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NEW CLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH q-ANALOGUE OF STRUVE FUNCTION

A. A. THOMBRE, D. N. CHATE, MALLIKARJUN G. SHRIGAN

Abstract. In this paper, we introduce and explore a new subclass of bi-univalent functions defined using q-analogue of Struve function in the open disk using quasi-subordination. For functions within this subclass, we provide estimates for the first two Taylor-Maclaurin coefficients. Some consequences of the main results are also observed.

1. Introduction

Let the set of all complex numbers be denoted by $\mathbb{C}$ and let the unit disk $(\varsigma \in \mathbb{C} : |\varsigma| < 1)$ be denoted by $\mathbb{U}$. Let $\mathcal{Q}$ be the class of functions $\mathfrak{l}(\varsigma)$ of the form

$$\mathfrak{l}(\varsigma) = \varsigma + \sum_{\mathrm{j}=2}^{\infty} a_{\mathrm{j}}\, \varsigma^{\mathrm{j}}, \quad (\varsigma \in \mathbb{U}), \tag{1}$$

which is analytic function in the unit disc $\mathbb{U}$ and $\mathcal{S}$ be the subclass of $\mathcal{Q}$ consisting of univalent functions.

For the functions $\mathfrak{l}_r(\varsigma) = \sum\limits_{\mathrm{j}=0}^{\infty} a_{\mathrm{j},r}\varsigma^{\mathrm{j}}$ where $r = 1, 2$ and both are analytic in $\mathbb{U}$, the Hadamard (or convolution) product of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ is defined as follows:

$$(\mathfrak{l}_1 * \mathfrak{l}_2)(\varsigma) := \varsigma + \sum_{\mathrm{j}=0}^{\infty} a_{\mathrm{j},1} a_{\mathrm{j},2}\, \varsigma^{\mathrm{j}},\ \varsigma \in \mathbb{U}. \tag{2}$$

For two functions $h, g \in \mathcal{Q}$, we say that h is subordinate to g, denoted as $h(\varsigma) \prec g(\varsigma)$, if there exists a Schwarz function $\xi(\varsigma) = \sum\limits_{\mathrm{j}=1}^{\infty} b_{\mathrm{j}}\varsigma^{\mathrm{j}}$ that is analytic in $\mathbb{U}$, satisfies $\xi(0) = 0$, and has the property $|\xi(z)| < 1$ for all $\varsigma \in \mathbb{U}$, such that $h(\varsigma) = g(\xi(\varsigma))$

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for all $\varsigma \in \mathbb{U}$. Further, if the function g is univalent in $\mathbb{U}$, then the next equivalence holds:

$$h(\varsigma) \prec g(\varsigma) \Leftrightarrow h(0) = g(0) \text{ and } h(\mathbb{U}) \subset g(\mathbb{U}).$$

The concept of *quasi-subordination* for two analytic functions is due to Robertson [27]. The function h is said to be *quasi-subordinate* to g, engraved as $h(\varsigma) \prec_{\varrho} g(\varsigma)$, if there exist the analytic functions ϱ and w, with $w(0) = 0$ such that $|\varrho(\varsigma)| \le 1$, $|w(\varsigma)| < 1$, and $h(\varsigma) = \varrho(\varsigma) g(w(\varsigma))$ for all $\varsigma \in \mathbb{U}$. Note that, if $\varrho(\varsigma) = 1$, then $h(\varsigma) = g(w(\varsigma))$, hence $h(\varsigma) \prec g(\varsigma)$ in $\varsigma \in \mathbb{U}$.

Let $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ represent the subclasses of $\mathcal{Q}$ consisting of functions that are, respectively, starlike and convex of order γ (where $0 \le \gamma < 1$) in the unit disk $\mathbb{U}$. Thus, we have (see, for details,[13], [28])

$$\mathcal{S}^*(\gamma) = \left\{ \mathfrak{l} : \mathfrak{l} \in \mathcal{Q} \text{ and } Re\left(\frac{\varsigma \mathfrak{l}'(\varsigma)}{\mathfrak{l}(\varsigma)} \right) > \gamma, (\varsigma \in \mathbb{U}; 0 \le \gamma < 1) \right\}, \tag{3}$$

$$\mathcal{C}(\gamma) = \left\{ \mathfrak{l} : \mathfrak{l} \in \mathcal{Q} \text{ and } Re\left(1 + \frac{\varsigma \mathfrak{l}''(\varsigma)}{\mathfrak{l}'(\varsigma)} \right) > \gamma, (\varsigma \in \mathbb{U}; 0 \le \gamma < 1) \right\}, \tag{4}$$

where, for convenience,

$$\mathcal{S}^*(0) = \mathcal{S}^*, \quad \mathcal{C}(0) = \mathcal{C}.$$

It is a well-established fact that

$$\mathfrak{l}(\varsigma) \in \mathcal{C}(\gamma) \Leftrightarrow \varsigma \mathfrak{l}'(\varsigma) \in \mathcal{S}^*(\gamma).$$

Srivastava [31] used different operators of q-calculus and fractional q-calculus and remembering the definition and notations. Let $0 < q < 1$, the *q-derivative* is defined as (also see [29])

$$\mathfrak{D}_q(\mathfrak{l}(\varsigma)) = \begin{cases} \frac{\mathfrak{l}(q\varsigma) - \mathfrak{l}(\varsigma)}{(q-1)\varsigma}, & \text{for } \varsigma \neq 0, \\ \mathfrak{l}'(0), & \text{for } \varsigma = 0. \end{cases} \tag{5}$$

We note that $\lim\limits_{q \to 1^-} \mathfrak{D}_q(\mathfrak{l}(\varsigma)) = \mathfrak{l}'(\varsigma)$. For function $\mathfrak{l}$ given by (1), we deduce that

$$\mathfrak{D}_q(\mathfrak{l}(\varsigma)) = 1 + \sum_{\mathfrak{j}=2}^{\infty} [\mathfrak{j}, q] a_{\mathfrak{j}} \, \varsigma^{\mathfrak{j}-1},$$

where

$$[\mathfrak{j}, q] = \frac{1 - q^{\mathfrak{j}}}{1 - q} = 1 + q + ... + q^{\mathfrak{j}} \longrightarrow \mathfrak{j}.$$

As $q \to 1^-$, $[\mathfrak{j}]_q \to \mathfrak{j}$. For a function $h(\varsigma) = \varsigma^{\mathfrak{j}}$, we observed that

$$\mathfrak{D}_q(h(\varsigma)) = \mathfrak{D}_q(\varsigma^{\mathfrak{j}}) = \frac{1 - q^{\mathfrak{j}}}{1 - q} \, \varsigma^{\mathfrak{j}-1} = [\mathfrak{j}, q] \, \varsigma^{\mathfrak{j}-1},$$

$$\lim_{q \to 1^-} (\mathfrak{D}_q(h(\varsigma))) = \mathfrak{j} \, \varsigma^{\mathfrak{j}-1} = h'(\varsigma),$$

where h' is the ordinary derivative and $[\mathfrak{j}, q]$ is the q–integer or basic number $\mathfrak{j}$.

The Koebe one-quarter theorem [13] guarantees that the image of the unit disk $\mathbb{U}$ under any univalent function $\mathfrak{l} \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Accordingly, every function $\mathfrak{l} \in \mathcal{S}$ has an inverse $\mathfrak{l}^{-1}$, which is defined by

$$\mathfrak{l}^{-1}(\mathfrak{l}(\varsigma)) = \varsigma, \quad (\varsigma \in \mathbb{U})$$

and

$$\mathfrak{l}(\mathfrak{l}^{-1}(w)) = w, \quad \left(|w| < r_0(\mathfrak{l}); r_0(\mathfrak{l}) \geq \frac{1}{4}\right),$$

where

$$g(w) = \mathfrak{l}^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{6}$$

We denote by Σ the class of all functions $\mathfrak{l}(\varsigma)$ that are bi-univalent in $\mathbb{U}$ and can be expressed through the Taylor-Maclaurin series expansion given in (1). Examples of functions in the class Σ are

$$\mathfrak{l}_1(\varsigma) = \frac{\varsigma}{1-\varsigma}, \qquad \mathfrak{l}_2(\varsigma) = -log(1-\varsigma), \qquad \mathfrak{l}_3(\varsigma) = \frac{1}{2} log\left(\frac{1+\varsigma}{1-\varsigma}\right).$$

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges [9] of the famous Bieberbach conjecture. There is an extensive literature dealing with geometric properties of different families of special functions, particularly the generalized hypergeometric functions [11, 20, 23, 24] and the Bessel functions [6, 7]. Someone can find more information about geometric properties of special functions in [1, 8, 14, 24]. Struve functions find applications in various fields, including water wave dynamics, ground wave analysis, unstable aerodynamics, resistive magnetohydrodynamic (MHD) instability theory, and optical directionality. More recently, Struve functions have appeared in many particles quantum dynamics studies of spin decoherence. We refer, in this connection, to the works of [30, 35].

The generalized Struve function of order p which is defined as

$$w_p(\varsigma) = \sum_{\mathfrak{j} \geq 0} \frac{(-c)^{\mathfrak{j}}}{\Gamma\left(\mathfrak{j} + \frac{3}{2}\right) \Gamma\left(p + \mathfrak{j} + \frac{b+2}{2}\right)} \left(\frac{\varsigma}{2}\right)^{2\mathfrak{j}+p+1}. \tag{7}$$

Although the series in the above definition is convergent everywhere in $\mathbb{U}$, the function $w_p(\varsigma)$ is generally not univalent in $\mathbb{U}$. We consider the function $u_p(\varsigma)$ defined by

$$u_p(\varsigma) = 2^p \sqrt{\pi}\, \Gamma\left(p + \frac{b+2}{2}\right) \varsigma^{-\frac{p+1}{2}} w_p(\sqrt{\varsigma}), \ (\sqrt{1} := 1). \tag{8}$$

Using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions by

$$(\lambda)_{\mathfrak{j}} = \frac{\Gamma(\lambda + \mathfrak{j})}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \mathfrak{j} = 0, \\ \lambda(\lambda+1)...(\lambda+\mathfrak{j}-1), & \text{if } \mathfrak{j} \in \mathbb{N}. \end{cases}$$

We obtain for the function $u_p(\varsigma)$ the following form

$$u_p(\varsigma) = \varsigma + \sum_{\mathfrak{j}=2}^{\infty} \frac{(-c/4)^{\mathfrak{j}}}{(3/2)_{\mathfrak{j}} (k)_{\mathfrak{j}}} \varsigma^{\mathfrak{j}}, \qquad (\varsigma \in \mathbb{U}), \tag{9}$$

where $k := p + \frac{b+2}{2} \neq 0, -1, -2, \cdots$. This function $u_p(\varsigma)$ is analytic in $\mathbb{U}$ (also refer [12]).

For $0 < q < 1$, the q-derivative operator for $u_p(\varsigma)$ is defined by:

$$\begin{aligned}\mathfrak{D}_{\mathfrak{q}} u_p(\varsigma) = \mathfrak{D}_{\mathfrak{q}}\left[\varsigma + \sum_{\mathrm{j}=2}^{\infty} \frac{(-c/4)^{\mathrm{j}}}{(3/2)_{\mathrm{j}}(k)_{\mathrm{j}}}\varsigma^{\mathrm{j}}\right] &:= \frac{u_p(q\varsigma) - u_p(\varsigma)}{\varsigma(q-1)} \\ &= 1 + \sum_{\mathrm{j}=2}^{\infty} \frac{(-c/4)^{\mathrm{j}}}{(3/2)_{\mathrm{j}}(k)_{\mathrm{j}}}[\mathrm{j}, q]\varsigma^{\mathrm{j}-1}, \quad (\varsigma \in \mathbb{U}),\end{aligned}$$

where

$$[\mathrm{j}, q] = \frac{1 - q^{\mathrm{j}}}{1 - q} = 1 + \sum_{l=1}^{l-1} q^l, \quad [0, q] := 0. \tag{10}$$

For $\delta > -1$ and $0 < q < 1$, define the function $\mathcal{I}_{k,c}^{\delta,q}(\varsigma) : \mathbb{U} \to \mathbb{C}$ by:

$$\mathcal{I}_{k,c}^{\delta,q}(\varsigma) = \varsigma + \sum_{\mathrm{j}=2}^{\infty} \frac{(-c/4)^{\mathrm{j}}}{(3/2)_{\mathrm{j}}(k)_{\mathrm{j}}} \frac{[\mathrm{j}, q]!}{[\delta + 1, q]_{\mathrm{j}-1}}\varsigma^{\mathrm{j}}, \quad (\varsigma \in \mathbb{U}). \tag{11}$$

where q-shifted factorial for any non-negative integer j is given by

$$[\mathrm{j}, q]! = \begin{cases} 1, & \text{if } \mathrm{j} = 0, \\ [1, q][2, q]...[\mathrm{j}, q], & \text{if } \mathrm{j} \in \mathbb{N} \end{cases}$$

and q-generalized Pochhammer symbol for $s > 0$ is defined by

$$[s, q]_{\mathrm{j}} = \begin{cases} 1, & \text{if } \mathrm{j} = 0, \\ [s, q][s + 1, q]...[s + \mathrm{j} - 1, q], & \text{if } \mathrm{j} \in \mathbb{N}. \end{cases}$$

Remark 1. *A simple computation shows that*

$$\mathcal{I}_{k,c}^{\delta,q}(\varsigma) * \mathcal{R}_{q,\delta+1}(\varsigma) = \varsigma\mathfrak{D}_{\mathfrak{q}} u_p(\varsigma), \varsigma \in \mathbb{U}.$$

where the function $\mathcal{R}_{q,\delta+1}(\varsigma)$ *is given by*

$$\mathcal{R}_{q,\delta+1}(\varsigma) = \varsigma + \sum_{\mathrm{j}=2}^{\infty} \frac{[\delta + 1, q]_{\mathrm{j}-1}}{[\mathrm{j} - 1, q]!}\varsigma^{\mathrm{j}}, \quad (\varsigma \in \mathbb{U}).$$

Now using idea of convolutions by El-Deeb and Bulboacă [15], we define a linear operator $\mathcal{W}_{k,c}^{\delta,q} : \mathcal{Q} \to \mathcal{Q}$ as

$$\begin{aligned}\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma) = \mathcal{I}_{k,c}^{\delta,q}(\varsigma) * \mathfrak{l}(\varsigma) &= \varsigma + \sum_{\mathrm{j}=2}^{\infty} \frac{(-c/4)^{\mathrm{j}}}{(3/2)_{\mathrm{j}}(k)_{\mathrm{j}}} \frac{[\mathrm{j}, q]!}{[\delta + 1, q]_{\mathrm{j}-1}} a_{\mathrm{j}}\varsigma^{\mathrm{j}} \\ &= \varsigma + \sum_{\mathrm{j}=2}^{\infty} \vartheta_{\mathrm{j}} a_{\mathrm{j}}\varsigma^{\mathrm{j}}, \quad (\delta > -1, 0 < q < 1 \text{ and } \varsigma \in \mathbb{U}),\end{aligned} \tag{12}$$

where coefficients of $\{\vartheta_{\mathrm{j}}\}_{\mathrm{j}=2}^{\infty}$ are given by

$$\vartheta_{\mathrm{j}} = \frac{(-c/4)^{\mathrm{j}}}{(3/2)_{\mathrm{j}}(k)_{\mathrm{j}}} \frac{[\mathrm{j}, q]!}{[\delta + 1, q]_{\mathrm{j}-1}}. \tag{13}$$

Remark 2. *Taking different particular cases for the coefficients* $\vartheta_{\mathfrak{j}}$ *we obtain the next special cases for the operator* $\mathcal{W}_{k,c}^{\delta,q}$

(i) $\vartheta_{\mathfrak{j}} = \left(\dfrac{1+\mathfrak{s}+\mu(\mathfrak{j}-1)}{1+\mathfrak{s}}\right)^m \dfrac{[\mathfrak{j},q]!}{[\delta+1,q]_{\mathfrak{j}-1}}, m \in \mathbb{Z}, \mathfrak{s} \geq 0, \mu \geq 0,$ *we obtain the operator* $\mathcal{W}_{k,c}^{\delta,q} := \mathcal{L}_{q,\mathfrak{s},\mu}^{\delta,m}$ *studied by* [26].

(ii) $\vartheta_{\mathfrak{j}} = \dfrac{(-1)^{\mathfrak{j}-1}\Gamma(\zeta+1)}{4^{\mathfrak{j}-1}(\mathfrak{j}-1)\Gamma(\mathfrak{j}+\zeta)} \dfrac{[\mathfrak{j},q]!}{[\delta+1,q]_{\mathfrak{j}-1}}, \zeta > 0,$ *we obtain the operator* $\mathcal{W}_{k,c}^{\delta,q} := \mathcal{H}_{\zeta,q}^{\delta}$ *studied by* [15].

(iii) $\vartheta_{\mathfrak{j}} = \left(\dfrac{n+1}{n+\mathfrak{j}}\right)^{\alpha} \dfrac{[\mathfrak{j},q]!}{[\delta+1,q]_{\mathfrak{j}-1}}, \alpha > 0, n \geq 0,$ *we obtain the operator* $\mathcal{W}_{k,c}^{\delta,q} := \mathcal{G}_{n,q}^{\delta,\alpha}$ *studied by* [16].

(iv) $\vartheta_{\mathfrak{j}} = \dfrac{m^{\mathfrak{j}-1}}{(\mathfrak{j}-1)} e^{-m} \dfrac{[\mathfrak{j},q]!}{[\delta+1,q]_{\mathfrak{j}-1}}, m > 0,$ *we obtain the operator* $\mathcal{W}_{k,c}^{\delta,q} := \mathcal{J}_q^{\delta,m}$ *studied by* [25].

(v) $\vartheta_{\mathfrak{j}} = \dfrac{[\mathfrak{j},q]!}{[\delta+1,q]_{\mathfrak{j}-1}},$ *we obtain the operator* $\mathcal{W}_{k,c}^{\delta,q} := \mathcal{S}_{n,q}^{\delta,\alpha}$ *studied by* [5].

Remark 3. *From the definition relation (12), we can easily verify that the next relations hold for all* $\mathfrak{l} \in \mathcal{Q}$

(i) $[\delta+1,q]\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma) = [\delta,q]\mathcal{W}_{k,c}^{\delta+1,q}\mathfrak{l}(\varsigma) + q^{\delta}\varsigma\,\mathfrak{D}_{\mathfrak{q}}\mathcal{W}_{k,c}^{\delta+1,q}\mathfrak{l}(\varsigma), \quad \varsigma \in \mathbb{U}.$

(ii) $\mathcal{M}_{k,c}^{\delta}\mathfrak{l}(\varsigma) = \lim\limits_{q\to 1^-} \mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma) = \varsigma + \sum\limits_{\mathfrak{j}=2}^{\infty} \dfrac{\mathfrak{j}!}{(\delta+1)_{\mathfrak{j}-1}} \dfrac{(-c/4)^{\mathfrak{j}}}{(3/2)_{\mathfrak{j}}(k)_{\mathfrak{j}}} a_{\mathfrak{j}}\varsigma^{\mathfrak{j}}, \ \varsigma \in \mathbb{U}.$

Recently El-Deeb et al. [18] investigated geometric characteristics of new subclasses connected with a q-analogue derivative and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients as well as Fekete–Szegö inequalities. It was followed by such works as those by Srivastava and El-Deeb [32], El-Deeb and Murugusunderamoorthy [17] and El-Deeb and Orhan [19] (see also [2, 3, 4, 21, 22, 33, 34]). Motivated by aforecited work, we introduce classes of analytic bi-univalent functions using the operator $\mathcal{W}_{k,c}^{\delta,q}$, as follows.

Definition 1.1. *A function* $\mathfrak{l}(\varsigma) \in \Sigma$ *given by (1) is said to be in the class* $\mathcal{N}_{\Sigma}^{q}(\psi,\nu,\varpi)$*, if the following conditions are satisfies:*

$$\frac{1}{|\psi|}\left[\frac{\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)+\nu\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}-1\right] \prec_{\varrho} (\varpi(\varsigma)-1)$$

and

$$\frac{1}{|\psi|}\left[\frac{w\left(\mathcal{W}_{k,c}^{\delta,q}g(w)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}g(w)+\nu w\left(\mathcal{W}_{k,c}^{\delta}g(w)\right)'}-1\right] \prec_{\varrho} (\varpi(w)-1),$$

where $0 \leq \nu < 1$, $\psi \in \mathbb{C}\setminus\{0\}$, $0 < q < 1$, $\varsigma \in \mathbb{U}$ *and the function* g *is given by (6).*

Remark 4. *For $q \to 1^-$, a function $\mathfrak{l}(\varsigma) \in \Sigma$ given by (1) is said to be in the class $\mathcal{N}_\Sigma(\psi, \nu, \varpi)$, if the following conditions are satisfies:*

$$\frac{1}{|\psi|}\left[\frac{\varsigma\left(\mathcal{W}_{k,c}^{\delta}\mathfrak{l}(\varsigma)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta}\mathfrak{l}(\varsigma)+\nu\varsigma\left(\mathcal{W}_{k,c}^{\delta}\mathfrak{l}(\varsigma)\right)'}-1\right] \prec_{\varrho} (\varpi(\varsigma)-1)$$

and

$$\frac{1}{|\psi|}\left[\frac{w\left(\mathcal{W}_{k,c}^{\delta}g(w)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta}g(w)+\nu w\left(\mathcal{W}_{k,c}^{\delta,q}g(w)\right)'}-1\right] \prec_{\varrho} (\varpi(w)-1),$$

where $0 \le \nu < 1, \psi \in \mathbb{C}\setminus\{0\}, \varsigma \in \mathbb{U}$ and the function g is given by (6).

Remark 5. *For $\nu = 0$, a function $\mathfrak{l}(\varsigma) \in \Sigma$ given by (1) is said to be in the class $\mathcal{N}_\Sigma(\psi, \varpi)$, if the following conditions are satisfies:*

$$\frac{1}{|\psi|}\left[\frac{\varsigma\left(\mathcal{W}_{k,c}^{\delta}\mathfrak{l}(\varsigma)\right)'}{\mathcal{W}_{k,c}^{\delta}\mathfrak{l}(\varsigma)}-1\right] \prec_{\varrho} (\varpi(\varsigma)-1)$$

and

$$\frac{1}{|\psi|}\left[\frac{w\left(\mathcal{W}_{k,c}^{\delta}g(w)\right)'}{\mathcal{W}_{k,c}^{\delta}g(w)}-1\right] \prec_{\varrho} (\varpi(w)-1),$$

where $\psi \in \mathbb{C}\setminus\{0\}, \varsigma \in \mathbb{U}$ and the function g is given by (6).

The well-known Carathéodory's class $\mathcal{P}$ of analytic functions in $\mathbb{U}$, normalized with $P(0) = 1$, and having positive real part in $\mathbb{U}$, that is $\mathrm{Re}P(\varsigma) > 0$ for all $\varsigma \in \mathbb{U}$ [10]. To prove our results, we need the following lemma.

Lemma 1.1. [13] *If $p \in \mathcal{P}$, then $|c_k| \le 2$, for each $k \ge 1$ and the inequality is sharp for each n. In particular, equality holds for all n for the function $p(\varsigma) = \frac{1+\varsigma}{1-\varsigma} = 1 + \sum\limits_{\mathrm{j}=1}^{\infty} 2\varsigma^j$.*

Definition 1.2. *A function $\mathfrak{l}(\varsigma)$ given by (1) and $\phi(\varsigma)$ and $\varphi(w)$ in $\mathcal{P}$ such that*

$$\phi(\varsigma) = 1 + \phi_1\varsigma + \phi_2\varsigma^2 + \dots, \tag{14}$$

$$\varphi(w) = 1 + \varphi_1 w + \varphi_2 w^2 + \dots, \tag{15}$$

then $\mathfrak{l}(\varsigma)$ is said to be in the class $\mathcal{N}_\Sigma^q(\phi, \varphi, \psi, \nu, \varpi)$, $0 \le \nu < 1$, $\psi \in \mathbb{C}\setminus\{0\}$, $0 < q < 1$, $\varsigma \in \mathbb{U}$, if the following conditions are satisfies:

$$\mathfrak{l} \in \Sigma \text{ and } \frac{1}{|\psi|}\left(\frac{\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)+\nu\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}-1\right) \in \phi(\mathbb{U}) \tag{16}$$

and

$$\frac{1}{|\psi|}\left(\frac{w\left(\mathcal{W}_{k,c}^{\delta,q}g(w)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}g(w)+\nu w\left(\mathcal{W}_{k,c}^{\delta}g(w)\right)'}-1\right)\in\varphi(\mathbb{U}) \tag{17}$$

where the function g is given by (6).

In this paper, we obtain the coefficient estimates for the functions of the classes $\mathcal{N}_{\Sigma}^{q}(\psi,\nu,\varpi)$ and $\mathcal{N}_{\Sigma}^{q}(\phi,\varphi,\psi,\nu,\varpi)$.

2. Coefficient Bounds for the Function Class $\mathcal{N}_{\Sigma}^{q}(\psi,\nu,\varpi)$.

Unless otherwise mentioned, we assume throughout the paper that ϖ is analytic in $\varsigma\in\mathbb{U}$ and $\varpi(0)=1$ and let

$$\varpi(\varsigma)=1+B_1\varsigma+B_2\varsigma^2+\ldots,\quad B_1>0. \tag{18}$$

and

$$\Xi(\varsigma)=A_0+A_1\varsigma+A_2\varsigma^2+\ldots,|\Xi(\varsigma)|\leq 1,\varsigma\in\mathbb{U}. \tag{19}$$

Theorem 2.1. *Let $\mathfrak{l}(\varsigma)$ be given by (1) belongs to the class $\mathcal{N}_{\Sigma}^{q}(\psi,\nu,\varpi)$, then*

$$|a_2|\leq\frac{|\psi||A_0|B_1\sqrt{B_1}}{\sqrt{2\psi a_0B_1^2(1-\nu)\vartheta_3-\{\psi a_0B_1^2(1-\nu^2)+(A_2-A_1)(1-\nu)^2\}\,\vartheta_2^2}}$$

and

$$|a_3|\leq\frac{|\psi A_0|^2B_1^2}{(1-\nu)^2\vartheta_2^2}+\frac{|\psi A_1|B_1}{2(1-\nu)\vartheta_3}+\frac{|\psi A_0|B_1}{2(1-\nu)\vartheta_3}.$$

where $0\leq\nu<1$, $\psi\in\mathbb{C}\setminus\{0\}$, $0<q<1$, $\varsigma\in\mathbb{U}$ and ϑ_{j}, $\mathrm{j}=\{2,3\}$, are given by (13).

Proof. If $\mathfrak{l}\in\mathcal{N}_{\Sigma}^{q}(\psi,\nu,\varpi)$ then there exists analytic functions $\mathfrak{u},\mathfrak{v}\colon\ \mathbb{U}\to\mathbb{U}$ with $\mathfrak{u}(0)=\mathfrak{v}(0)=0$, $|\mathfrak{u}|<1$, $|\mathfrak{v}|<1$ such that

$$\frac{1}{|\psi|}\left[\frac{\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)+\nu\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}-1\right]=\Xi(\varsigma)(\varpi(\mathfrak{u}(\varsigma))-1) \tag{20}$$

and

$$\frac{1}{|\psi|}\left[\frac{w\left(\mathcal{W}_{k,c}^{\delta,q}g(w)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}g(w)+\nu w\left(\mathcal{W}_{k,c}^{\delta}g(w)\right)'}-1\right]=\Xi(w)(\varpi(\mathfrak{v}(w))-1). \tag{21}$$

We set the functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ in $\mathcal{P}$ such that

$$\mathfrak{f}_1(\varsigma):=\frac{1+\mathfrak{u}(\varsigma)}{1-\mathfrak{u}(\varsigma)}=1+s_1\varsigma+s_2\varsigma^2+\cdots$$

$$\mathfrak{f}_2(w):=\frac{1+\mathfrak{v}(w)}{1-\mathfrak{v}(w)}=1+t_1w+t_2w^2+\cdots.$$

It follows that

$$\mathfrak{u}(\varsigma)=\frac{\mathfrak{f}_1(\varsigma)-1}{\mathfrak{f}_1(\varsigma)+1}=\frac{s_1}{2}\varsigma+\left(s_2-\frac{s_1^2}{2}\right)\frac{\varsigma^2}{2}+\cdots \tag{22}$$

and

$$\mathfrak{v}(w) = \frac{\mathfrak{f}_2(w) - 1}{\mathfrak{f}_2(w) + 1} = \frac{t_1}{2} w + \left(t_2 - \frac{t_1^2}{2}\right) \frac{w^2}{2} + \cdots . \tag{23}$$

Using (22) and (23) together with (18) and (19) in right hands of the relations (20) and (21), we obtain

$$\begin{aligned} \Xi(\varsigma)(\varpi(\mathfrak{u}(\varsigma)) - 1) &= \frac{1}{2} A_0 B_1 s_1 \varsigma \\ &+ \left(\frac{1}{2} A_1 B_1 s_1 + \frac{1}{2} A_0 B_1 \left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4} A_0 B_2 s_1^2\right) \varsigma^2 + \cdots \end{aligned} \tag{24}$$

and

$$\begin{aligned} \Xi(w)(\varpi(\mathfrak{v}(w)) - 1) &= \frac{1}{2} A_0 B_1 t_1 w \\ &+ \left(\frac{1}{2} A_1 B_1 t_1 + \frac{1}{2} A_0 B_1 \left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4} A_0 B_2 t_1^2\right) w^2 + \cdots . \end{aligned} \tag{25}$$

In light of (18) and (19), we get

$$\frac{(1-\nu)\vartheta_2}{\psi} a_2 = \frac{A_0 B_1 s_1}{2}, \tag{26}$$

$$\frac{2(1-\nu)\vartheta_3 a_3 - (1-\nu^2)\vartheta_2^2 a_2^2}{\psi} = \frac{A_1 B_1 s_1}{2} + \frac{A_0 B_1}{2}\left(s_2 - \frac{s_1^2}{2}\right) + \frac{A_0 B_2}{4} s_1^2 \tag{27}$$

and

$$-\frac{(1-\nu)\vartheta_2}{\psi} a_2 = \frac{A_0 B_1 t_1}{2}, \tag{28}$$

$$\frac{2(1-\nu)\vartheta_3 (2a_2^2 - a_3) - (1-\nu^2)\vartheta_2^2 a_2^2}{\psi} = \frac{A_1 B_1 t_1}{2} + \frac{A_0 B_1}{2}\left(t_2 - \frac{t_1^2}{2}\right) + \frac{A_0 B_2}{4} t_1^2. \tag{29}$$

Using (26) and (28), we get

$$s_1 = -t_1 \tag{30}$$

and

$$8(1-\nu)^2 \vartheta_2^2 a_2^2 = \psi^2 A_0^2 B_1^2 \left(s_1^2 + t_1^2\right). \tag{31}$$

Adding (27) and (29), we get

$$\frac{4(1-\nu)\vartheta_3 - 2(1-\nu^2)\vartheta_2^2}{\psi} a_2^2 = \frac{A_0 B_1 (s_2 + t_2)}{2} + \frac{A_0 (A_2 - A_1)(s_1^2 + t_1^2)}{4}. \tag{32}$$

By using (30), (31) and Lemma 1.1 in (32), we get

$$|a_2| \le \frac{|\psi||A_0| B_1 \sqrt{B_1}}{\sqrt{2\psi a_0 B_1^2 (1-\nu)\vartheta_3 - \left\{\psi a_0 B_1^2 (1-\nu^2) + (A_2 - A_1)(1-\nu)^2\right\} \vartheta_2^2}}.$$

Next, to find bound on $|a_3|$, by subtracting (29) and (27), we obtain

$$\frac{4(1-\nu)\vartheta_3}{\psi}\left(a_3 - a_2^2\right) = \frac{A_0 B_1 (s_2 - t_2)}{2} + \frac{A_1 B_1 (s_1 - t_1)}{2}. \tag{33}$$

Using (30), (31) and (33), we get

$$|a_3| = \frac{\psi A_0^2 B_1^2 (s_1^2 + t_1^2)}{8(1-\nu)^2 \vartheta_2^2} + \frac{\psi A_1 B_1 (s_1 - t_1)}{8(1-\nu)\vartheta_3} + \frac{\psi A_0 B_1 (s_2 - t_2)}{8(1-\nu)\vartheta_3}. \tag{34}$$

Applying Lemma 1.1 once again for the coefficients s_1, t_1, s_2 and t_2, we get

$$|a_3| \leq \frac{|\psi A_0|^2 B_1^2}{(1-\nu)^2 \vartheta_2^2} + \frac{|\psi A_1| B_1}{2(1-\nu)\vartheta_3} + \frac{|\psi A_0| B_1}{2(1-\nu)\vartheta_3}.$$

This completes the proof. □

If we take $\nu = 0$ in Theorem 2.1 we obtain the following result

Corollary 2.5. *If the function* $\mathfrak{l}(\varsigma)$ *given by (1) belongs to the class* $\mathcal{N}_\Sigma^q(\psi, \varpi)$*, then*

$$|a_2| \leq \frac{|\psi||A_0| B_1 \sqrt{B_1}}{\sqrt{2\psi a_0 B_1^2 \vartheta_3 - \{\psi a_0 B_1^2 + (A_2 - A_1)\} \vartheta_2^2}}$$

and

$$|a_3| \leq \frac{|\psi A_0|^2 B_1^2}{\vartheta_2^2} + \frac{|\psi A_1| B_1}{2\vartheta_3} + \frac{|\psi A_0| B_1}{2\vartheta_3}.$$

where $\psi \in \mathbb{C} \setminus \{0\}$, $0 < q < 1$, $\varsigma \in \mathbb{U}$ *and* ϑ_{j}, $\mathrm{j} = \{2, 3\}$, *are given by (13).*

3. General Coefficient Bounds for the Function Class $\mathcal{N}_\Sigma^q(\phi, \varphi, \psi, \nu, \varpi)$.

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{N}_\Sigma^q(\phi, \varphi, \psi, \nu, \varpi)$.

Theorem 3.2. *Let* $\mathfrak{l}(\varsigma)$ *given by (1) belongs to the class* $\mathcal{N}_\Sigma^q(\phi, \varphi, \psi, \nu, \varpi)$*, then*

$$|a_2| \leq min\left\{\sqrt{\frac{|\psi|^2 \left(|\phi'(0)|^2 + |\varphi'(0)|^2\right)}{8(1-\nu)^2\vartheta_2^2}}, \sqrt{\frac{|\psi| \left(|\phi''(0)| + |\varphi''(0)|\right)}{4\{2(1-\nu)\vartheta_3 - (1-\nu^2)\vartheta_2^2\}}}\right\}$$

and

$$|a_3| \leq \frac{|\psi|^2 \{|\phi'(0)|^2 + |\varphi'(0)|^2\}}{4(1-\nu)^2\vartheta_2^2} + \frac{|\psi| \{|\phi''(0)| + |\varphi''(0)|\}}{8(1-\nu)\vartheta_3}.$$

where $0 \leq \nu < 1$, $\psi \in \mathbb{C} \setminus \{0\}$, $0 < q < 1$, $\varsigma \in \mathbb{U}$ *and* ϑ_{j}, $\mathrm{j} = \{2, 3\}$, *are given by (13).*

Proof. : We write the argument inequalities in (16) and (17) in their equivalent forms as follows:

$$\frac{1}{|\psi|}\left[\frac{\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma) + \nu\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'} - 1\right] = \phi(\varsigma) \quad (\varsigma \in \mathbb{U}),$$

and

$$\frac{1}{|\psi|}\left[\frac{w\left(\mathcal{W}_{k,c}^{\delta,q}g(w)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}g(w) + \nu w\left(\mathcal{W}_{k,c}^{\delta}g(w)\right)'} - 1\right] = \varphi(w) \quad (w \in \mathbb{U}),$$

where the function g is given by (6). The functions $\phi(\varsigma)$ and $\varphi(w)$ have the following Taylor-Maclaurin series expensions:

$$\phi(\varsigma) = 1 + \phi_1\varsigma + \phi_2\varsigma^2 + ...$$

and

$$\varphi(w) = 1 + \varphi_1 w + \varphi_2 w^2 + ...$$

Now, upon equating the coefficients of

$$\frac{1}{|\psi|}\left[\frac{\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)+\nu\varsigma\left(\mathcal{W}_{k,c}^{\delta,q}\mathfrak{l}(\varsigma)\right)'}-1\right]$$

with those of $\phi(\varsigma)$ and the coefficients of

$$\frac{1}{|\psi|}\left[\frac{w\left(\mathcal{W}_{k,c}^{\delta,q}g(w)\right)'}{(1-\nu)\mathcal{W}_{k,c}^{\delta,q}g(w)+\nu w\left(\mathcal{W}_{k,c}^{\delta}g(w)\right)'}-1\right]$$

with those of $\varphi(w)$, we get

$$\frac{(1-\nu)\vartheta_2}{\psi}a_2=\phi_1, \tag{35}$$

$$\frac{2(1-\nu)\vartheta_3a_3-(1-\nu^2)\vartheta_2^2a_2^2}{\psi}=\phi_2 \tag{36}$$

and

$$-\frac{(1-\nu)\vartheta_2}{\psi}a_2=\varphi_1, \tag{37}$$

$$\frac{2(1-\nu)\vartheta_3(2a_2^2-a_3)-(1-\nu^2)\vartheta_2^2a_2^2}{\psi}=\varphi_2. \tag{38}$$

Using (35) and (37), we get

$$\phi_1=-\varphi_1 \tag{39}$$

and

$$8(1-\nu)^2\vartheta_2^2a_2^2=\psi^2\left(\phi_1^2+\varphi_1^2\right). \tag{40}$$

Adding (36) and (38), we get

$$\frac{4(1-\nu)\vartheta_3-2(1-\nu^2)\vartheta_2^2}{\psi}a_2^2=\phi_2+\varphi_2. \tag{41}$$

Therefore, we find from the equations (40) and (41) that

$$|a_2|^2\leq\frac{|\psi|^2\left(|\phi_1'(0)|^2+|\varphi_1'(0)|^2\right)}{8(1-\nu)^2\vartheta_2^2}.$$

and

$$|a_2|^2\leq\frac{|\psi|\left(|\phi_2''(0)|+|\varphi_2''(0)|\right)}{4\{2(1-\nu)\vartheta_3-(1-\nu^2)\vartheta_2^2\}}.$$

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (38) from (36) and upon using (40). We thus get

$$a_3=\frac{|\psi|^2\{\phi_1^2+\varphi_1^2\}}{4(1-\nu)^2\vartheta_2^2}+\frac{\psi(\varphi_2-\phi_2)}{4(1-\nu)\vartheta_3}.$$

We thus find that

$$|a_3|\leq\frac{|\psi|^2\{|\phi'(0)|^2+|\varphi'(0)|^2\}}{4(1-\nu)^2\vartheta_2^2}+\frac{|\psi|\{|\phi''(0)|+|\varphi''(0)|\}}{8(1-\nu)\vartheta_3}.$$

□

This completes the proof.

If we take $\nu=0$ in Theorem (3.2) we obtained following result

Corollary 3.5. *Let* $\mathfrak{l}(\varsigma)$ *given by (1) belongs to the class* $\mathcal{N}_{\Sigma}^{q}(\phi,\varphi,\psi,\varpi)$*, then*

$$|a_2| \leq min\left\{\sqrt{\frac{|\psi|^2\left(|\phi'(0)|^2+|\varphi'(0)|^2\right)}{8\vartheta_2^2}},\sqrt{\frac{|\psi|\left(|\phi''(0)|+|\varphi''(0)|\right)}{4\{2\vartheta_3-\vartheta_2^2\}}}\right\}$$

and

$$|a_3| \leq \frac{|\psi|^2\{|\phi'(0)|^2+|\varphi'(0)|^2\}}{4\vartheta_2^2}+\frac{|\psi|\{|\phi''(0)|+|\varphi''(0)|\}}{8\vartheta_3}.$$

where $\psi \in \mathbb{C}\setminus\{0\}$*,* $0<q<1$*,* $\varsigma\in\mathbb{U}$ *and* ϑ_{j}*,* $j=\{2,3\}$*, are given by (13).*

4. Concluding Remark

Throughout the paper, we used the definition of q-difference operator and Struve functions to introduce the classes $\mathcal{N}_{\Sigma}^{q}(\psi,\nu,\varpi)$ and $\mathcal{N}_{\Sigma}^{q}(\phi,\varphi,\psi,\nu,\varpi)$. Moreover, we calculate the coefficient estimates for functions belong to these classes.

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A. A. Thombre,

Research Scholar, Department of Mathematics, Swami Ramanand Teerth Marathawada University, Nanded, India.

Email address: `ashokthombre123@gmail.com`

D. N. Chate
Department of Mathematics, Sanjeevanee Mahavidyalaya, Chapoli, India
Email address: dhananjayachate@gmail.com

Mallikarjun G. Shrigan
Department of Mathematics, School of Computational Sciences, JSPM University Pune,, Pune, India.
Email address: mgshrigan@gmail.com