RELATIONS FOR MOMENTS OF ORDER STATISTICS FROM INDEPENDENT AND NON-IDENTICAL GENERAL CLASS OF DISTRIBUTIONS RANDOM VARIABLES AND APPLICATIONS

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key words and phrases : order statistics , outliers , single moments , product moments , relations , doubly truncated , general class of distributions , permanents

Abstract

This paper deals with obtaining relations for single and product moments of the order statistics from independent and non-identical general class of distributions random variables in two cases , the untruncated and the truncated cases . The relations for moments in presence of p outliers are obtained as special cases.

1 INTRODUCTION

The moments of order statistics have assumed considerable interest in the recent years and, in fact, have been tabulated quite extensively for many distributions. The recurrence relations and identities for moments of order statistics are very important since they reduce the computations process and usefully express the higher moments of order statistics as terms of the lower moments which make the calculation is very easy.

Recurrence relations for moments from non-identical distributions (I.NI.D) are devloped in the recent 20 years or so by many authors see, for example Balakrishnan (1988) has obtained recurrence relations for moments for order statistics from n independent and non-identical distributed random variables. Balakrishnan (1994a,b) has obtained recurrence relations for the single and the product moments from (I.NI.D) exponential distribution and its right truncated. Balakrishnan and Balsubramanian (1995) gave recurrence relations for moments from (I.NI.D) power function distribution. Childs and Balakrishnan (1998) have obtained recurrence relations for moments from (I.NI.D) Pareto distribution. Childs and et al. (2001) gave recurrence relations for the single and product moments from (I.NI.D) right truncated Lomax distribution. Moshref (2000) has established recurrence relations for moments from (I.NI.D) generalized power function distribution. Ashour and Afify (2002) have established order statistics from non-identical doubly truncated generalized Weibull random variables. Ashour and et al. (2002) have established recurrence relations from multiple outliers from non-identical doubly truncated generalized Weibull. Mahmoud and et al. (2006)

have obtained recurrence relations for moments of order statistics from non-identical generlaized Pareto random variables . Finally recurrence relations for moments for Logistic from (I.NI.D) random variables have obtained by Childs and Balakrishnan (2006).

Let $X_1, X_2, ..., X_n$ be independent random variables having probability density functions (p.d.f) $f_1(x), f_2(x), ..., f_n(x)$ and cumulative distribution functions (c.d.f) $F_1(x), F_2(x), ..., F_n(x)$ respectively. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics. Then the probability density function (p.d.f) of $X_{r:n}$ and the joint probability density function (j.p.d.f) of $(X_{r:n}, X_{s:n})$ can be written as David (1981) respectively :-

$$f_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \sum_p \prod_{a=1}^{r-1} F_{ia}(x) f_{ir}(x) \prod_{b=r+1}^{n} (1 - F_{ib}(x)), \quad 1 \leq r \leq n \tag{1.1}$$

$$f_{r,s:n}(x,y) = \frac{1}{(r-1)!(s-r-1)!(n-s)!} \sum_p \prod_{a=1}^{r-1} F_{ia}(x) f_{ir}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y) \times \prod_{c=s+1}^{n} (1 - F_{ic}(y)), \quad 1 \leq r < s \leq n . \tag{1.2}$$

Alaternative forms of the densities (1.1) and (1.2) in terms of permanents of matrices are obtained by Veghan and Venables (1972) as follows :-

$$f_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \times$$

$$\begin{vmatrix} F_1(x) & F_2(x) & ... & F_n(x) \\ f_1(x) & f_2(x) & ... & f_n(x) \\ 1-F_1(x) & 1-F_2(x) & ... & 1-F_n(x) \end{vmatrix}^{+} \begin{matrix} \} \, r-1 \text{ rows} \\ \}1 \\ \} \, n-r \text{ rows} \end{matrix} , \tag{1.3}$$

$$f_{r,s:n}(x,y) = \frac{1}{(r-1)!(s-r-1)!(n-s)!} \times$$

$$\begin{vmatrix} F_1(x) & F_2(x) & ... & F_n(x) \\ f_1(x) & f_2(x) & ... & f_n(x) \\ F_1(y)-F_1(x) & F_2(y)-F_2(x) & ... & F_n(y)-F_n(x) \\ f_1(y) & f_2(y) & ... & f_n(y) \\ 1-F_1(y) & 1-F_2(y) & ... & 1-F_n(y) \end{vmatrix}^{+} \begin{matrix} \} \, r-1 \text{ rows} \\ \}1 \\ \} \, s-r-1 \text{ rows} \\ \}1 \\ \} \, n-s \text{ rows} \end{matrix} \tag{1.4}$$

The main theme of this paper is deriving relations for moments for order statistics from (I.NI.D) for general class of distributions and deducing the relations for moments when the sample contains p-outliers as an application. We consider the case when the variables $X_{i,s}$ are independent with cumulative distribution functions :

$$F_i(x) = 1 - be^{-m_i \lambda(x)} \quad , \quad \beta \leq x \leq \delta \quad , b \text{ is a scale parameter} \tag{1.5}$$

$\lambda(x)$ is continious, positive and differentiable ,

and probability density functions:

$$f_i(x) = m_i b \lambda'(x) e^{-m_i \lambda(x)} \quad , \tag{1.6}$$

It is easy to show that

$$f_i(x) = m_i \lambda'(x)(1 - F_i(x)) \; , \quad i = 1, 2, .., n \; , \tag{1.7}$$

and

$$m_i\lambda''(x)(1-F_i(x)) = \frac{\lambda''(x)}{\lambda'(x)}f_i(x) \ , \qquad \lambda'(x) \neq 0 \quad . \qquad (1.8)$$

Now we present the notations which we need through this paper .

2 NOTATIONS

$c_{r:n}$	$\frac{1}{(r-1)!(n-r)!}$
$c_{r,s:n}$	$\frac{1}{(r-1)!(s-r-1)!(n-s)!}$
$\sum_p$	the summation over $n!$ permutations $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$.
$\lvert A\rvert^+$	the permanent of matrix A.
I.I.D	independent and identical distributions
I.NI.D	independent and non-identical distributions.
(p.d.f)	probability density function
(c.d.f)	cumulative distribution function
$\mu_{r:n}^{(k)}$	$E(X_{r:n}^k)$
$\mu_{r,s:n}^{(k,l)}$	$E(X_{r:n}^k\ X_{s:n}^l)$
$\lambda'(x)$	$\frac{d\lambda(x)}{dx}$
$\alpha_{r:n}^{(k)}$	$E(X_{r:n}^k\ \lambda'(X_{r:n}))$
$\eta_{r,s:n}^{(k,l)}$	$E(X_{r:n}^k\ X_{s:n}^l\lambda'(X_{r:n}))$
$\xi_{r,s:n}^{(k,l)}$	$E(X_{r:n}^k\ X_{s:n}^l\lambda'(X_{s:n}))$
$\alpha_{r,s:n}^{(k,l)}$	$E(X_{r:n}^k\ X_{s:n}^l\lambda'(X_{r:n})\lambda'(X_{s:n}))$
$\Delta_{r:n}^{(k+1)}$	$E(X_{r:n}^{(k+1)}(\frac{\lambda''(X_{r:n})}{\lambda'(X_{r:n})}))$.
$\Upsilon_{r,s:n}^{(k,l)}$	$E(X_{r:n}^{(k)}X_{s:n}^{(l)}(\frac{\lambda''(X_{s:n})}{\lambda'(X_{s:n})})\lambda'(X_{r:n}))$
$\tau_{r,s:n}^{(k,l)}$	$E(X_{r:n}^{(k)}X_{s:n}^{(l)}(\frac{\lambda''(X_{r:n})}{\lambda'(X_{r:n})})\lambda'(X_{s:n}))$
$\alpha_{r:n}^{[i](k)}$	the single moments arising from $(n-1)$ variables obtained by deleting X_i from the original n variables $X_1, X_2, ..., X_n$.
$\alpha_{r:n}^{(k)}[p-1]$	the single moments when the sample of size $(n-1)$ consists of $(p-1)$ outliers .
$\alpha_{r,s:n}^{(k,l)}[p-1]$	the product moments when the sample of size $(n-1)$ consists of $(p-1)$ outliers.

Remark (2.1)

when $\lambda'(x) = 1$ we have:-

i: $\alpha_{r:n}^{(k)} = \mu_{r:n}^{(k)}, \alpha_{r,s:n}^{(k,l)} = \eta_{r,s:n}^{(k,l)} = \xi_{r,s:n}^{(k,l)} = \mu_{r,s:n}^{(k,l)}$

ii: $\Delta_{r:n}^{(k+1)} = \Upsilon_{r,s:n}^{(k,l)} = \tau_{r,s:n}^{(k,l)} = 0$

3 RELATIONS FOR SINGLE MOMENTS

In this section we shall present relations for the single moments of order statistics.

Relation (3.1) For all $n > 1$, $k = 0, 1, 2, \ldots$.

$$\alpha_{1:n}^{(k+1)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \left\{ (k+1)\mu_{1:n}^{(k)} + \Delta_{1:n}^{(k+1)} + \beta^{k+1}(\sum_{i=1}^{n} m_i)(\lambda'(\beta)) \right\}.$$

Relation (3.2) For all $1 < r \le n$, $k, l = 0, 1, 2, \ldots$.

$$\alpha_{r:n}^{(k+1)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \left\{ (k+1)\mu_{r:n}^{(k)} + \Delta_{r:n}^{(k+1)} + \sum_{i=1}^{n} m_i \alpha_{r-1:n-1}^{[i](k+1)} \right\}.$$

4 RELATIONS FOR PRODUCT MOMENTS

In this section we shall obtain five relations for moments for the product moments of order statistics :-

Relation (4.1) For all $x < y$,$1 < r < s < n$, $k, l = 0, 1, 2, \ldots$.

$$\alpha_{r,s:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{ k\xi_{r,s:n}^{(k-1,l)} + l\eta_{r,s:n}^{(k,l-1)} + \Upsilon_{r,s:n}^{(k,l)} + \tau_{r,s:n}^{(k,l)} + \sum_{i=1}^{n} m_i \alpha_{r-1,s-1:n-1}^{[i](k,l)} \}.$$

Relation (4.2) For all $1 < r < n$, $k, l = 0, 1, 2, \ldots$.

$$\alpha_{r,r+1:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{ k\xi_{r,r+1:n}^{(k-1,l)} + l\eta_{r,r+1:n}^{(k,l-1)} + \Upsilon_{r,r+1:n}^{(k,l)} + \tau_{r,r+1:n}^{(k,l)} + \sum_{i=1}^{n} m_i \alpha_{r-1,r:n-1}^{[i](k,l)} \}.$$

Relation (4.3) For all $1 < s < n$, $k, l = 0, 1, 2, \ldots$.

$$\alpha_{1,s:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{ k\xi_{1,s:n}^{(k-1,l)} + l\eta_{1,s:n}^{(k,l-1)} + \Upsilon_{1,s:n}^{(k,l)} + \tau_{1,s:n}^{(k,l)} + \beta^k \lambda'(\beta) \sum_{i=1}^{n} m_i \alpha_{s-1:n-1}^{[i](l)} \}.$$

Relation (4.4) For all $1 < n$, $k, l = 0, 1, 2, \ldots$.

$$\alpha_{1,n:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{ k\xi_{1,n:n}^{(k-1,l)} + l\eta_{1,n:n}^{(k,l-1)} + \Upsilon_{1,n:n}^{(k,l)} + \tau_{1,n:n}^{(k,l)} + \beta^k \lambda'(\beta) \sum_{i=1}^{n} m_i \alpha_{n-1:n-1}^{[i](l)} \}.$$

Relation (4.5) For all $1 < r < n$

$$\alpha_{r,n:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{ k\xi_{r,n:n}^{(k-1,l)} + l\eta_{r,n:n}^{(k,l-1)} + \Upsilon_{r,n:n}^{(k,l)} + \tau_{r,n:n}^{(k,l)} + \sum_{i=1}^{n} m_i \alpha_{r-1,n-1:n-1}^{[i](k,l)} \}.$$

Remark (4.1)

i : Relations (3.1), (3.2) and (4.1)-(4.5) will enable to compute all single and product moments.

ii: Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = 0$, $\delta = \infty$, $\lambda(x) = x$, $k = 1$, $l = 1$, $m_i = \frac{1}{\theta_i}$ the results for exponential in non-identical case are obtained (Balakrishnan 1994a).

iii: Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = 0$, $\delta = \infty$, $\lambda(x) = x^p$, $k = 1$, $l = 1$, $m_i = \theta_i$ the results for Weibull in non-identical case are obtained.

iv : Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = 1$, $\delta = \infty$, $\lambda(x) = \ln(x)$, $k = 2$, $l = 2$, $m_i = \nu_i$ the results for pareto in non-identical case are obtained (Childs 1996 Ph. D Thesis).

v : Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = 0$, $\delta = \infty$, $\lambda(x) = x^2$, $k = 1$, $l = 1$, $m_i = \theta_i$, $b = 1$ the results for Rayleigh in non-identical case are obtained.

vi: Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = 0$, $\delta = \infty$, $\lambda(x) = \ln x$, $k = 2$, $l = 2$, $m_i = \theta_i$, $b = -1$ the results for Burr III in non-identical case are obtained.

5 GENERALIZATION TO THE TRUNCATED CASE

Let us consider the case when $X_{i,s}$ are independent having doubly truncated general class of distributions with density functions:-

$$f_i(x) = m_i A_i \lambda'(x) e^{-m_i\lambda(x)} , \qquad \beta \le c \le x \le d \le \delta , \tag{5.1}$$

where $$A_i = \frac{1}{e^{-m_i\lambda(c)} - e^{-m_i\lambda(d)}} , \tag{5.2}$$

and cumulative distribution functions:

$$F_i(x) = w_i - m_i A_i e^{-m_i\lambda(x)} , \tag{5.3}$$

where $$w_i = A_i e^{-m_i\lambda(c)} = \frac{e^{-m_i\lambda(c)}}{e^{-m_i\lambda(c)} - e^{-m_i\lambda(d)}} , \quad c < x < d \ , i = 1, 2, .., n \ . \tag{5.4}$$

It is clear from Eq. (5.1) and Eq. (5.3) that the distribution satisfies the differential equations:-

$$f_i(x) = m_i \lambda'(x)(w_i - F_i(x)) \tag{5.5}$$

$$m_i \lambda''(x)(w_i - F_i(x)) = \frac{\lambda''(x)}{\lambda'(x)} f_i(x) \tag{5.6}$$

Making use of (5.1)-(5.6) we can obtain the following relations for moments for the single and the product moments of order statistics :-

Relation (5.1) For all $n > 1$, $k = 0, 1, 2, ...$.

$$\alpha_{1:n}^{(k+1)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{(k+1)\mu_{1:n}^{(k)} + \Delta_{1:n}^{(k+1)} + c^{k+1}\lambda'(c)(\sum_{i=1}^{n} m_i w_i) - \sum_{i=1}^{n} m_i(w_i - 1)\alpha_{1:n-1}^{[i](k+1)}\}.$$

Relation (5.2) For all $1 < r < n$, $k = 0, 1, 2, ...$.

$$\alpha_{r:n}^{(k+1)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{(k+1)\mu_{r:n}^{(k)} + \Delta_{r:n}^{(k+1)} + \sum_{i=1}^{n} m_i w_i \alpha_{r-1:n-1}^{[i](k+1)} - \sum_{i=1}^{n} m_i(w_i - 1)\alpha_{r:n-1}^{[i](k+1)})\}.$$

Relation (5.3) For all $n > 1$, $k = 0, 1, 2, ...$.

$$\alpha_{n:n}^{(k+1)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{(k+1)\mu_{n:n}^{(k)} + \Delta_{n:n}^{(k+1)} + \sum_{i=1}^{n} m_i w_i \alpha_{n-1:n-1}^{[i](k+1)} - d^{k+1}(\lambda'(d)(\sum_{i=1}^{n} m_i(w_i - 1))\}.$$

Relation (5.4) For all $1 < r < n$, $k, l = 0, 1, 2, ...$.

$$\alpha_{r,r+1:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)} \{i\xi_{r,r+1:n}^{(k-1,l)} + l\eta_{r,r+1:n}^{(k,l-1)} + \Upsilon_{r,r+1:n}^{(k,l)} + \tau_{r,r+1:n}^{(k,l)} + \sum_{i=1}^{n} m_i w_i \alpha_{r-1,r:n-1}^{[i](k,l)}$$

$$-\sum_{i=1}^{n} m_i(w_i-1)\alpha_{r,r+1:n-1}^{[i](k,l)}\}.$$

Relation (5.5) For all $1 < s < n,\ k,l = 0,1,2,\ldots$.

$$\alpha_{1,s:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)}\{k\xi_{1,s:n}^{(k-1,l)} + l\eta_{1,s:n}^{(k,l-1)} + \Upsilon_{1,s:n}^{(k,l)} + \sum_{i=1}^{n} m_i(w_i-1)\alpha_{1,s:n-1}^{[i](k,l)} + \tau_{1,s:n}^{(k,l)}$$

$$+c^k\lambda'(c)\sum_{i=1}^{n} m_i w_i \alpha_{s-1:n-1}^{[i](l)}\}.$$

Relation (5.6) For all $n > 1,\ k,l = 0,1,2,\ldots$.

$$\alpha_{1,n:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)}\{k\xi_{1,n:n}^{(k-1,l)} + l\eta_{1,n:n}^{(k,l-1)} + \Upsilon_{1,n:n}^{(k,l)} - d^l\lambda'(d)\sum_{i=1}^{n} m_i(w_i-1)\alpha_{1:n-1}^{[i](k)}(\lambda'(x))$$

$$+\tau_{1,n:n}^{(k,l)} + c^k\lambda'(c)\sum_{i=1}^{n} m_i w_i \alpha_{n-1:n-1}^{[i](k)}\}.$$

Relation (5.7) For all $1 < r < n,\ k,l = 0,1,2,\ldots$.

$$\alpha_{r,n:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)}\{k\xi_{r,n:n}^{(k-1,l)} + l\eta_{r,n:n}^{(k,l-1)} + \Upsilon_{r,n:n}^{(k,l)} - d^l\ \lambda'(d)\sum_{i=1}^{n} m_i(w_i-1)\alpha_{r:n-1}^{[i](k)}$$

$$+\tau_{r,n:n}^{(k,l)} + \sum_{i=1}^{n} m_i w_i \alpha_{r-1,n-1:n-1}^{[i](k,i)}\}.$$

Relation (5.8) For all $1 < r < s < n,\ k,l = 0,1,2,\ldots$.

$$\alpha_{r,s:n}^{(k,l)} = \frac{1}{(\sum_{i=1}^{n} m_i)}\{i\xi_{r,s:n}^{(k-1,l)} + l\eta_{r,s:n}^{(k,l-1)} + \Upsilon_{r,s:n}^{(k,l)} + \tau_{r,s:n}^{(k,l)} + \sum_{i=1}^{n} m_i w_i \alpha_{r-1,s-1:n-1}^{[i](k,l)}$$

$$-\sum_{i=1}^{n} m_i(w_i-1)\alpha_{r,s:n-1}^{[i](k,l)}\}.$$

Remark (5.1)

i :The relations (5.1)-(5.8) will enable us to compute the single and the product moments of order statistics in a simple recursive manner for any specified values m_i $(i = 1,2,\ldots,n)$ and truncation points (c,d) for any distribution satisfy the class conditions.

ii : When $c = \beta$, $d = \delta$ we have $w_i = 1$ (nontruncated case).

iii : If $\beta = 0$, $\delta = T$, $\lambda(x) = x$, $m_i = \frac{1}{\theta_i}$ the results for exponential in non-identical right truncated case are obtained (Balakrishnan 1994b).

iv : If $\beta = t$, $\delta = u$, $\lambda(x) = \ln(x)$, $m_i = \nu_i$ the results for pareto in non-identical doubly truncated case are obtained (Childs 1996 Ph. D Thesis).

v: Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = t$, $\delta = u$, $\lambda(x) = x^p$, $k = 1$, $l = 1$, $m_i = \theta_i$ the results for Weibull in non-identical doubly truncated case are obtained.

vi: Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = 0$, $\delta = u$, $\lambda(x) = x^p$, $k = 1$, $l = 1$, $m_i = \theta_i$ the results for Weibull in non-identical right truncated case are obtained.

vii: Relations (3.1), (3.2) and (4.1)-(4.5) if we put $\beta = t$, $\delta = \infty$, $\lambda(x) = x^p$, $k = 1$, $l = 1$, $m_i = \theta_i$ the results for Weibull in non-identical left truncated case are obtained.

viii : The results for identical case is obtained by letting $m_1 = m_2 = ... = m_n$.

6 RESULTS FOR P-OUTLIER MODEL

In this section we assume that $X_1, X_2, X_3, ..., X_{n-p}$ are independent general class of distributions with shape parameter (m) random variables, and $X_{n-p+1}, ..., X_n$ are independent general class of distributions with shape parameter (m^*) random variables. Then we obtain the following relations for moments which are deduced as special cases from relations (3.1),(3.2) and (4.1)-(4.5).

Relation (6.1) For all $n > 1$, $k = 0, 1, 2, ...$ $., p < \frac{n}{2}$

$$\alpha_{1:n}^{(k+1)}[p] = \frac{1}{((n-p)m+pm^*)}\{(k+1)\mu_{1:n}^{(k)}[p] + \Delta_{1:n}^{(k+1)}[p] + (n-p)m\ \beta^{k+1}(\lambda'(\beta)) +pm^*\ \beta^{k+1}(\lambda'(\beta))\}.$$

Relation (6.2) For all $1 < r \leqslant n$

$$\alpha_{r:n}^{(k+1)}[p] = \frac{1}{((n-p)m+pm^*)}\{(k+1)\mu_{r:n}^{(k)}[p] + \Delta_{r:n}^{(k+1)}[p] + (n-p)m\alpha_{r-1:n-1}^{(k+1)}[p] +pm^*\alpha_{r-1:n-1}^{(k+1)}[p-1]\ \}.$$

Relation (6.3) For all $1 < r < s < n$

$$\alpha_{r,s:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{r,s:n}^{(k-1,l)}[p] + l\eta_{r,s:n}^{(k,l-1)}[p] + \Upsilon_{r,s:n}^{(k,l)}[p] + \tau_{r,s:n}^{(k,l)}[p] +(n-p)m\alpha_{r-1,s-1:n-1}^{(k,l)}[p] + pm^*\alpha_{r-1,s-1:n-1}^{(k,l)}[p-1]\}.$$

Relation (6.4) For all $1 < r < n$

$$\alpha_{r,r+1:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{r,r+1:n}^{(k-1,l)}[p] + l\eta_{r,r+1:n}^{(k,l-1)}[p] + \Upsilon_{r,r+1:n}^{(k,l)}[p] + \tau_{r,r+1:n}^{(k,l)}[p] +(n-p)m\alpha_{r-1,r:n-1}^{(k,l)}[p] + pm^*\alpha_{r-1,r:n-1}^{(k,l)}[p-1]\}.$$

Relation (6.5) For all $1 < s < n$

$$\alpha_{1,s:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{1,s:n}^{(k-1,l)}[p] + l\eta_{1,s:n}^{(k,l-1)}[p] + \Upsilon_{1,s:n}^{(k,l)}[p] + \tau_{1,s:n}^{(k,l)}[p] +\beta^k\lambda'(\beta)(n-p)m\alpha_{s-1:n-1}^{(l)}[p] + pm^*\beta^k\lambda'(\beta)\alpha_{s-1:n-1}^{(l)}[p-1]\}.$$

Relation (6.6) For all $n > 1$

$$\alpha_{1,n:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{1,s:n}^{(k-1,l)}[p] + l\eta_{1,s:n}^{(k,l-1)}[p] + \Upsilon_{1,n:n}^{(k,l)}[p] + \tau_{1,n:n}^{(k,l)}[p] +\beta^k\lambda'(\beta)(n-p)m\alpha_{n-1:n-1}^{(l)}[p] + pm^*\beta^k\lambda'(\beta)\alpha_{n-1:n-1}^{(l)}[p-1]\}.$$

Relation (6.7) For all $1 < r < n$

$$\alpha_{r,n:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{r,n:n}^{(k-1,l)}[p] + l\eta_{r,n:n}^{(k,l-1)}[p] + \Upsilon_{r,n:n}^{(k,l)}[p] + \tau_{r,n:n}^{(k,l)}[p] +(n-p)m\alpha_{r-1,n-1:n-1}^{(k,l)}[p] + pm^*\alpha_{r-1,n-1:n-1}^{(k,l)}[p-1]\}.$$

7 RESULTS FOR P-OUTLIER MODEL (TRUNCATED CASE)

In this section we obtain the following relations for moments which are deduced as special cases from relations (5.1)-(5.8).

Relation (7.1) For all $n > 1$

$$\alpha_{1:n}^{(k+1)}[p] = \frac{1}{((n-p)m+pm^*)}\{(k+1)\mu_{1:n}^{(k)}[p] + \Delta_{1:n}^{(k+1)}[p] + c^{k+1}\lambda'(c)(n-p)mw_m$$
$$+c^{k+1}\lambda'(c)pm^*w_{m^*} - (n-p)m(w_i-1)\alpha_{1:n-1}^{(k+1)}(\lambda'(x))[p] - pm^*(w_{m^*}-1)\alpha_{1:n-1}^{(k+1)}[p-1]\}.$$

Relation (7.2) For all $1 < r < n$

$$\alpha_{r:n}^{(k+1)}[p] = \frac{1}{((n-p)m+pm^*)}\{(k+1)\mu_{r:n}^{(k)}[p] + \Delta_{r:n}^{(k+1)}[p] + (n-p)mw_m\alpha_{r-1:n-1}^{(k+1)}[p]$$
$$+pm^*w_{m^*}\alpha_{r-1:n-1}^{(k+1)}[p-1] - (n-p)m(w_m-1)\alpha_{r:n-1}^{(k+1)}[p]$$
$$-pm^*(w_{m^*}-1)\alpha_{r:n-1}^{(k+1)}[p-1]\}.$$

Relation (7.3) For all $n > 1$

$$\alpha_{n:n}^{(k+1)}[p] = \frac{1}{((n-p)m+pm^*)}\{(k+1)\mu_{n:n}^{(k)}[p] + \Delta_{n:n}^{(k+1)}[p] + (n-p)mw_m\alpha_{n-1:n-1}^{(k+1)}[p]$$
$$+pm^*w_{m^*}\alpha_{n-1:n-1}^{(k+1)}[p-1] - d^{k+1}(n-p)m(w_m-1)(\lambda'(d))$$
$$-d^{k+1}pm^*(w_{m^*}-1)(\lambda'(d))\}.$$

Relation (7.4) For all $1 < r < n$

$$\alpha_{r,r+1:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{r,r+1:n}^{(k-1,l)}[p] + l\eta_{r,r+1:n}^{(k,l-1)}[p] + \Upsilon_{r,r+1:n}^{(k,l)}[p] + \tau_{r,r+1:n}^{(k,l)}[p]$$
$$+(n-p)mw_m\alpha_{r-1,r:n-1}^{(k,l)}[p] + pm^*w_{m^*}\alpha_{r-1,r:n-1}^{(k,l)}[p-1]$$
$$-(n-p)m(w_m-1)\alpha_{r,r+1:n-1}^{(k,l)}[p] - pm^*(w_{m^*}-1)\alpha_{r,r+1:n-1}^{(k,l)}[p-1]\}$$

Relation (7.5) For all $1 < s < n$

$$\alpha_{1,s:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{1,s:n}^{(k-1,l)}[p] + l\eta_{1,s:n}^{(k,l-1)}[p] + \Upsilon_{1,s:n}^{(k,l)}[p]$$
$$+ (n-p)m(w_m-1)\alpha_{1,s:n-1}^{(k,l)}[p] + pm^*(w_{m^*}-1)\alpha_{1,s:n-1}^{(k,l)}[p-1] + \tau_{1,s:n}^{(k,l)}[p]$$
$$+c^k\lambda'(c)(n-p)mw_m\alpha_{s-1:n-1}^{(l)}[p] + c^k\lambda'(c)pm^*w_{m^*}\alpha_{s-1:n-1}^{(l)}[p-1]\}.$$

Relation (7.6) For all $n > 1$

$$\alpha_{1,n:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{1,n:n}^{(k-1,l)}[p] + l\eta_{1,n:n}^{(k,l-1)}[p] + \Upsilon_{1,n:n}^{(k,l)}[p]$$
$$-d^l\lambda'(d)(n-p)m(w_m-1)\alpha_{1:n-1}^{(k)}[p] - d^l\lambda'(d)pm^*(w_{m^*}-1)\alpha_{1:n-1}^{(k)}[p-1] + \tau_{1,n:n}^{(k,l)}[p]$$
$$+c^k\lambda'(c)(n-p)mw_m\alpha_{n-1:n-1}^{(l)}[p] + c^k\lambda'(c)pm^*w_{m^*}\alpha_{n-1:n-1}^{(l)}[p-1]\}.$$

Relation (7.7) For all $1 < r < n$

$$\alpha_{r,n:n}^{(k,l)}[p] = \frac{1}{((n-p)m+pm^*)}\{k\xi_{r,n:n}^{(k-1,l)}[p] + l\eta_{r,n:n}^{(k,l-1)}[p] + \Upsilon_{r,n:n}^{(k,l)}[p]$$
$$-d^l\lambda'(d)(n-p)m(w_m-1)\alpha_{r:n-1}^{(k)}[p] - d^l\lambda'(d)pm^*(w_{m^*}-1)\alpha_{r:n-1}^{(k)}[p-1]$$

$$+(n-p)mw_m\alpha_{r-1,n-1:n-1}^{(k,i)}[p]+pm^{*}w_{m^{*}}\alpha_{r-1,n-1:n-1}^{(k,i)}[p-1]\}.$$

Relation (7.8) For all $1 < r < s < n$

$$\alpha_{r,s:n}^{(k,l)}[p]=\frac{1}{((n-p)m+pm^{*})}\{k\xi_{r,s:n}^{(k-1,l)}[p]+l\eta_{r,s:n}^{(k,l-1)}[p]+\Upsilon_{r,s:n}^{(k,l)}[p]+\tau_{r,s:n}^{(k,l)}[p]$$
$$+(n-p)mw_m\alpha_{r-1,s-1:n-1}^{(k,l)}[p]+pm^{*}w_{m^{*}}\alpha_{r-1,s-1:n-1}^{(k,l)}[p-1]$$
$$-(n-p)m(w_m-1)\alpha_{r,s:n-1}^{(k,l)}[p]-pm^{*}(w_{m^{*}}-1)\alpha_{r,s:n-1}^{(k,l)}[p-1]\}.$$

8 CONCLUSION

The identities and relations for single and the product moments of order statistics from (I.NI.D) general class of distributions have been obtained. Recurrence relations for some distributions in order statistics for example see, Balakrishnan (1994a,b) and (Childs 1996) . Various results for another distributions such as Weibull, Burr III, Pareto and Rayleigh distributions in (I.NI.D) can be deduced as special cases from the general class of distributions. The results for the above distributions in (I.I.D) case can be also obtained as special cases. Finally the applications for p-outlier model are obtained.

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10 APPENDIX A

Proof. of Relation (3.2)

From Eq.(1.1) , we have

$$\mu_{r:n}^{(k)} = c_{r:n} \sum_{P} \int_{\beta}^{\delta} x^k \prod_{a=1}^{r-1} F_{ia}(x) f_{ir}(x) \prod_{b=r+1}^{n} (1 - F_{ib}(x)) dx \, . \tag{10.1}$$

By using Eq. (1.7), we obtain

$$\mu_{r:n}^{(k)} = c_{r:n} \sum_{p} m_{ir} \int_{\beta}^{\delta} x^k \prod_{a=1}^{r-1} F_{ia}(x)\lambda'(x)(1-F_{ir}(x)) \prod_{b=r+1}^{n} (1-F_{ib}(x))dx . \quad (10.2)$$

Integration by parts, yeilds

$$(k+1)\ \mu_{r:n}^{(k)} = c_{r:n} \sum_{p} m_{ir}\{-\sum_{j=1}^{r-1} \int_{\beta}^{\delta} x^{k+1} \prod_{\substack{a=1\\a\neq j}}^{r-1} F_{ia}(x) f_{ij}(x)\lambda'(x)(1-F_{ir}(x))\times \prod_{b=r+1}^{n} (1-F_{ib}(x))dx$$

$$-\int_{\beta}^{\delta} x^{k+1} \prod_{a=1}^{r-1} F_{ia}(x)\lambda''(x)(1-F_{ir}(x) \prod_{b=r+1}^{n} (1-F_{ib}(x))dx$$

$$+\int_{\beta}^{\delta} x^{k+1} \prod_{a=1}^{r-1} F_{ia}(x)\lambda'(x) f_{ir}(x) \prod_{b=r+1}^{n} (1-F_{ib}(x))dx$$

$$+\sum_{j=r+1}^{n} \int_{\beta}^{\delta} x^{k+1} \prod_{a=1}^{r-1} F_{ia}(x)\lambda'(x)(1-F_{ir}(x)) \prod_{\substack{b=r+1\\b\neq j}}^{n} (1-F_{ib}(x)) f_{ij}(x)dx\} . \quad (10.3)$$

Upon splitting the first term in integral $(1-F_{ir}(x))$ in two integrations and from Eq. (1.8), we can write

$$(k+1)\ \mu_{r:n}^{(k)} = -c_{r:n} \sum_{p} m_{ir} \sum_{j=1}^{r-1} \int_{\beta}^{\delta} x^{k+1} \prod_{\substack{a=1\\a\neq j}}^{r-1} F_{ia}(x) f_{ij}(x)\lambda'(x) \prod_{b=r+1}^{n} (1-F_{ib}(x))dx$$

$$+c_{r:n} \sum_{p} m_{ir} \sum_{j=1}^{r-1} \int_{\beta}^{\delta} x^{k+1} \prod_{\substack{a=1\\a\neq j}}^{r-1} F_{ia}(x) F_{ir}(x) f_{ij}(x)\lambda'(x) \prod_{b=r+1}^{n} (1-F_{ib}(x))dx$$

$$-c_{r:n} \sum_{p} \int_{\beta}^{\delta} x^{k+1} \prod_{a=1}^{r-1} F_{ia}(x)\left(\frac{\lambda''(x)}{\lambda'(x)}\right) f_{ir}(x) \prod_{b=r+1}^{n} (1-F_{ib}(x))dx$$

$$+c_{r:n} \sum_{p} m_{ir} \int_{\beta}^{\delta} x^{k+1} \prod_{a=1}^{r-1} F_{ia}(x)\lambda'(x) f_{ir}(x) \prod_{b=r+1}^{n} (1-F_{ib}(x))dx$$

$$+c_{r:n} \sum_{p} m_{ir} \sum_{j=r+1}^{n} \int_{\beta}^{\delta} x^{k+1} \prod_{a=1}^{r-1} F_{ia}(x)\lambda'(x)(1-F_{ir}(x) \prod_{\substack{b=r+1\\b\neq j}}^{n} (1-F_{ib}(x)) f_{ij}(x)dx. \quad (10.4)$$

Relation (3.2) is derived simply by rewriting Eq. (10.4). □

11 APPENDIX B

Proof. of Relation (4.1)

From Eq.(1.2) , we have

$$\xi_{r,s:n}^{(k-1,l)} = c_{r,s:n} \sum_{p} \int_{\beta}^{\delta} \int_{\beta}^{y} x^{k-1} y^{l} \lambda'(y) \prod_{a=1}^{r-1} F_{ia}(x) f_{ir}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y)\times \prod_{c=s+1}^{n} (1-F_{ic}(y))dxdy . \quad (11.1)$$

From Eq. (1.7), we find

$$\xi_{r,s:n}^{(k-1,l)} = c_{r,s:n} \sum_{p} m_{ir} \int_{\beta}^{\delta} \int_{\beta}^{y} x^{k-1} y^{l} \lambda'(y) \lambda'(x) \prod_{a=1}^{r-1} F_{ia}(x)(1 - F_{ir}(x)) \times$$
$$\prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is(y)} \prod_{c=s+1}^{n} (1 - F_{ic}(y)) dx dy \quad . \qquad (11.2)$$

Let $J(y) = \int_{\beta}^{y} x^{k-1} \prod_{a=1}^{r-1} F_{ia}(x) \lambda'(x)(1 - F_{ir}(x)) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) dx \quad . \qquad (11.3)$

Integration by parts, yeilds

$$kJ(y) = -\sum_{j=1}^{r-1} \int_{\beta}^{y} x^{k} \prod_{\substack{a=1 \\ a \neq j}}^{r-1} F_{ia}(x) \lambda'(x) f_{ij}(x)(1 - F_{ir}(x)) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) dx$$
$$- \int_{\beta}^{y} x^{k} \prod_{a=1}^{r-1} F_{ia}(x) \lambda''(x)(1 - F_{ir}(x)) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) dx$$
$$+ \int_{\beta}^{y} x^{k} \prod_{a=1}^{r-1} F_{ia}(x) \lambda'(x) f_{ir}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) dx$$
$$+ \sum_{j=r+1}^{s-1} \int_{\beta}^{y} x^{k} \prod_{a=1}^{r-1} F_{ia}(x) \lambda'(x)(1 - F_{ir}(x)) f_{ij}(x) \prod_{\substack{b=r+1 \\ b \neq j}}^{s-1} (F_{ib}(y) - F_{ib}(x)) dx \quad . \qquad (11.4)$$

Upon splitting the first term in integral $(1 - F_{ir}(x))$ in two integrations ,we find

$$k\xi_{r,s:n}^{(k-1,l)} = c_{r,s:n} \sum_{p} m_{ir} \{ -\sum_{j=1}^{r-1} \int_{\beta}^{\delta} \int_{\beta}^{y} x^{k} y^{l} \prod_{\substack{a=1 \\ a \neq j}}^{r-1} F_{ia}(x) \lambda'(x) \lambda'(y) f_{ij}(x) \times$$
$$\prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y) \prod_{c=s+1}^{n} (1 - F_{ic}(y)) dx dy$$
$$+ \sum_{j=1}^{r-1} \int_{\beta}^{\delta} \int_{\beta}^{y} x^{k} y^{l} \prod_{\substack{a=1 \\ a \neq j}}^{r} F_{ia}(x) \lambda'(x) \lambda'(y) f_{ij}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) \times$$
$$f_{is}(y) \prod_{c=s+1}^{n} (1 - F_{ic}(y)) dx dy$$
$$- \int_{\beta}^{\delta} \int_{\beta}^{y} x^{k} y^{l} \prod_{a=1}^{r-1} F_{ia}(x) \lambda''(x) \lambda'(y)(1 - F_{ir}(x)) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) \times$$
$$f_{is}(y) \prod_{c=s+1}^{n} (1 - F_{ic}(y)) dx dy$$
$$+ \int_{\beta}^{\delta} \int_{\beta}^{y} x^{k} y^{l} \prod_{a=1}^{r-1} F_{ia}(x) \lambda'(x) \lambda'(y) f_{ir}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y) \prod_{c=s+1}^{n} (1 - F_{ic}(y)) dx dy$$
$$+ \sum_{j=r+1}^{s-1} \int_{\beta}^{\delta} \int_{\beta}^{y} x^{k} y^{l} \prod_{a=1}^{r-1} F_{ia}(x) \lambda'(x) \lambda'(y)(1 - F_{ir}(x)) f_{ij}(x) \prod_{\substack{b=r+1 \\ b \neq j}}^{s-1} (F_{ib}(y) - F_{ib}(x)) \times$$
$$f_{is}(y) \prod_{c=s+1}^{n} (1 - F_{ic}(y)) dx dy \} . \qquad (11.5)$$

From Eq. (1.8) , we have

$$k\xi_{r,s:n}^{(k-1,l)} = c_{r,s:n}\{-\sum_{p} m_i \sum_{j=1}^{r-1} \int_{\beta}^{\delta}\int_{\beta}^{y} x^k y^l \prod_{\substack{a=1\\a\neq j}}^{r-1} F_{ia}(x)\lambda'(x)\lambda'(y) f_{ij}(x)\times$$
$$\prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y) \prod_{c=s+1}^{n} (1 - F_{ic}(y))dxdy$$
$$+\sum_{p} m_{ir} \sum_{j=1}^{r-1} \int_{\beta}^{\delta}\int_{\beta}^{y} x^k y^l \prod_{\substack{a=1\\a\neq j}}^{r} F_{ia}(x)\lambda'(x)\lambda'(y) f_{ij}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x))\times$$
$$f_{is}(y) \prod_{c=s+1}^{n} (1 - F_{ic}(y))dxdy$$
$$-\sum_{p} \int_{\beta}^{\delta}\int_{\beta}^{y} x^k y^l \prod_{a=1}^{r-1} F_{ia}(x) f_{ir}(x)\left(\frac{\lambda''(x)}{\lambda'(x)}\right)\lambda'(y) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y)\times$$
$$\prod_{c=s+1}^{n} (1 - F_{ic}(y))dxdy$$
$$+\sum_{p} m_{ir} \int_{\beta}^{\delta}\int_{\beta}^{y} x^k y^l \prod_{a=1}^{r-1} F_{ia}(x)\lambda'(x)\lambda'(y) f_{ir}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y)\times$$
$$\prod_{c=s+1}^{n} (1 - F_{ic}(y))dxdy$$
$$+\sum_{p} m_i \sum_{j=r+1}^{s-1} \int_{\beta}^{\delta}\int_{\beta}^{y} x^k y^l \prod_{a=1}^{r-1} F_{ia}(x)\lambda'(x)\lambda'(y)(1 - F_{ir}(x)) f_{ij}(x)\times$$
$$\prod_{\substack{b=r+1\\b\neq j}}^{s-1} (F_{ib}(y)-F_{ib}(x))\, f_{is}(y) \prod_{c=s+1}^{n} (1-F_{ic}(y))dxdy\,\}. \tag{11.6}$$

On other hand

$$\eta_{r,s:n}^{(k,l-1)} = c_{r,s:n} \sum_{p} \int_{\beta}^{\delta}\int_{x}^{\delta} x^k y^{l-1} \lambda'(x) \prod_{a=1}^{r-1} F_{ia}(x) f_{ir}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{is}(y)\times$$
$$\prod_{c=s+1}^{n} (1 - F_{ic}(y))dydx\ . \tag{11.7}$$

From Eq. (1.7), we have

$$\eta_{r,s:n}^{(k,l-1)} = c_{r,s:n} \sum_{p} m_{is} \int_{\beta}^{\delta}\int_{x}^{\delta} x^k y^{l-1} \prod_{a=1}^{r-1} F_{ia}(x) f_{ir}(x) \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x))\lambda'(y)\lambda'(x)\times$$
$$(1-F_{is}(y)) \prod_{c=s+1}^{n} (1-F_{ic}(y))dydx \tag{11.8}$$

Let $I(x) = \int_{x}^{\delta} y^{l-1} \prod_{b=r+1}^{s-1} (F_{ib}(y) - F_{ib}(x))\lambda'(y)(1 - F_{is}(y)\times$

$$\prod_{c=s+1}^{n} (1-F_{ic}(y))dy\ . \tag{11.9}$$

Integration by parts, yeilds

$$lI(x) = -\sum_{j=r+1}^{s-1} \int_{x}^{\delta} y^l \prod_{\substack{b=r+1\\b\neq j}}^{s-1} (F_{ib}(y) - F_{ib}(x)) f_{ij}(y)\lambda'(y)(1 - F_{is}(y))\times$$

$$\prod_{c=s+1}^{n}(1-F_{ic}(y))dy$$

$$-\int_{x}^{\delta} y^{l}\lambda^{''}(y)\prod_{b=r+1}^{s-1}(F_{ib}(y)-F_{ib}(x))(1-F_{is}(y))\prod_{c=s+1}^{n}(1-F_{ic}(y))dy$$

$$+\int_{x}^{\delta} y^{l}\lambda^{'}(y)\prod_{b=r+1}^{s-1}(F_{ib}(y)-F_{ib}(x))f_{is}(y)\prod_{c=s+1}^{n}(1-F_{ic}(y))dy$$

$$+\sum_{j=s+1}^{n}\int_{x}^{\delta} y^{l}\lambda^{'}(y)\prod_{b=r+1}^{s-1}(F_{ib}(y)-F_{ib}(x))(1-F_{is}(y))f_{ij}(y)\times \prod_{\substack{c=s+1\\c\neq j}}^{n}(1-F_{ic}(y))dy \quad . \qquad (11.10)$$

By using (1.8) we have

$$l\eta_{r,s:n}^{(k,l-1)} = -c_{r,s:n}\sum_{p} m_{is}\sum_{j=r+1}^{s-1}\int_{\beta}^{\delta}\int_{x}^{\delta} x^{k}y^{l}\prod_{a=1}^{r-1}F_{ia}(x)f_{ir}(x)\lambda^{'}(x)\times \prod_{\substack{b=r+1\\b\neq j}}^{s-1}(F_{ib}(y)-F_{ib}(x))\ f_{ij}(y)\lambda^{'}(y)\ (1-F_{is}(y))\prod_{c=s+1}^{n}(1-F_{ic}(y))dydx$$

$$-c_{r,s:n}\sum_{p}\int_{\beta}^{\delta}\int_{x}^{\delta} x^{k}y^{l}\prod_{a=1}^{r-1}F_{ia}(x)f_{ir}(x)\lambda^{'}(x)\left(\frac{\lambda^{''}(y)}{\lambda^{'}(y)}\right)f_{is}(y)\prod_{b=r+1}^{s-1}(F_{ib}(y)-F_{ib}(x))\times \prod_{c=s+1}^{n}(1-F_{ic}(y))dydx$$

$$+c_{r,s:n}\sum_{p} m_{is}\int_{\beta}^{\delta}\int_{x}^{\delta} x^{k}y^{l}\prod_{a=1}^{r-1}F_{ia}(x)f_{ir}(x)\lambda^{'}(x)\lambda^{'}(y)\prod_{b=r+1}^{s-1}(F_{ib}(y)-F_{ib}(x))f_{is}(y)\times \prod_{c=s+1}^{n}(1-F_{ic}(y))dydx$$

$$+c_{r,s:n}\sum_{p} m_{is}\sum_{j=s+1}^{n}\int_{\beta}^{\delta}\int_{x}^{\delta} x^{k}y^{l}\prod_{a=1}^{r-1}F_{ia}(x)f_{ir}(x)\lambda^{'}(x)\lambda^{'}(y)\prod_{b=r+1}^{s-1}(F_{ib}(y)-F_{ib}(x))\times (1-F_{is}(y))f_{ij}(y)\prod_{\substack{c=s+1\\c\neq j}}^{n}(1-F_{ic}(y))dydx \quad . \qquad (11.11)$$

Adding (11.6), (11.11) and simplifying the resulting expression, We obtain the relation (4.1) □