THE INVERTED LINEAR EXPONENTIAL DISTRIBUTION AS A LIFE TIME DISTRIBUTION

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Abstract

In this paper, we introduce the inverted linear exponential (ILE) distribution. Some statistical properties of this distribution are studied, such as the mean, the variance, the mode, the median, the failure rate, and mean residual time. The maximum likelihood estimates for the parameters, the reliability function and the failure rate function of this distribution are also obtained.

Key Words. Inverted linear exponential distribution, linear exponential distribution, reliability function, failure rate, mean residual life time.

§. 1. Introduction.

In the field of reliability, the distribution with non-monotonic failure rate is of considerable interest modeling, since it is widely applicable in life time studies. So, it is very important to introduce and study the statistical properties of a new inverted distribution.

In texts there are many inverted distributions such as inverse Gaussian, inverted exponential, inverse Weibull and inverted Gamma distributions.

Chhikara and Folks (1977) suggested the use of inverse Gaussian distribution for a model of such lifetime behavior and discussed different reliability features of the distribution. They showed that its failure rate is non-monotonic. For further details on the inverse Gaussian distribution refer to Tweedie (1957) and Johnson, Kotz and Balakrishnan (1994). Lin, C. T., et.al (1989) introduced and studied some statistical properties of inverted exponential and inverted gamma distributions.

Mohie El-Din et. al (1997) studied the order statistics from doubly truncated linear exponential distribution. Mahmoud et.al (2003a) derived some recurrence relations for the single and the product moments from the inverse Weibull distribution and doubly truncated inverse Weibull distribution. Firally they characterized the inverse Weibull distribution using some of these recurrence relations. Mahmoud et.al (2003b) derived the exact expression for the single moments of order statistics from the inverse Weibull distribution. The variance and the covariance of the order statistics are calculated and the best linear unbiased estimates (BLUEs) for the location and the scale parameters of the inverse Weibull distribution are obtained. These BLUEs are applied to draw inference for the location and scale parameters of the underlying model. Finally, they carried out a simulation study to illustrate the theoretical results.

In this paper, we introduce the inverted linear exponential (ILE) distribution and study some of its statistical properties.

The probability density function (p.d.f) of the linear exponential distribution with parameters λ and θ is given by the following:

$$g(t;\lambda,\theta) = (\lambda+\theta t)e^{-(\lambda t+\frac{\theta t^2}{2})}, \qquad t>0, \qquad \theta,\lambda \geq 0. \tag{1.1}$$

Setting $X=\frac{1}{T}$, (1.1) reduces to the inverted linear exponential (ILE) distribution which is given as follows:

$$f(x;\lambda,\theta) = \frac{1}{x^3}(\lambda x+\theta)e^{-\frac{1}{2x^2}(2\lambda x+\theta)}, \qquad \lambda>0,\theta>0,x>0. \tag{1.2}$$

From (1.2) one can show that the cummulative distribution function (cdf) is as follows:

$$F(x;\lambda,\theta) = e^{-\frac{1}{2x^2}(2\lambda x+\theta)}, \qquad x>0, \tag{1.3}$$

and the reliability function

$$R_1(t) = 1-e^{-\frac{1}{2x^2}(2\lambda x+\theta)}. \tag{1.4}$$

From (1.2) and (1.3), we get

$$x^3 f(x;\lambda,\theta) = (\lambda x+\theta)F(x;\lambda,\theta). \tag{1.5}$$

From (1.2), special distributions can be obtained:

(i) For $\theta=0$, (1.2) reduces to

$$f(x;\lambda) = \frac{\lambda}{x^2}e^{-\frac{\lambda}{x}}, \qquad x>0,\lambda>0,$$

which is the p.d.f of the inverted exponential distribution Lin et al (1989).

(ii) For $\lambda=0$, (1.2) can be written in the following form:

$$f(x;\theta) = \frac{\theta}{x^3}e^{-\frac{\theta}{2x^2}}, \qquad x>0,\theta>0,$$

which is the p.d.f of the inverted Rayleigh distribution.

§. 2. Some Statistical Properties.

The statistical properties play an important role in characterization of the distributions, so some statistical properties for the ILE distribution are handled.

§. 2.1 The Mean and the variance.

Using (1.1), the mean μ_1 and the variance σ_1^2 of the linear exponential (L.E) distribution are given by:

$$\mu_1 = \sqrt{\frac{2\pi}{\theta}} e^{\frac{\lambda^2}{2\theta}} \bar{\Phi}(\frac{\lambda}{\sqrt{\theta}}), \tag{2.1}$$

$$\sigma_1^2 = \frac{2}{\theta} - \mu_1 \left(\frac{2\lambda}{\theta} + \mu_1 \right), \tag{2.2}$$

where $\Phi(.)$ is cdf of the standard normal distribution.

Since the mean μ and the variance σ^2 of the ILE distribution can not be obtained in closed forms, then using Lindly formula (See Lindly 1980, pp.), we get

$$\mu \simeq \psi(\mu_1) + \frac{1}{2}\sigma_1^2 \psi''(\mu_1), \tag{2.3}$$

and

$$\sigma^2 \simeq \sigma_1^2 \psi'(\mu)^2, \tag{2.4}$$

where $\psi(t) = \frac{1}{t}$.

§. 2.2 The Mode.

It is easy to show that the mode of the ILE distribution can be obtained by solving the following equation

$$2\lambda x^3 + (3\theta - \lambda^2)x^2 - 2\lambda\theta x - \theta^2 = 0. \tag{2.5}$$

Using Mathematica program to solve Equation (2.5), yields one real root and two complex roots. This leads that this function is a uncinal function.

The real root is given by

$$x_1 = \frac{\lambda^2 - 3\theta}{6\lambda} + \frac{2}{6\lambda}\text{Re}[\xi + i\eta]^{\frac{1}{3}},$$

where

$$\xi = \lambda^6 + 9\theta\lambda^4 + 27\theta^2\lambda^2 - 27\theta^3$$

$$\eta = 6\sqrt{3}\sqrt{27\theta^5\lambda^2 + 9\theta^4\lambda^4 + \theta^3\lambda^6},$$

$$i = \sqrt{-1}$$

This leads that the inverted linear exponential distribution is a unimodal distribution.

Remark (1).

(i) For $\theta = 0$, it is easly to show that the mode $\tilde{x} = \frac{\lambda}{2}$, which is the mode of the inverted linear exponential with one parameter.

(ii) For $\lambda = 0$, Eq. (2.5) reduces to $30x^2 - \theta^2 = 0$, which leads to the mode $\tilde{x} = \sqrt{\frac{\theta}{3}}$ of the inverted rayleigh distribution with one parameter.

§. 2.3 The Quantile and the Median.

It easy to show that the quantile t_q of an ILE random variable given by:

$$t_q = \frac{\theta}{\sqrt{\lambda^2 - 2\theta \ln q} - \lambda}, \qquad 0 < q < 1. \tag{2.6}$$

For $q = 0.5$, the median of the ILE random variable is obtained as follows

$$\text{Med} = \frac{\theta}{\sqrt{\lambda^2 + 2\theta \ln 2} - \lambda}. \tag{2.7}$$

Remark (2).

(i) For $\lambda = 0$, the quantile and the median of the inverted Rayleigh distribution are obtained as follows:

$$t_q = \sqrt{\frac{\theta}{2 \ln \frac{1}{q}}}$$

$$\text{Med} = \sqrt{\frac{\theta}{2 \ln 2}}$$

(ii) As $\theta \to 0$, we obtain the quantile and the median for the inverted exponential distribution as follows:

$$\tilde{\tilde{t}}_q = \frac{\lambda}{\ln \frac{1}{q}}$$

$$\text{Med} = \frac{\lambda}{\ln 2}$$

§. 2.4 The failure rate.

The failure rate $r_1(x)$ of the ILE (λ, θ) can be given in the form

$$r_1(x) = \frac{\lambda x + \theta}{x^3 \left[e^{\frac{1}{2x^2}(2\lambda x + \theta)} - 1 \right]}. \tag{2.8}$$

It is easy to show that

$$\lim_{x \to 0} r_1(x) = 0, \tag{2.9}$$

and

$$\lim_{x \to \infty} r_1(x) = 0. \tag{2.10}$$

From (2.9) and (2.10) and since $r_1(x)$ is a non-negative function, one can see that $r_1(x)$ is a non-monotonic function. This property makes the ILE distribution widely applicable. Figures (1a) and (1b) represent $f(x)$, while Figures (2a) and (2b) represent the failure rate function $r_1(x)$ for various values of λ and θ.

§. 2.5 The Maximum Likelihood Estimation.

Let $X_1, X_2, \cdots, X_n$ be a random sample from (1.2), then the likelihood function is:

$$L(\lambda;\theta;x_1,x_2,\cdots,x_n)=\prod_{i=1}^{n}\frac{1}{x_i^2}\left(\lambda+\frac{\theta}{x_i}\right)\exp\left[-\sum_{i=1}^{n}\frac{1}{x_i}\left(\lambda+\frac{\theta}{2x_i}\right)\right].$$

Then

$$\ln L = 2\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}\ln\left(\lambda+\frac{\theta}{x_i}\right)-\sum_{i=1}^{n}\frac{1}{x_i}\left(\lambda+\frac{\theta}{2x_i}\right). \tag{2.11}$$

Differentiating (2.11) w.r.t λ and θ respectively, we obtain

$$\sum_{i=1}^{n}\left[\frac{1}{\hat{\lambda}+\frac{\hat{\theta}}{x_i}}-\frac{1}{x_i}\right]=0, \tag{2.12}$$

and

$$\sum_{i=1}^{n}\left[\frac{1}{\hat{\theta}+\hat{\lambda}x_i}-\frac{1}{2x_i^2}\right]=0, \tag{2.13}$$

One can obtain $\hat{\lambda}$ and $\hat{\theta}$ by solving (2.12) and (2.13) numerically.

Equation (2.12) can be written as follows

$$\frac{n}{\hat{\lambda}}-\frac{\hat{\theta}}{\hat{\lambda}}\sum\left[\frac{1}{\hat{\lambda}x_i+\hat{\theta}}\right] \tag{2.14}$$

Making use of Eq. (2.13) and (2.14) yields.

$$\hat{\lambda}=\frac{n-\hat{\theta}\sum_{i=1}^{n}\frac{1}{2x_i^2}}{\sum_{i=1}^{n}\frac{1}{x_i}} \tag{2.15}$$

From (2.15), we can obtain

$$0\le\hat{\lambda}\le\frac{n}{\sum\frac{1}{x_i}} \tag{2.16}$$

$$0\le\hat{\theta}\le\frac{n}{\sum\frac{1}{2x_i^2}} \tag{2.17}$$

§. 2.5.1 Illustrative example .

The following maintenance date were reported on active repair times (hours) for an airborne communication transceiver [Von Alven 1964, page 156], 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0.

Using the inverted linear exponential model for the data, the MLE of λ and θ are

$$\hat{\lambda} = 0.944 \quad , \quad \hat{\theta} = 0.061.$$

Using the inverted exponential model $g(x) = \frac{\beta}{x^2} \exp\left(-\frac{\beta}{x}\right)$, $x > 0$ for the data, the MLE of β is

$$\hat{\beta} = 1.001$$

Considering the inverse Gaussian model

$$f(x) = \left(\frac{\alpha}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left[\frac{-\lambda(x-\gamma)^2}{2\gamma^2 x}\right], \qquad x > 0$$

for the data, the MLE of α and μ are

$$\hat{\alpha} = 1.87 \quad , \quad \hat{\gamma} = 2.095.$$

all the previous result are calculated using Mathcad 2001.

The calculated value of the Kolomogorov-Smirnov test are 0.14 for the inverted linear exponential 0.16 for the inverted exponential and 0.05 for the inverse Gaussian and these values are smaller than their corresponding values expected at 5% significance level, which is 0.22 at $n = 40$. This means that the three distributions can be good models for the above data.

The estimated failure rates of the inverted linear exponential and inverse Gaussian are $\hat{r}_1(x), \hat{r}_2(x)$ and $\hat{r}_3(x)$ respectively, where

$$\hat{r}_2(x) = \frac{\hat{\beta}}{x^2\left(e^{\hat{\beta}}/x - 1\right)},$$

and

$$\hat{r}_3(x) = \left(\frac{\hat{\alpha}}{2\pi x^3}\right)^{\frac{1}{2}} \frac{e^{-\frac{\alpha(x-\hat{\mu})^2}{2\hat{\mu}^2 x}}}{\Phi\left(\sqrt{\frac{\hat{\alpha}}{x}}(1-\frac{x}{\hat{\mu}})\right) - e^{2\hat{\alpha}/\hat{\mu}}\Phi\left(\sqrt{\frac{\hat{\alpha}}{x}}(1-\frac{x}{\hat{\mu}})\right)}$$

It can be seen from Fig (4.a) that for small values of t, say less than are equal 1, the inverted linear exponential has the advantage to model the empirical failure rate comparing with the inverted exponential and the inverse Gaussian distributions. However, for large t, the inverse Gaussian distribution seems to hold the advantage.

For comparison of reliability features estimated from the three distributions, Fig (4.b) shows that the reliability curves for the inverted linear $\hat{R}_1(x)$ exponential, inverted exponential $\hat{R}_2(x)$ (See Lin 89) and inverse Gaussian $\hat{R}_3(x)$ (See Chkikara and Folks ,1977) are almost over lapping for small values. This implies that the inverted linear exponential distribution provides as good a fit as do, the inverted exponential and inverse Gaussian distributions for small values but for large values one can see that the inverse Gaussian is the best.

§. 2.6. The Mean residual life time.

Let $m(t)$ denote the mean residual life time of ILE distribution. Then

$$\begin{aligned} m(t) &= E[X - t|X > t] \\ &= \frac{1}{R_1(t)} \int_t^{\infty} (x - t) f(x) dx, \end{aligned} \tag{2.18}$$

where $f(x)$ and $R_1(t)$ are the p.d.f and reliability functions respectively.

Equation (2.18) can be put in the following form:

$$m(t) = \frac{1}{R_1(t)} \int_t^{\infty} R_1(t) dt. \tag{2.19}$$

The mean residual life time can be expressed in terms of failure rates as:

$$m(t) = \int_0^{\infty} \exp\left(- \int_t^{t+x} r_1(y) dy \right) dx. \tag{2.20}$$

(See Watson and Wells 1961). Figures (3a) and (3b) represent the mean residual life time for various values of λ and θ.

From (2.20) it is clear that $m(t)$ decreases (increases) monotonically for the life distribution with IFR (DFR). However, if $r_1(x)$ first increases and then starts to decrease monotonically at some time t_0, as in the case of the inverted linear exponential distribution, we have $m(t_2) > m(t_1)$ for $t_2 > t_1 > t_0$. Thus the mean residual life time for the inverted linear exponential distribution may increase from some time on ward. The increase in the mean residual life time can be described in terms of physical behavior of a product. For example on excluding the units with high deficiency rate after they have failed, the failure rate of the surviving units would be relatively small and their average life time would be higher longevity.

§. 3 Conclusion.

The failure rate of many devices is expected to be non-monotonic, first increasing and later decreasing. In such situations the inverted linear exponential distribution provides a suitable choice for a life time model. Hence, we found that it is beneficial to introduce the inverted linear exponential distribution and discuss some of its statistical properties. The MLEs of the parameters λ, θ, the reliability function and the failure rate of this distribution are also obtained. An illustrative example is given using real data to compare between the estimated failure rates and reliability functions of the inverse Gaussian, inverted exponential and inverted linear exponential.

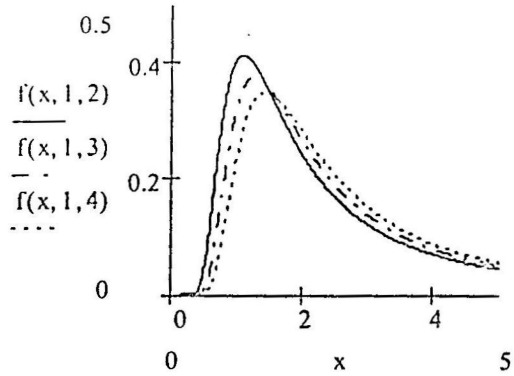

Fig. (1a)
The density function
$\lambda = 1$ and $\theta = 2,3,4$

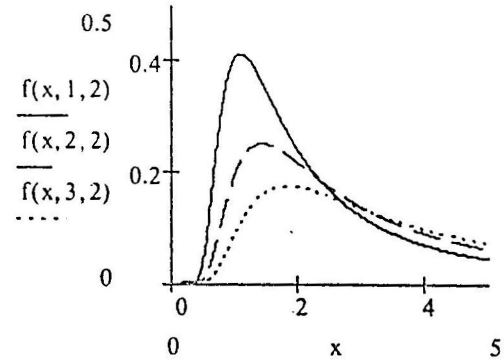

Fig. (1b)
The density function
$\lambda = 1,2,3$ and $\theta = 2$

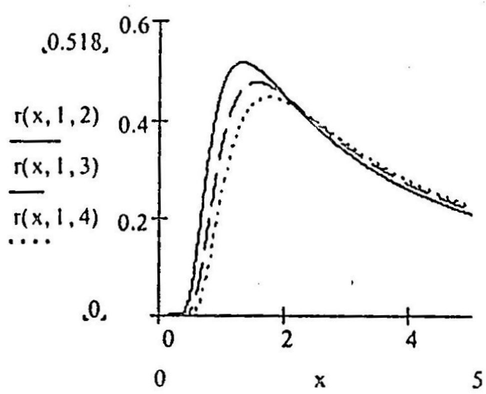

Fig. (2a)
The failure rate function
$\lambda = 1$ and $\theta = 2,3,4$

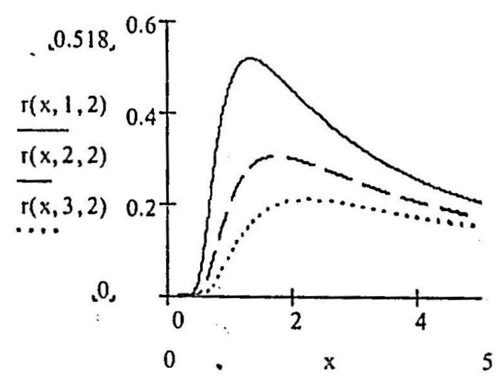

Fig. (2b)
The failure rate function
$\lambda = 1,2,3$ and $\theta = 2$

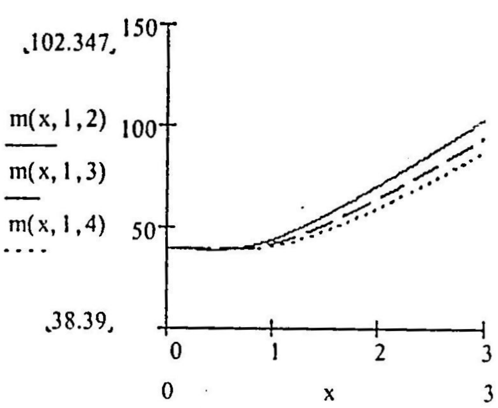

Fig. (3a)
The mean residual time distribution
$\lambda = 1$ and $\theta = 2,3,4$

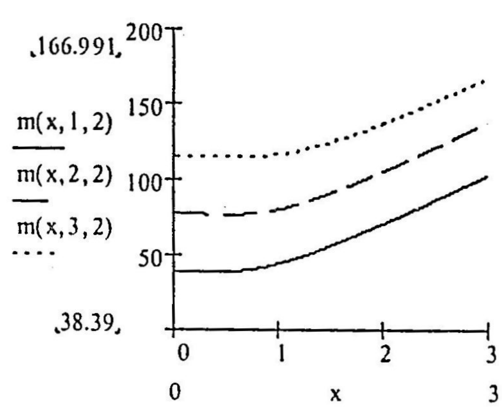

Fig. (3b)
The mean residual time distribution
$\lambda = 1,2,3$ and $\theta = 2$

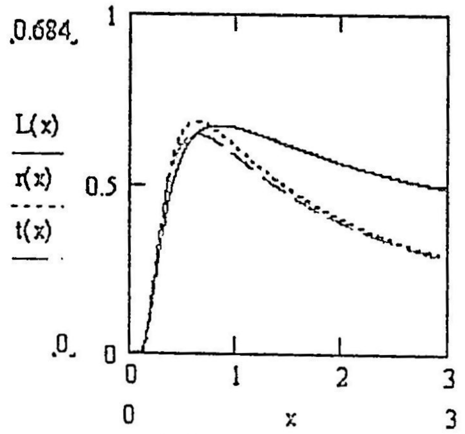

Fig. (4a)
The estimated failure rates functions
$\lambda = 1$ and $\theta = 2,3,4$

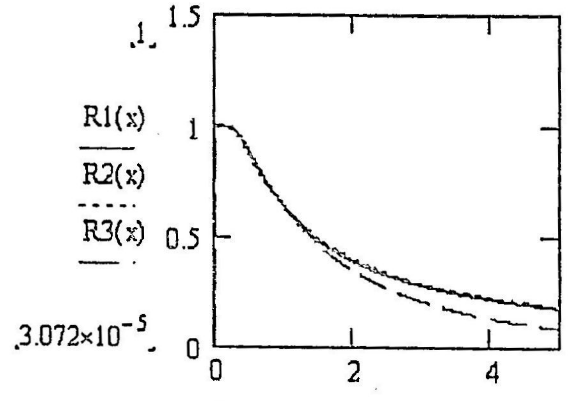

Fig. (4b)
The estimated failure rates functions
$\lambda = 1,2,3$ and $\theta = 2$

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