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Modified Atangana-Baleanu-Caputo Derivative for Non-Linear Hyperbolic Coupled System

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ABSTRACT

This study presents the fractional modified Atangana-Baleanu-Caputo derivative for the solution of a non-homogeneous nonlinear coupled system of hyperbolic partial differential equations. The system has also been solved in the Atangana-Baleanu-Caputo derivative to prove that it is effective for these kinds of problems. The system has been fractional in space-time, and it has been demonstrated through research that the suggested approach is second-order convergent in both space and time and conditionally stable. The numerical method non-standard finite difference has been provided toward the conclusion to compare the exact and numerical results to the problem. The stability of the current system was explained by applying Von Neumann analysis. The effectiveness and reliability of the theoretical estimations are demonstrated by the numerical solutions.

1. Introduction

Fractional differential equations have drawn a lot of interest in applied mathematics and engineering over the past 20 years. In addition to being a hot topic in mathematics, fractional calculus has applications in a wide range of other fields, including engineering, chemistry, aerodynamics, control theory, physics, biology, continuum, and statistical mechanics. This fraction may be seen as a function in any variable, including time, space and other variables. Therefore, fractional derivatives authors benefit from displaying such unusual behaviors to explain various processes. Such operators reveal the distinguished characteristics of extended relationships, which the criterion integer order differential equation can't prove. The solvability of boundary value problems (BVPs) for nonlinear fractional differential equations has been investigated in recent years, fixed point theorems are typically used in these kinds of issues to explore the existence and multiplicity of solutions [1]-[3]. The fractional order calculus is a logical progression from the constant order calculus.

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Although literature now offers several definitions for fraction derivatives, the most widely used are Riemann-Liouville, Caputo, and Atangana-Baleanu derivatives, we refer the reader to basic books [4]-[7]. Fractional differentiation is always evolving to address practical issues, and compared to integer-order derivatives, fractional derivatives are more advantageous because they may characterize memory and the inherited characteristics of physical materials [8].

Differential equations can also be solved numerically using the nonstandard finite difference technique (NSFDM), numerous issues, including linear and non-linear partial differential equations, have been resolved with its help. This approach can be used for a region that contains a variety of materials, problems with arious boundary forms, and numerous kinds of boundary conditions [9]-[12].

One of these issues that draws in a lot of scientists is the hyperbolic partial differential equations, which are very useful in physics and mathematics and have numerous applications. A review of numerical methods for non-linear partial differential equations was given by Polyanin [13] and Tadmor [14]. Nonlinear hyperbolic partial differential equations have been applied in different fields, such as in hypoelastic solids [15], astrophysics [16], electromagnetic theory [17], propagation of heat waves [18], [19], and other disciplines. Numerous authors in relevant domains like biology, physics, electrical networks, fluid flows, and viscoelasticity attempt to model these occurrences as coupled systems [20]-[23]. Furthermore, because coupled systems of fractional differential equations are found in many scientific applications, the study of these systems has garnered a lot of attention (we refer to [24], [25], [26]). With the use of the NSFDM and Taylor's expansion of function, a numerical method for discretizing the modified Atangana-Baleanu-Caputo derivative (MABC) derivative has been created in this study. Regarding the solution of the hyperbolic partial differential equation [27], consider the following space-time fraction for the non-linear coupled system. In recent years, mathematical systems could be depicted suitability and more accurately by employing the fractional order derivative. More recently, Atangana-Baleanu-Caputo sense (ABC) defined a modified Caputo fractional derivative by introducing generalized Mittag-Leffler function as the non-local and non-singular kernel [28], [29]. These new types of derivatives have been used in the modeling of real-life applications in different fields.

The paper is organized In the following way: In Section (1), the introduction, the model is presented together with their fractional form. Section (2) contains the fraction calculus definitions. The method for non-linear coupled systems is described in Section (3), additionally, it explains the fractional derivative schemes for ABC and MABC derivatives. Section (4) presents truncation errors for the proposed model. Section (5) presents stability assessments and their conditions. Section (6) contains numerical graphics and results. Section (7) conclusion.

$$u_{tt} - u_{xx} - \frac{1}{x} u_x - v u_x = f(x,t), \quad t \in [0, T] \; and \; x \in [a,b],$$

$$v_{tt} - v_{xx} - \frac{1}{x} v_x - u\, v_x = g(x,t), \quad t \in [0, T] \; and \; x \in [a,b]. \qquad (1)$$

with initial and boundary conditions,

$$u(x,0) = f_1(x) \qquad v(x,0) = g_1(x), \qquad x \epsilon [a,b],$$

$$u_t(x,0) = f_2(x) \qquad v_t(x,0 = g_2(x), \qquad x \epsilon [a,b],$$

$$u(a,t) = f_3(t) \qquad u(b,t) = f_4(t), \qquad x \epsilon [a,b],$$

$$v(a,t) = g_3(t) \qquad v(b,t) = g_4(t), \qquad x \epsilon [a,b]. \qquad (2)$$

Where $u(x,t), v(x,t)$ are unknown functions. The time fraction of orders $1 < \alpha \leq 2$ for the equation (1) is given,

$${}^{MABC}D_t^{\alpha} \; u_{tt}(x,t) - {}^{MABC}D_x^{\beta} u_{xx}(x,t) - \frac{1}{x} u_x(x,t) - v(x,t) u_x(x,t) - f(x,t) = R_1,$$

$${}^{MABC}D_t^{\alpha} v_{tt}(x,t) - {}^{MABC}D_x^{\beta} v_{xx}(x,t) - \frac{1}{x} v_x(x,t) - u(x,t) v_x(x,t) - g(x,t) = R_2. \qquad (3)$$

Where ${}^{MABC}D_t^{\alpha}$ and ${}^{MABC}D_x^{\beta}$ are the modified Atangana-Baleanu-Caputo MABC fractional operator for time and space [28] for the same boundary and initial conditions.

2. Fractional Calculus Definitions

There are many definitions of fractional calculus of order α the most basic and relevant definitions are discussed in this section (see [5], [31]), such as Riemann-Liouville's definition, Caputo's fractional derivative, Atangana-Baleanu fractional derivative in Caputo sense and the modified ABC fractional operator,

- The Riemann-Liouville fractional derivative of order α, $0 > \alpha > 1$ of a function $f(t)$ is define as:

$${}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(r-\alpha)} \frac{d^r}{dt^r} \int_0^t (t-\tau)^{r-\alpha-1} f(\tau) d\tau. \qquad (4)$$

- The Caputo-Fabrizio derivative with fractional order when $f(x,t)$ be a function in $H^1(a,b), b > a$ and $\alpha \in C$, $0 > \alpha \leq 1$ see [32]-[34], will define as,

$$ {}^{CF}_{a}D^{\alpha}_{t} f(t) = \frac{\alpha\, B(\alpha)}{(1-\alpha)} \int_0^t \big(f(x,t) - f(x,\tau)\big)\, e^{\frac{-\alpha(1-\tau)}{(1-\alpha)}} d\tau, \quad 0 < \alpha < 1 \tag{5}$$

where, $B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ is normalization function such that $B(0) = B(1) = 1$.

- The Atangana-Baleanu-Caputo fractional derivative [35] is defined as,

$$ {}^{ABC}_{a}D^{\alpha}_{t} f(t) = \frac{B(\alpha)}{(1-\alpha)} = \int_0^t E_\alpha\left(-\alpha \frac{(t-s)^\alpha}{(1-\alpha)}\right) \dot{f}(s) ds\,, \quad 0 < \alpha < 1, \tag{6}$$

where E_α is Mittag-Leffler function where, $E_{\alpha,\beta}(K) = \sum_0^\infty \frac{K^r}{\Gamma(r\alpha+\beta)}$ it is a modified form of Caputo-Fabrizio that presents the ideal properties of non-singularity, and non-locality of the kernel.

- The modified Atangana-Baleanu-Caputo MABC fractional operator in $L^1(0,T)$ in Caputo sense [28] was defined as,

$$ {}^{MABC}D^{\delta}_{t} f(t) = \frac{B(\alpha)}{(1-\alpha)} = [f^{n-1}(t) - E_{\alpha(-\mu_\alpha t^\alpha)} f^{n-1}(0) - \mu_\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t-s)^\alpha)\, f^{n-1}(s) ds]. \tag{7}$$

Where $\mu_\alpha = \frac{\alpha}{(1-\alpha)}$. The derivative is defined for $1 < \alpha \le 2$ and $n - 1 < \delta < n$, where $\delta = \alpha + n - 1$. The MABC fractional operator leads to new solutions of several fractional differential equations and a description of the dynamics of fractional processes.

3. The Method for Non-linear Coupled Systems

Let's consider that the solution domain of our problem is $x_0 = 0, x_N = L$ and $x_i = ih$ such that $i = 0,1,2, \ldots, N$ and $h = dx = \frac{N}{L}$. Let $t_0 = 0,\ t_M = t_{max}$ where $j = 0,1,2, \ldots, M,\ t_j = jk$ and $k = dt = \frac{t_{max}}{M}$ is divided into intervals having equal lengths h in the x direction and k for the t direction. The values of the solution u_i^j are given by $u(x_i, t_j)$. Expand $u(x_i, t_j)$ using the Taylor's expansion around t_j for $t \in (t_j, t_{j+1})$ to get the NSFDM approximations for the terms u_x and v_x as follows:

$$ u_x = \frac{u^{j+1}_{i+1} - u^{j+1}_{i-1}}{2\phi(h)}, \qquad v_x = \frac{v^{j+1}_{i+1} - v^{j+1}_{i-1}}{2\phi(h)}. \tag{8}$$

Use the following equations to obtain the space-time discretization for the model (1) by the Atangana-Baleanu-Caputo and the Modified Atangana-Baleanu-Caputo derivatives:

$$ {}^{ABC}D^{\alpha}_{tt}\, u(x_i, t_j) - {}^{ABC}D^{\beta}_{xx} u(x_i, t_j) - \frac{1}{x_i}\left(\frac{(u^{j+1}_{i+1} - u^{j+1}_{i-1})}{2\phi(h)}\right) - v_i^j \left(\frac{(u^{j+1}_{i+1} - u^{j+1}_{i-1})}{2\phi(h)}\right) = f_i^j,$$

$$ {}^{ABC}D^{\alpha}_{tt}\, v(x_i, t_j) - {}^{ABC}D^{\beta}_{xx} v(x_i, t_j) - \frac{1}{x_i}\left(\frac{(v^{j+1}_{i+1} - v^{j+1}_{i-1})}{2\phi(h)}\right) - u_i^j \left(\frac{(v^{j+1}_{i+1} - v^{j+1}_{i-1})}{2\phi(h)}\right) = g_i^j, \tag{9}$$

$$ {}^{MABC}D^{\alpha}_{tt}\, u(x_i, t_j) - {}^{MABC}D^{\beta}_{xx} u(x_i, t_j) - \frac{1}{x_i}\left(\frac{(u^{j+1}_{i+1} - u^{j+1}_{i-1})}{2\phi(h)}\right) - v_i^j \left(\frac{(u^{j+1}_{i+1} - u^{j+1}_{i-1})}{2\phi(h)}\right) = f_i^j,$$

$$ {}^{MABC}D^{\alpha}_{tt}\, v(x_i, t_j) - {}^{MABC}D^{\beta}_{xx} v(x_i, t_j) - \frac{1}{x_i}\left(\frac{(v^{j+1}_{i+1} - v^{j+1}_{i-1})}{2\phi(h)}\right) - u_i^j \left(\frac{(v^{j+1}_{i+1} - v^{j+1}_{i-1})}{2\phi(h)}\right) = g_i^j. \tag{10}$$

3.1 ABC Derivative Discretization

For getting the time fractional derivative scheme in ABC derivatives put n=2 in equation (6) and $1 < \alpha \le 2$ for the function $u(x_i, t_j)$ we will get,

$$ {}^{ABC}D^{\alpha}_{tt}\, u(x_i, t_j) = \frac{B(\alpha)}{(1-\alpha)} \int_0^t E_\alpha(-\mu_\alpha (t-s)^\alpha)\, u'(x_i, s_i) ds,$$

where $\mu_\alpha = \frac{\alpha}{1-\alpha}$ and let $M(\alpha) = \frac{B(\alpha)}{1-\alpha}$,

$$ {}^{ABC}D_{tt}^{\alpha} u_i^j = M(\alpha)(\sum_{k=0}^{j-1} \frac{(u_i^{j+1-k} - u_i^{j-1-k})}{2O(k)} - o(k^2)) \int_0^t E_\alpha(-\mu_\alpha(\mathrm{t}-s)^\alpha)\, ds $$

$$ = M(\alpha)(\sum_{k=0}^{j-1} \frac{(u_i^{j+1-k} - u_i^{j-1-k})}{2O(k)} - o(k^2))\ [(t_j - t_k - 1)E_{\alpha,1}(-\mu_\alpha(t_j - t_k - 1)^\alpha) - (t_j - t_k) $$

$$ E_{\alpha,1}\ (-\mu_\alpha(t_j - t_k)^\alpha)] + R_m. \qquad \text{for } t > 0. \qquad (11) $$

For getting the space fractional derivative scheme in ABC derivatives put n=2 in equation (6) and $1 < \beta \leq 2$ for the function $u(x_i, t_j)$ we will get,

$$ {}^{ABC}D_{xx}^{\beta} u(x_i, t_j) = \frac{B(\beta)}{(1-\beta)} \int_0^x E_\beta(-\mu_\beta(\mathrm{x} - q)^\beta)\ u'(q_i, \mathrm{t}_i) dq, $$

$$ {}^{ABC}D_{xx}^{\beta} u_i^j = M(\beta)(\sum_{h=0}^{i-1} \frac{(u_{i+1-h}^j - u_{i-1-h}^j)}{2\phi(h)} - \phi(h^2)) \int_0^x E_\beta(-\mu_v(\mathrm{x}-q)^\beta)\ dq $$

$$ = M(\beta)(\sum_{h=0}^{i-1} \frac{(u_{i+1-h}^j - u_{i-1-h}^j)}{2\phi(h)} - \phi(h^2))[(x_i - x_h - 1)E_{\beta,1}(-\mu_\beta(x_i - x_h - 1)^\beta) - $$

$$ (x_i - x_h)E_{\beta,1}\ (-\mu_\beta(x_i - x_h)^\beta)] + R_n, \quad \text{for } x > 0. \qquad (12) $$

Where $M(\beta) = \frac{B(\beta)}{1-\beta}$ and R_m, R_n are the truncation errors for the non-linear coupled system equations. To derive the discretization schemes for the system equations (9) to get the time-space fractional schemes as follows:

$$ = M(\alpha)(\sum_{k=0}^{j-1} \frac{(u_i^{j+1-k} - u_i^{j-1-k})}{2O\,(k)})\ [(t_j - t_k - 1)E_{\alpha,1}(-\mu_\alpha(t_j - t_k - 1)^\alpha) - (t_j - t_k)E_{\alpha,1}\ (-\mu_\alpha(t_j - t_k)^\alpha)] - $$

$$ M(\beta)(\sum_{h=0}^{i-1} \frac{(u_{i+1-h}^j - u_{i-1-h}^j)}{2\phi(h)} - \phi(h^2))\,[(x_i - x_h - 1)E_{\beta,1}(-\mu_\beta(x_i - x_h - 1)^\beta) - (x_i - x_h)E_\beta\ (-\mu_\beta(x_i - x_h)^\beta)] - $$

$$ \frac{1}{x_i^j}\left(\frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\phi(h)}\right) - v_i^j \left(\frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\phi(h)}\right) - f_i^j = R_1. \qquad (13) $$

By taking the same steps for the equation $v(x_i, t_j)$ in the coupled system, to get the time-space fractional derivative schemes,

$$ M(\alpha)(\sum_{k=0}^{j-1} \frac{(v_i^{j+1-k} - v_i^{j-1-k})}{2O\,(k)})\ [(t_j - t_k - 1)E_{\alpha,1}(-\mu_\alpha(t_j - t_k - 1)^\alpha) - (t_j - t_k)E_{\alpha,1}\ (-\mu_\alpha(t_j - t_k)^\alpha)] $$

$$-M(\beta)(\sum_{h=0}^{i-1}\frac{(v_{i+1-h}^{j}-v_{i-1-h}^{j})}{2\phi(h)}-\phi(h^2))\left[(x_i-x_h-1)E_{\beta,1}(-\mu_\beta(x_i-x_h-1)^\beta)-(x_i-x_h)E_\beta\left(-\mu_\beta(x_i-x_h)^\beta\right)\right]-\frac{1}{x_i^j}\left(\frac{v_{i+1}^{j+1}-v_{i-1}^{j+1}}{2\phi(h)}\right)-u_i^j\left(\frac{v_{i+1}^{j+1}-v_{i-1}^{j+1}}{2\phi(h)}\right)-g_i^j=R_2. \quad (14)$$

3.2 MABC Derivative Discretization

For getting the time fractional derivative scheme in MABC derivatives put n=2 in equation (2) and $1<\alpha\le 2$ for the function $u(x_i,t_j)$ we will get,

$$^{MABC}D_{tt}^{\alpha}u_i^j=\frac{B(\alpha)}{(1-\alpha)}\left[u_t(x_i,t_j)-E_\alpha(-\mu_\alpha t^\alpha)u_t(x_i,t_j)|_{t=0}-\mu_\alpha\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_\alpha(t-s)^\alpha)u'(x_i,s_j)ds\right]=$$
$$M(\alpha)\left[\sum_{k=0}^{j-1}\frac{u_i^{j+1-k}-u_i^{j-1-k}}{2O(k)}-O(k^2)\right]-E_\alpha(-\mu_\alpha t^\alpha)w(x)-\mu_\alpha\int_0^t\left[\sum_{k=0}^{j-1}\frac{u_i^{j+1-k}-u_i^{j-1-k}}{2O(k)}\right.$$
$$\left.-O(k^2)\right](t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_\alpha(t-s)^\alpha)ds],\quad for\ t>0,$$

where w(x) is function,

$$^{MABC}D_{tt}^{\alpha}u_i^j=M(\alpha)\sum_{k=0}^{j-1}\frac{u_i^{j+1-k}-u_i^{j-1-k}}{2O(k)}-M(\alpha)E_\alpha(-\mu_\alpha t^\alpha)w(x)-M(\alpha)\,\mu_\alpha\sum_{k=0}^{j-1}\frac{u_i^{j+1-k}-u_i^{j-1-k}}{2O(k)}$$
$$\left[(t_j-t_k)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha(t_j-t_k)^\alpha)-(t_j-t_{k+1})^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha(t_j-t_{k+1})^\alpha)\right]+R_m. \quad (15)$$

For getting the space fractional derivative scheme in MABC derivatives put $n=2$ in equation (2) and $1<\beta\le 2$ for the function $u(x_i,t_j)$ we will get,

$$^{MABC}D_{xx}^{\beta}u_i^j=\frac{B(\beta)}{(1-\beta)}\left[u_x(x_i,t_j)-E_\beta(-\mu_\beta x^\beta)u_x(x_i,t_j)|_{x=0}-\mu_\beta\int_0^x(x-q)^{\beta-1}E_{\beta,\beta}(-\mu_\beta(x-q)^\beta)u'(q,t_j)dq\right]=$$
$$M(\beta)\sum_{h=0}^{i-1}\frac{u_{i+1-h}^{j}-u_{i-1-h}^{j}}{2\phi(h)}-M(\beta)E_\beta(-\mu_\beta x^\beta)w(t)-M(\beta)\,\mu_\beta\sum_{h=0}^{i-1}\frac{u_{i+1-h}^{j}-u_{i-1-h}^{j}}{2\phi(h)}$$
$$\left[(x_i-x_h)^\beta E_{\beta,\beta+1}(-\mu_\beta(x_i-x_h)^\beta)-(x_i-x_{h+1})^\beta E_{\beta,\beta+1}(-\mu_\beta(x_i-x_{h+1})^\beta)\right]+R_n. \quad (16)$$

Where $w(t)$ is function and R_m,R_n are the truncation errors for the nonlinear coupled system equations. To derive the discretization schemes for the system equations (10), for $t_j=jk$ and $x_i=ih$,

$$M(\alpha)\sum_{k=0}^{j-1}\frac{u_i^{j+1-k}-u_i^{j-1-k}}{2O(k)}-M(\alpha)E_\alpha(-\mu_\alpha t^\alpha)w(x)-M(\alpha)\,\mu_\alpha\sum_{k=0}^{j-1}\frac{u_i^{j+1-k}-u_i^{j-1-k}}{2O(k)}$$
$$\left[(t_j-t_k)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha(t_j-t_k)^\alpha)-(t_j-t_{k+1})^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha(t_j-t_{k+1})^\alpha)\right]-$$
$$M(\beta)\sum_{h=0}^{i-1}\frac{u_{i+1-h}^{j}-u_{i-1-h}^{j}}{2\phi(h)}-M(\beta)E_\beta(-\mu_\beta x^\beta)w(t)-M(\beta)\,\mu_\beta\sum_{h=0}^{i-1}\frac{u_{i+1-h}^{j}-u_{i-1-h}^{j}}{2\phi(h)}$$

$$\left[(x_i-x_h)^\beta E_{\beta,\beta+1}(-\mu_\beta(x_i-x_h)^\beta)-(x_i-x_{h+1})^\beta E_{\beta,\beta+1}(-\mu_\beta(x_i-x_{h+1})^\beta)\right]$$
$$-\frac{1}{x_i^j}\left(\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2\phi(h)}\right)-v_i^j\left(\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2\phi(h)}\right)-f_i^j=R_1. \quad (17)$$

By taking the same steps for the equation $v(x_i,t_j)$ in the coupled system, to get the time-space fractional derivative schemes,

$$M(\alpha)\sum_{k=0}^{j-1}\frac{v_i^{j+1-k}-v_i^{j-1-k}}{2O(k)}-M(\alpha)E_\alpha(-\mu_\alpha t^\alpha)w(x)-M(\alpha)\,\mu_\alpha\sum_{k=0}^{j-1}\frac{v_i^{j+1-k}-v_i^{j-1-k}}{2O(k)}$$
$$\left[(t_j-t_k)^\alpha E_{\alpha,\alpha+1}\left(-\mu_\alpha(t_j-t_k)^\alpha\right)-(t_j-t_{k+1})^\alpha E_{\alpha,\alpha+1}\left(-\mu_\alpha(t_j-t_{k+1})^\alpha\right)\right]-$$
$$M(\beta)\sum_{h=0}^{i-1}\frac{v_{i+1-h}^{j}-v_{i-1-h}^{j}}{2\phi(h)}-M(\beta)E_\beta(-\mu_\beta x^\beta)w(t)-M(\beta)\,\mu_\beta\sum_{h=0}^{i-1}\frac{v_{i+1-h}^{j}-v_{i-1-h}^{j}}{2\phi(h)}$$

$$\left[(x_i-x_h)^\beta E_{\beta,\beta+1}\left(-\mu_\beta(x_i-x_h)^\beta\right)-(x_i-x_{h+1})^\beta E_{\beta,\beta+1}\left(-\mu_\beta(x_i-x_{h+1})^\beta\right)\right]$$
$$-\frac{1}{x_i^j}\left(\frac{v_{i+1}^{j+1}-v_{i-1}^{j+1}}{2\phi(h)}\right)-u_i^j\left(\frac{v_{i+1}^{j+1}-v_{i-1}^{j+1}}{2\phi(h)}\right)-g_i^j=R_2. \quad (18)$$

For simplicity, we will write the system equation in the form:

$$A\sum_{k=0}^{j-1}\left(\chi_i^{j+1-k}-\chi_i^{j-1-k}\right)-A_1E_\alpha(-\mu_\alpha t^\alpha)-A\,\mu_\alpha\sum_{k=0}^{j-1}\left(\chi_i^{j+1-k}-\chi_i^{j-1-k}\right)\delta_\alpha$$
$$-A_2\sum_{h=0}^{i-1}\left(\chi_{i+1-h}^{j}-\chi_{i-1-h}^{j}\right)-A_3E_\beta(-\mu_\beta x^\beta)-A_2\,\mu_\beta\sum_{h=0}^{i-1}\left(\chi_{i+1-h}^{j}-\chi_{i-1-h}^{j}\right)\delta_\beta$$
$$-\chi_1\left(\chi_{i+1}^{j+1}-\chi_{i-1}^{j+1}\right)-F=G. \quad (19)$$

Where $X, X1, F$ and G are square block matrices. Also, if the matrix X is invertible, then by writing this system in a matrix form as follows:

$$X=\begin{pmatrix}u\\v\end{pmatrix},\ X_1=\begin{pmatrix}\frac{1}{2\phi(h)\,x_i}+\frac{v_i^j}{2\phi(h)} & 0\\ 0 & \frac{1}{2\phi(h)\,x_i}+\frac{u_i^j}{2\phi(h)}\end{pmatrix},\ F=\begin{pmatrix}f_i^j & 0\\ 0 & g_i^j\end{pmatrix},\ G=\begin{pmatrix}R_1 & 0\\ 0 & R_2\end{pmatrix},$$

$$A=\frac{M(\alpha)}{2O(k)},\quad A_1=M(\alpha)w(x),\quad A_2=\frac{M(\beta)}{2\phi(h)},\qquad A_3=M(\beta)w(t)$$

and

$$\delta_\alpha=\left[(t_j-t_k)^\alpha E_{\alpha,\alpha+1}\left(-\mu_\alpha(t_j-t_k)^\alpha\right)-(t_j-t_{k+1})^\alpha E_{\alpha,\alpha+1}\left(-\mu_\alpha(t_j-t_{k+1})^\alpha\right)\right],$$

$$\delta_\beta=\left[(x_i-x_h)^\beta E_{\beta,\beta+1}\left(-\mu_\beta(x_i-x_h)^\beta\right)-(x_i-x_{h+1})^\beta E_{\beta,\beta+1}\left(-\mu_\beta(x_i-x_{h+1})^\beta\right)\right].$$

Where R_1 and R_2 are the estimated truncation errors for the system.

4. Truncation Error

We will estimate the truncation error for the proposed numerical methods in (3.2) from the definition of truncation error given by [37],

$$R_1 = -M(\alpha)\sum_{k=0}^{j-1} O(k^2) + A\sum_{k=0}^{j-1} O(k^2)\,\mu_\alpha \int_{t_k}^{t_{k+1}} (t_j - s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t_j - s)^\alpha) ds + M(\beta)\sum_{h=0}^{i-1} \phi(h^2) - A_2 \sum_{h=0}^{i-1} \phi(h^2) \int_{x_h}^{x_{h+1}} (x_i - q)^{\beta-1} E_{\beta,\beta}(-\mu_\beta (x_i - q)^\beta) dq$$

$$= -M(\alpha)\sum_{k=0}^{j-1} O(k^2) + A\sum_{k=0}^{j-1} O(k^2)\ \mu_\alpha\, k^\alpha\left[(j-k-1)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_k + 1)^\alpha) - (j-k)^\alpha\, E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_k)^\alpha)\right] + M(\beta)\sum_{h=0}^{i-1} \phi(h^2) - A_2 \sum_{h=0}^{i-1} \phi(h^2)\, h^\beta\left[(i-h-1)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_h + 1)^\beta) - (i-h)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_h)^\beta)\right],$$

where $A = M(\alpha)\mu_\alpha$ and $A^* = M(\beta)\mu_\beta$, for the time step $m = n$ let G, C and F are constants where $t_j = jk$ and $x_i = ih$ are constants as defined,

$$R(1) = -M(\alpha)C_0 - M(\alpha)C_1 - AC_1(1)^\alpha[(j-1)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_1)^\alpha) - (j-2)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_2)^\alpha)] - (j-2)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_2)^\alpha)] - M(\alpha)C_2 - AC_2(2)^\alpha[(j-2)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_2)^\alpha) - (j-3)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_3)^\alpha)] + \cdots$$

$$-M(\alpha)(C_j - 2) - AC_j - 2(2)^\alpha\left[(2)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_j - 2)^\alpha) - (1)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_j - 1)^\alpha)\right] - M(\alpha)(C_j - 1) - AC_j - 1(1)^\alpha\left[(1)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_j - 1)^\alpha) - (0)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha (t_j - t_j)^\alpha)\right]$$

$$+M(\beta)F_0 + M(\beta)F_1 + A^*F_1(1)^\beta[(i-1)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_1)^\beta) - (i-2)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_2)^\beta)] + M(\beta)F_2 + A^*F_2(2)^\beta[(i-2)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_2)^\beta) - (i-3)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_3)^\beta)] + \cdots$$
$$+M(\beta)(F_i - 2) + A^*F_i - 2(2)^\beta\left[(2)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_i - 2)^\beta) - (1)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_i - 1)^\beta)\right]$$
$$+M(\beta)(F_i - 1) + A^*F_i - 1(1)^\beta\left[(1)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_i - 1)^\beta) - (0)^\beta E_{\beta,\beta+1}(-\mu_\beta (x_i - x_i)^\beta)\right],$$

we can get the global truncation errors as follows,

$$|R_1| \le C_k M(\alpha) O(k^2) - F_h M(\beta)\phi(h^2). \tag{20}$$

Since $|R_1| = |R_2|$ the space derivative of the first and second orders are approximated as,

$$u_x = \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\phi(h)} - (\phi(h))^2, \qquad v_x = \frac{v_{i+1}^{j+1} - v_{i-1}^{j+1}}{2\phi(h)} - (\phi(h))^2.$$

Which yields an accuracy of order

$$\left(O(k^2) + 3\,\phi(h^2)\right). \tag{21}$$

5. Stability

The stability of the schemes in (3.2) were examined using a technique of Von-Neumann method (see [38]) by considering $R_1 = R_2 = 0$ which can be written in the form,

$$\begin{aligned} {}^{MABC}D_t^\alpha\ u_{tt} - {}^{MABC}D_x^\beta\ u_{xx} - \frac{1}{x_i} u_x - v\,u_x - f_i^j &= 0, \\ {}^{MABC}D_t^\alpha\ v_{tt} - {}^{MABC}D_x^\beta\ v_{xx} - \frac{1}{x_i} v_x - u\,v_x - g_i^j &= 0, \end{aligned} \tag{22}$$

Let X, Y_1, Y_2, Y_3 and Y_4 are square block matrices. Also, the matrix X is invertible, then By writing this system in a matrix form as follows:

$$Y_1\ {}^{MABC}D_t^\alpha X_{tt} - Y_2\ {}^{MABC}D_x^\beta X_{xx} - Y_3\ X_x - Y_4 = 0, \tag{23}$$

$$X = \begin{pmatrix} u \\ v \end{pmatrix},\ Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ Y_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ Y_3 = \begin{pmatrix} \left(\frac{1}{x_i} + v\right) & 0 \\ 0 & \left(\frac{1}{x_i} + u\right) \end{pmatrix},\ Y_4 = \begin{pmatrix} f_i^j & 0 \\ 0 & g_i^j \end{pmatrix},$$

$$Y_1\ (M(\alpha)\sum_{k=0}^{j-1}\frac{\left(x_i^{j+1-k}-x_i^{j-1-k}\right)}{2O(k)} - M(\alpha)E_\alpha(-\mu_\alpha t^\alpha)w(x) - M(\alpha)\mu_\alpha\sum_{k=0}^{j-1}\frac{\left(x_i^{j+1-k}-x_i^{j-1-k}\right)}{2O(k)}$$
$$[(t_j-t_k)^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha(t_j-t_k)^\alpha)-(t_j-t_{k+1})^\alpha E_{\alpha,\alpha+1}(-\mu_\alpha(t_j-t_{k+1})^\alpha)]]-$$
$$Y_2\ (M(\beta)\sum_{h=0}^{i-1}\frac{\left(x_{i+1-h}^{j}-x_{i-1-h}^{j}\right)}{2\phi(h)} - M(\beta)E_\beta(-\mu_\beta x\wedge\beta)w(t) - M(\beta)\mu_\beta\sum_{h=0}^{i-1}\frac{\left(x_{i+1-h}^{j}-x_{i-1-h}^{j}\right)}{2\phi(h)}$$
$$[(x_i-x_h)^\beta$$
$$E_{\beta,\beta+1}(-\mu_\beta(x_i-x_h)^\beta)-(x_i-x_{h+1})^\beta E_{\beta,\beta+1}(-\mu_\beta(x_i-x_{h+1})^\beta)]]-Y_3\ \left(\frac{X_{i+1}^{j+1}-X_{i-1}^{j+1}}{2\phi(h)}\right)-\ Y_4\ =0.$$

By applying the mathematical required steps for the above system, we will get the form,

$$a\sum_{k=0}^{j-1}\frac{\left(x_i^{j+1-k}\right)}{2O(k)}-a\sum_{k=0}^{j-1}\frac{\left(x_i^{j-1-k}\right)}{2O(k)}-d\sum_{k=0}^{j-1}\frac{\left(x_i^{j+1-k}\right)}{2O(k)}+d\sum_{k=0}^{j-1}\frac{\left(x_i^{j-1-k}\right)}{2O(k)}-\mathrm{b}-\mathrm{a}^*\sum_{h=0}^{i-1}\frac{\left(x_{i+1-h}^{j}\right)}{2\phi(h)}-$$
$$a^*\sum_{h=0}^{i-1}\frac{\left(x_{i-1-h}^{j}\right)}{2\phi(h)}-d^*\sum_{h=0}^{i-1}\frac{\left(x_{i+1-h}^{j}\right)}{2\phi(h)}+d^*\sum_{h=0}^{i-1}\frac{\left(x_{i-1-h}^{j}\right)}{2\phi(h)}-b^*-H\left(X_{i+1}^{j+1}-X_{i-1}^{j+1}\right)-P=0, \tag{24}$$

where $a,b,c,d,a*,b*,c*,d*,H$ and P are constants where $k\ =\ 1,2,\ldots,j-1,\ h\ =\ 1,2,\ldots,i-1,$

$$\mathrm{a}=\ \mathrm{Y_1M}(\alpha),\ \mathrm{a}^*=\mathrm{Y_1M}(\beta),\ \mathrm{b}=\mathrm{Y_1}\ \mathrm{M}(\alpha)\mathrm{E}_\alpha(-\mu_\alpha \mathrm{t}^\alpha)\mathrm{w(x)},\ \ \mathrm{b}^*=\mathrm{Y_1}\ \mathrm{M}(\beta\ )\mathrm{E}_\beta\left(-\mu_\beta \mathrm{x}^\beta\right)\mathrm{w(t)},$$
$$\mathrm{c}\ =\left[\left(\mathrm{t_j}-\mathrm{t_k}\right)^\alpha \mathrm{E}_{(\alpha,\alpha+1)\left(-\mu_\alpha(\mathrm{t_j}-\mathrm{t_k})^\alpha\right)}-\left(\mathrm{t_j}-\mathrm{t_{k+1}}\right)^\alpha \mathrm{E}_{\alpha,\alpha+1}\left(-\mu_\alpha\left(\mathrm{t_j}-\mathrm{t_{k+1}}\right)^\alpha\right)\right],$$
$$\mathrm{c}^*\ =\ [(\mathrm{x_i}\ -\mathrm{x_h})^\beta \mathrm{E}_{\beta,\beta+1}\left(-\mu_\beta(\mathrm{x_i}-\mathrm{x_h})^\beta\right)-(\mathrm{x_i}-\mathrm{x_{h+1}}\ )^\beta \mathrm{E}_{\beta,\beta+1}\left(-\mu_\beta(\mathrm{x_i}-\mathrm{x_{h+1}})^\beta\right)],$$
$$\mathrm{d}\ =\ \mathrm{a\,c}\,\mu_\alpha,\qquad \mathrm{d}^*\ =\ \mathrm{a}^*\mathrm{c}^*\,\mu_\beta,\qquad \mathrm{H}=\frac{\mathrm{Y_3}}{2\phi(\mathrm{h})},\qquad \mathrm{P}=\mathrm{Y_4}.$$

Applying the Von-Neumann stability analysis by assuming that $\chi_i^j=\lambda^j e^{I\Upsilon ki}$ into the equations system (24) where $I=\sqrt{-1}$ as follows. Divide the deduced equation by $\lambda^j e^{I\Upsilon ki}$ and put every $\frac{\lambda^{j+1}}{\lambda^j}=\eta$. Using the Euler formulas $\left(e^{i\theta}-e^{-i\theta}\right)=2i\sin(\theta)$ and $\left(e^{i\theta}+e^{-i\theta}\right)=\cos(\theta)$ [38] and making some necessary arrangements we will have that,

$$\mathrm{a}\sum_{k=0}^{j-1}\frac{(\lambda^{\{1-k\}}-\lambda^{-(1+k)})}{2O(K)}-\mathrm{d}\sum_{k=0}^{j-1}\frac{(\lambda^{\{1-k\}}-\lambda^{-(1+k)})}{2O(K)}-\mathrm{b}-\mathrm{a}^*\sum_{h=0}^{i-1}\frac{Ie^{-I\Upsilon kh}\sin(\Upsilon k)}{\phi(h)}-\mathrm{d}^*\sum_{h=0}^{i-1}\frac{Ie^{-I\Upsilon kh}\cos(\Upsilon k)}{\phi(h)}$$
$$-\mathrm{b}^*\ -\ 2I\,\mathrm{H}\,\eta\sin(\Upsilon k)-\ \mathrm{P}\ =0,$$

Therefore, given the conditions, schemes in (3.2) are stable if,

$$|\eta|\leq\ 1.$$

$$\eta=\frac{1}{2I\ H\sin(\Upsilon k)}\left|(a-d)\sum_{k=0}^{j-1}\frac{(\lambda^{\{1-k\}}-\lambda^{-(1+k)})}{2O(K)}-b-a^{\{*\}}\sum_{h=0}^{i-1}\frac{Ie^{-I\Upsilon kh}\sin(\Upsilon k)}{\phi(h)}-d^*\sum_{h=0}^{i-1}\frac{e^{-I\Upsilon kh}\cos(\Upsilon k)}{\phi(h)}-b^{\{*\}}-P\right|\leq\ 1. \tag{25}$$

6. Numerical Discussion

In the following, NSFDM is introduced to study the fractional coupled hyperbolic system model (1), to illustrate the efficiency of MABC derivative, we investigate the following example [36]. All values of the parameters are given in tables (1)-(4) throughout this section we used $\phi(h) = 0.5\sinh\left(\frac{dx}{2}\right)$ and $O(k) = 0.001\left(1 - e^{(-dt)}\right)$ at values of $h = 0.2$ and $k = 0.2$. Figures (1) show the numerical results for Ex (6) for the NSFDM to the functions u and v at $h = k = 0.1$ using MABC derivative. Figures (2) show the error analysis for NSFDM at the same functions at h=k=0.1. Figures (4) show the numerical results for Ex (6) for the NSFDM to the functions u and v at $h = k = 0.1$ using

ABC derivative. Figures (5) show the error analysis between the numerical results and the exact solution for the same functions at $h = k = 0.1$. Figures (3), (6) show how the numerical solution for the function u and v are compatible with the exact solution. Consider the coupled system of hyperbolic partial differential equation:

Example 1. Let us consider the exact solution for the functions $u(x,t) = x^2\sin(t)$ and $v(x,t) = x^2\cos(t)$ of the non-linear coupled system of hyperbolic PDEs (1), with the initial and boundary conditions [36] as follows:

$u(x,0) = 0 \qquad v(x,0) = x^2, \quad x \in [0,1],$

$u_t(x,0) = x^2 \qquad v_t(x,0) = 0, \quad x \in [0,1],$

$u(a,t) = 0 \qquad u(l,t) = \sin(t), \quad x \in [0,1],$

$v(a,t) = 0 \qquad v(L,t) = \cos(t), \quad x \in [0,1].$

When

$$f(x,t) = -x^2\sin(t) - 2x^3\sin(t)\cos(t) - 4\sin(t),$$
$$g(x,t) = -x^2\cos(t) - 2x^3\sin(t)\cos(t) - 4\cos(t).$$

Table 1: Comparison between numerical, exact solutions and their difference in error by using NSFDM and MABC derivative for the function u at dx=dt=0.2.

at dx=dt=0.2,	Numerical (CPU = 12.620 s)	Exact	Error
x=0	0	0	0
0.2	3.1175×10^{-2}	2.86942×10^{-2}	2.4813×10^{-3}
0.4	1.2105×10^{-1}	1.14776×10^{-1}	6.2771×10^{-3}
0.6	2.6086×10^{-1}	2.58248×10^{-1}	2.6166×10^{-3}
0.8	4.5932×10^{-1}	4.59107×10^{-1}	2.1951×10^{-4}
1	7.1735×10^{-1}	7.17356×10^{-1}	0

Table 2: Comparison between numerical, exact solutions and their difference in error by using NSFDM and MABC derivative for the function v at dx=dt=0.2.

at dx=dt=0.2,	Numerical (CPU = 18.277 s)	Exact	Error
x=0	0	0	0
0.2	3.2394×10^{-2}	2.7868×10^{-2}	4.5266×10^{-3}
0.4	1.2130×10^{-1}	1.1147×10^{-1}	9.8318×10^{-3}
0.6	2.5275×10^{-1}	2.5081×10^{-1}	1.9401×10^{-3}
0.8	4.3685×10^{-1}	4.4589×10^{-1}	9.0358×10^{-3}
1	6.9670×10^{-1}	6.9670×10^{-1}	0

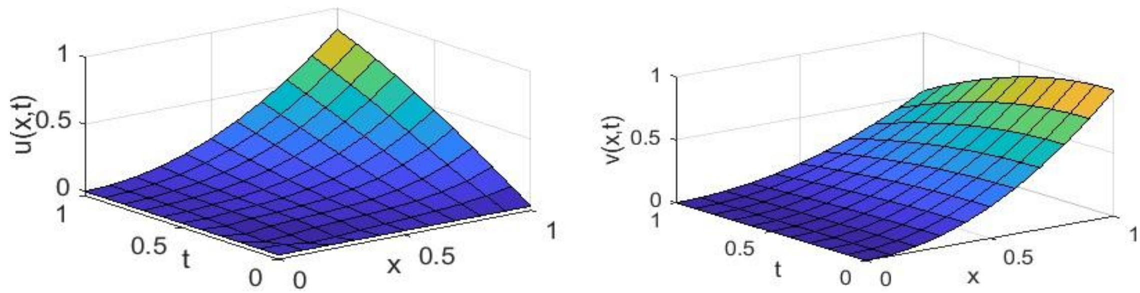

Figure 1: The numerical analysis for EX.1 using MABC derivative for the functions u and v at h=0.1 and t=0.1.

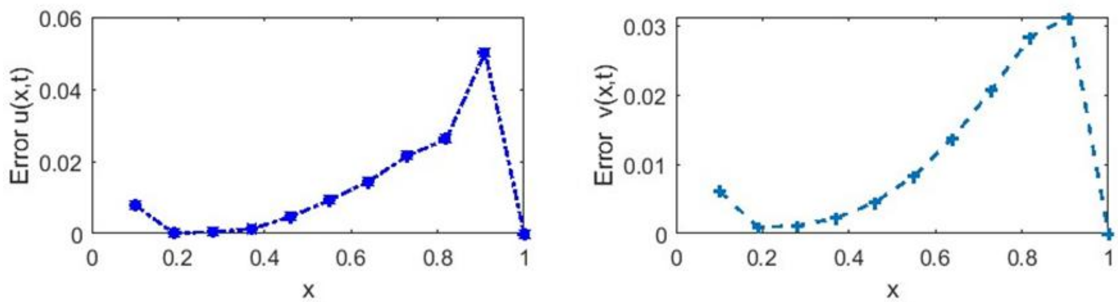

Figure 2: The error analysis for Ex.1 using MABC derivative for the functions u and v at h=0.1 and t=0.1.

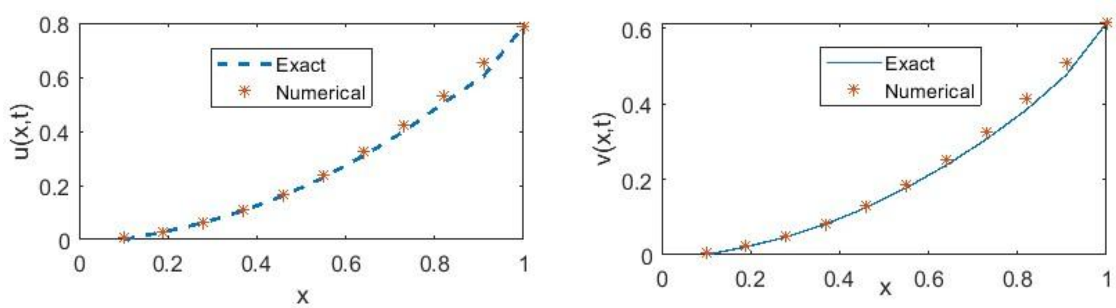

Figure 3: Numerical and exact analysis for EX.1 using MABC derivative for the functions u and v at h=0.1 and t=0.1.

Table 3: Comparison between numerical solutions, the exact solution and their difference in error by using NSFDM and ABC derivative for the function u at dx=dt=0.2.

for n=m=5,	Numerical (CPU = 9.253 s)	Exact	Error
x=0	0	0	0
0.2	2.2610×10^{-2}	1.5576×10^{-2}	7.0337×10^{-3}
0.4	7.7378×10^{-2}	6.2306×10^{-2}	1.5072×10^{-2}
0.6	1.4147×10^{-1}	1.4019×10^{-1}	1.2893×10^{-3}
0.8	2.3799×10^{-1}	2.4922×10^{-1}	1.1232×10^{-2}
1	3.8941×10^{-1}	3.8941×10^{-1}	0

Table 4: Comparison between numerical solutions, the exact solution and their difference in error by using NSFDM and ABC derivative for the function v at dx=dt=0.2.

for n=m=5,	Numerical (CPU = 15.911 s)	Exact	Error
x=0	0	0	0
0.2	6.8871×10^{-2}	3.6842×10^{-2}	3.2029×10^{-2}
0.4	2.1239×10^{-1}	1.4736×10^{-1}	6.5029×10^{-2}
0.6	3.5324×10^{-1}	3.3158×10^{-1}	2.1666×10^{-2}
0.8	5.5856×10^{-1}	5.8947×10^{-1}	3.0917×10^{-2}
1	9.2106×10^{-1}	9.2106×10^{-1}	0

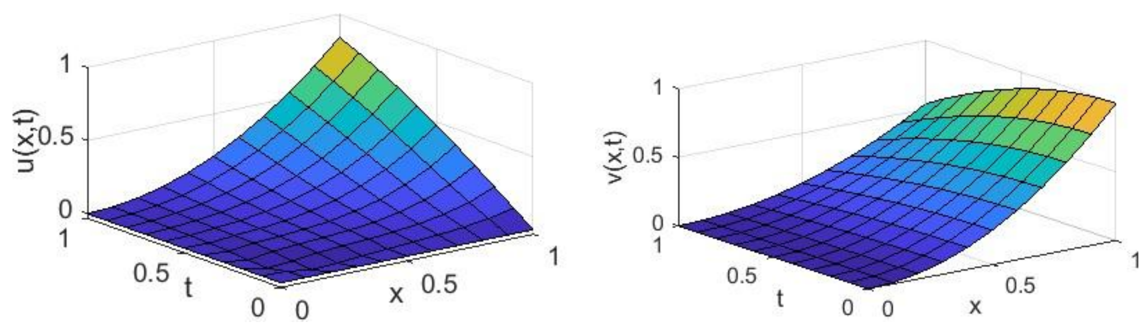

Figure 4: The numerical analysis for EX.1 using ABC derivative for the functions u and v at h=0.1 and t=0.1.

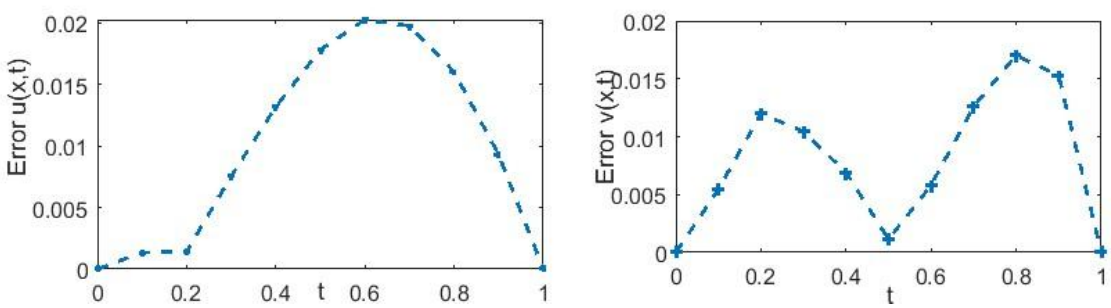

Figure 5: The error analysis for Ex.1 using ABC derivative for the functions u and v at h=0.1 and t=0.1.

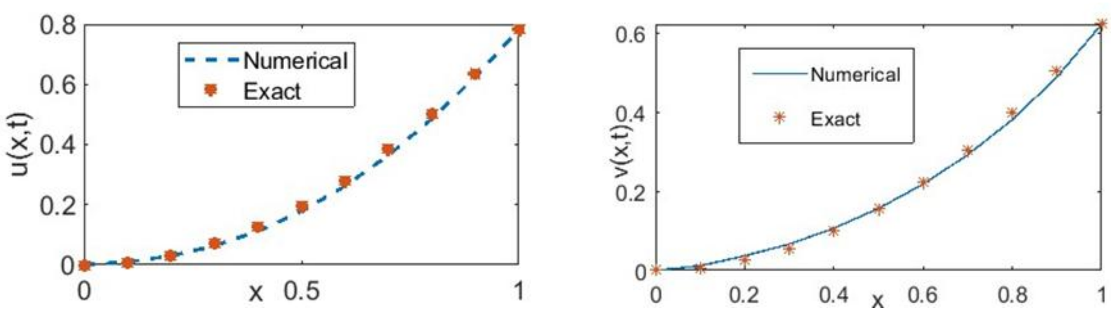

Figure 6: Numerical and exact analysis for EX.1 using ABC derivative for the functions u and v at h=0.1 and t=0.1

Conclusions

In this research, we formulated the numerical solutions for the ABC and MABC operators successfully for the suggested non-linear coupled system of hyperbolic partial differential equation. The MABC operator was utilized to fractionalize the coupled system. Theoretical analysis like the non-standard finite difference method was confirmed by numerical results, which were presented in diverse graphs and tables. The following are the key conclusions:

- Numerical results showed that the NSFD fraction method gives accurate results compared with the exact solution to the proposed problem.
- By using the John Von Neumann stability analysis approach, the constancy analysis of the referred-to variable order was put to the test.
- The results in the tables and the numerical figures display that the schemes attained from applying the submitted numerical methods are compatible with the exact solution.
- We can apply the MABC operator to an enormous number of problems defined and encountered in technology and science.
- Truncation errors were calculated.
- The results indicate by the comparison between the ABC and MABC derivatives that the proposed approach is highly accurate and very effective for these kinds of problems.

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