivalent to those established by Joshi(1978).

– From(4.17)and(4.18): For $1 \leq i < j \leq n$

$$\mu_{i,j:n} = \frac{1}{(n-j+1)}\mu_{i:n} + \mu_{i,j-1:n}.$$

Which is the relation established by Joshi(1982).

Now, we will adopt the following technique depending on Order statistics to confirm the above results.

Let X be a continuous random variable having a cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $X_1, X_2, \ldots X_n$ be a random sample of size n from this distribution and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics obtained by arranging the $X_i's$ in the ascending order of magnitude. Then the pdf of $X_{i:n}$ $(1 \leq i \leq n)$ is given by [see David (1981, pp.9) and Arnold, Balakrishnan and Nagaraja (1992, pp. 10)]

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x). \quad -\infty < x < \infty, \tag{4.24}$$

and the joint density of $X_{i:n}$ and $X_{j:n}$ $(1 \leq i < j \leq n)$ is given by [see David (1981, pp.10) and Arnold, Balakrishnan and Nagaraja (1992, pp. 16)]

$$f_{i,j:n}(x,y) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j}$$
$$f(x)f(y). \quad -\infty < x < y < \infty. \tag{4.25}$$

Let us denote the single moment $E(X_{i:n}^k)$ by $\mu_{i:n}^k$ and the product moment $E(X_{i:n}X_{j:n})$ by $\mu_{i,j:n}$. Then from the density function of $X_{i:n}$ in (4.24) we have

$$\mu_{i:n}^{(k)} = \frac{n!}{(i-1)!(n-i)!}\int_{-\infty}^{\infty} x^k\,[F(x)]^{i-1}\,[1-F(x)]^{n-i}\,f(x)dx, i = 1, \ldots, n, k \geq 1. \tag{4.26}$$

Then from the joint density function of $X_{i:n}$ and $X_{j:n}$ we have

$$\mu_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_{-\infty}^{\infty}\int_{x}^{\infty} xy[F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j}$$
$$f(x)f(y)dydx, 1 \leq i < j \leq n \tag{4.27}$$

Theorem 4.1 *For Weibull population with pdf as in (1.3), we have*

$$\mu_{1:n}^{(k+\delta)} = \frac{k+\delta}{n\delta}\mu_{1:n}^{(k)}, \quad n \geq 2, \quad k \geq 0 \tag{4.28}$$

Proof We have from (4.26) that

$$\mu_{1:n}^{(k)} = n\int_0^{\infty} x^k\,[1-F(x)]^{n-1}\,f(x)dx. \tag{4.29}$$

By using (1.5) in the right hand side of (4.29) treating x^k for integration and the rest of integral for differentiation, we obtain for $n \geq 2$ and $k \geq 0$

$$\mu_{1:n}^{(k)} = \frac{n\delta}{k+\delta}[n\int_0^{\infty} x^{k+\delta}\,[1-F(x)]^{n-1}\,f(x)dx]. \tag{4.30}$$

By simplifying (4.30) we get (4.28).

Theorem 4.2 *For Weibull population with pdf as in (1.3), we have*

$$\mu_{i:n}^{(k+\delta)} = \mu_{i-1:n}^{(k+\delta)} + \frac{k+\delta}{(n-i+1)\delta}\mu_{i:n}^{(k)}, \quad 1 \leq i \leq n, \quad k \geq 0 \tag{4.31}$$

Proof By using (1.5) in the right hand side of (4.26) treating x^k for integration and the rest of integral for differentiation, we obtain

$$\mu_{i:n}^{(k)} = \frac{\delta}{k+\delta}\frac{n!}{(i-1)!(n-i)!}[-(i-1)\int_0^{\infty} x^{k+\delta}[F(x)]^{i-2}\,[1-F(x)]^{n-i+1}\,f(x)dx$$
$$+(n-i+1)\int_0^{\infty} x^{k+\delta}[F(x)]^{i-1}\,[1-F(x)]^{n-i}\,f(x)dx]. \tag{4.32}$$

By simplifying (4.32) we get (4.31).

Theorem 4.3 *For Weibull population with pdf as in (1.3), we have*

$$\mu_{i,j:n}^{(k+\delta)} = \frac{1}{(n-j+1)}\mu_{i:n}^{(k)} + \mu_{i,j-1:n}^{(k+\delta)}, \quad 1 \leq i < j \leq n. \tag{4.33}$$

Proof By considering the expression of $f_{i,j:n}(x,y)$ in (4.25) and using relation (1.5), we may write for $1 \leq i < j \leq n$,

$$\mu_{i:n}^{(k)} = E(X_{i:n}^k X_{j:n}^0) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_{-\infty}^{\infty} x[F(x)]^{i-1}I(x)f(x)dx \tag{4.34}$$

where

$$I(x) = \int_x^{\infty}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j}f(y)dy \tag{4.35}$$

by using (1.5) in the right hand side of (4.35), we obtain

$$I(x) = (n-j+1)\int_x^{\infty} y^{\delta}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j}f(y)dy$$
$$-(j-i-1)\int_x^{\infty} y^{\delta}[F(y)-F(x)]^{j-i-2}[1-F(y)]^{n-j+1}f(y)dy. \tag{4.36}$$

Substituting of the expression for $I(x)$ in (4.36) into (4.34) gives equation (4.33).

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