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On the solutions of a rational system of difference equations

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ABSTRACT

Difference equations is one of the fundamental topics of mathematics that is the used to understand the behavior of models defined in discrete time domain. Difference equations are used to approximate differential equations. The history of difference equations dates back to very old times. Although they are seen as discrete structures of differential equations, they have a much older history. In this paper, we study the admissible solutions of the system of difference equations $x_{n+1} = \frac{x_n}{y_n}$, $y_{n+1} = \frac{x_n}{ax_n+by_n}$, $n = 0,1, \ldots$, where a,b are nonnegative real number such that (a+b≠0) and the initial values x_0, y_0 are nonzero real numbers. We show that the equilibrium point (b/(1-a),1) of the abovementioned system is locally asymptotically stable when |a|<1 We show also that the equilibrium point (b/(1-a),1) is globally asymptotically stable. When |a|>1, the equilibrium point is unstable (saddle point) . and finally, it is nonhyperbolic point when |a|=1. We shall also introduce the forbidden set and provided some illustrative examples.

1. Introduction

In [13], Kudlak et al. studied the existence of unbounded solutions of the system
of difference equations

$$x_{n+1} = \frac{x_n}{y_n}, \qquad y_{n+1} = x_n + \gamma_n y_n, \qquad n = 0,1, \ldots,$$

where $0 < \gamma_n < 1$ and the initial values are positive real numbers. In [1], Camouzis et al., studied the global behavior of the system of difference equations

$$x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n} \text{ and } y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}, n = 0,1, \ldots,$$

with nonnegative parameters and positive initial conditions. They studied the boundedness character of the system (1.1) in its special cases. Although, a difference equation may be simple, the solutions of it may have complicated behaviors. In this paper, we study the global behavior of the admissible solutions of the system of difference equations

$$x_{n+1} = \frac{x_n}{y_n}, \qquad y_{n+1} = \frac{x_n}{ax_n + by_n}, \quad n = 0,1, \ldots, \tag{1}$$

where a, b are real number and the initial values x_0, y_0 are nonzero real numbers.

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For more on systems of difference equations that are solved in closed form (see [2]-[12], [14], [16]-[19]) and the references therein.

2. Linearized stability and solution of system (1)

In this section, we investigate the local asymptotic behavior of the equilibrium point of system (1) and derive its solution. We also show that the unique equilibrium point of system (1) is globally asymptotically stable.
It is clear that the unique equilibrium point of system (1) is $(\frac{b}{1-a},1)$. To study the linearized stability of the unique equilibrium, point of system (1), we consider the transformation

$$F\binom{x}{y}=\begin{pmatrix}\frac{x}{y}\\ \frac{x}{ax+by}\end{pmatrix}.$$

The linearized system associated with system (1) about an equilibrium point $(\bar{x},\bar{y})$ is

$$Z_{n+1}=J_{F(\bar{x},\bar{y})}Z_n, n=0,1,...,$$

where

$$Z_n=\binom{x_n}{y_n}, n=0,1,...,$$

and

$$J_{F(\bar{x},\bar{y})}=\begin{pmatrix}\frac{1}{\bar{y}} & -\frac{\bar{x}}{\bar{y}^2}\\ b\frac{\bar{y}^3}{\bar{x}^2} & -b\frac{\bar{y}^2}{\bar{x}}\end{pmatrix}.$$

For more on Linearized stability, see [15].

2.1. Theorem 1.

Consider system (1) and assume that $a\neq 1$. Then the equilibrium point $(\frac{b}{1-a},1)$ of system (1) is

1) locally asymptotically stable if $|a|<1$,
2) unstable (saddle point) if $|a|>1$,
3) Nonhyperbolic point if $|a|=1$.

Proof.

The Jacobian matrix $J_{F(\bar{x},\bar{y})}$ about the equilibrium point $(\frac{b}{1-a},1)$, becomes

$$J_{F(\bar{x},\bar{y})}=\begin{pmatrix}1 & -\frac{b}{1-a}\\ \frac{(1-a)^2}{b} & -(1-a)\end{pmatrix}.$$

It is sufficient to see that the eigenvalues of the matrix (2.1) are $\lambda_1=0,\lambda_2=a$ and the result follows.

Special cases

When $a=0$, system (1) becomes

$$x_{n+1}=\frac{x_n}{y_n},\qquad y_{n+1}=\frac{x_n}{by_n},\qquad n=0,1,....$$

Then

$$x_{n+1}=by_{n+1}, n=0,1,....$$

Therefore,

$$\begin{cases}x_n=b\\ y_n=1\end{cases}, \quad n=2,3,....$$

When $b=0$, system (1) becomes

$$x_{n+1}=\frac{x_n}{y_n},\qquad y_{n+1}=\frac{1}{a},\qquad n=0,1,....$$

Then

$$x_{n+1}=ax_n, n=1,2,....$$

$$\begin{cases}x_n=a^{n-1}\frac{x_0}{y_0}\\ y_n=\frac{1}{a}\end{cases}, \quad n=1,2,....$$

Now, suppose that a and b are positive real numbers. For system (1), we can write

$$u_{n+1}=au_n+b, n=0,1,..., \qquad (2)$$

where $u_n=\frac{x_n}{y_n}$, with $u_0=\frac{x_0}{y_0}$.

Solving (2), we obtain the solution for system (1).
When $a=1$, the solution to system (1) is

$$\begin{cases}x_n=\frac{x_0}{y_0}+b(n-1)\\ y_n=\frac{x_0+b(n-1)y_0}{x_0+bny_0}\end{cases}, \quad n=1,..., \qquad (3)$$

2.2. Theorem 2.

Assume that $\{(x_n,y_n)\}_{n=0}^{\infty}$ is an admissible solution for system (1). If $a\neq 1$, then the solution of system (1) is

$$\begin{cases}x_n=\frac{a^{n-1}\gamma+b}{1-a}\\ y_n=\frac{a^{n-1}\gamma+b}{a^n\gamma+b}\end{cases}, \quad n=1,..., \qquad (4)$$

where $\gamma=\frac{x_0}{y_0}(1-a)-b$.

Proof.

The proof is by induction on n.
For $n=1$, using formula (4), we get

$$x_1=\frac{\gamma+b}{1-a}=\frac{\frac{x_0}{y_0}(1-a)}{1-a}=\frac{x_0}{y_0}.$$

Also,

$$y_1=\frac{\gamma+b}{a\gamma+b}=\frac{\frac{x_0}{y_0}(1-a)}{a(\frac{x_0}{y_0}(1-a)-b)+b}=\frac{x_0}{ax_0+by_0},$$

As expected,
Now, assume that formula (4) is true for a certain $n_0\in\mathbb{N}$.
Then

$$x_{n_0+1}=\frac{x_{n_0}}{y_{n_0}}=\left(\frac{a^{n_0-1}\gamma+b}{1-a}\right)\left(\frac{a^{n_0}\gamma+b}{a^{n_0-1}\gamma+b}\right)=\frac{a^{n_0}\gamma+b}{1-a}.$$

Also,

$$y_{n_0+1} = \frac{x_{n_0}}{ax_{n_0} + by_{n_0}} = \frac{\left(\frac{a^{n_0-1}\gamma+b}{1-a}\right)}{a\left(\frac{a^{n_0-1}\gamma+b}{1-a}\right) + b\left(\frac{a^{n_0-1}\gamma+b}{a^{n_0}\gamma+b}\right)}$$

$$= \frac{\left(\frac{1}{1-a}\right)}{\frac{a}{1-a} + \frac{b}{a^{n_0}\gamma+b}} = \frac{a^{n_0}\gamma + b}{a^{n_0+1}\gamma + b}.$$

This completes the proof.

Remark

The forbidden set for system (1) when a and b are positive real numbers is

$$\mathcal{F} = \cup_{n=0}^{\infty} \{(x_0, y_0) \in \mathbb{R}^2 : y_0 = -\frac{1}{\sum_{i=0}^{n} \left(\frac{1}{a}\right)^i} x_0\}.$$

2.3. Theorem 3.

The unique equilibrium point $(\frac{b}{1-a}, 1)$ of system (1) is globally asymptotically stable.

Proof.

Using formula (4), we get

$$(x_n, y_n) \to (\frac{b}{1-a}, 1), \text{ as } n \to \infty.$$

In view of Theorem (1), the equilibrium point $(\frac{b}{1-a}, 1)$ is globally asymptotically stable.

3. Illustrative examples:

Example 1.

Consider system (1), with $a = 0.7$, $b = 0.5$ and let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1) with $x_0 = -1$ and $y_0 = 0.8$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ converges to the Equilibrium point $(\frac{b}{1-a}, 1)$ which is $(1.67,1)$ (see Fig. 1).

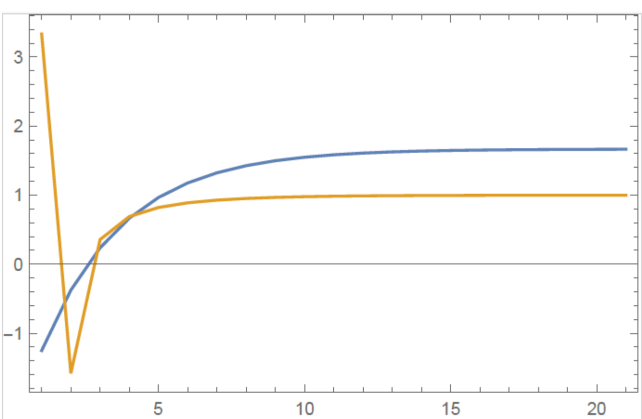

Fig. (1). $x_{n+1} = \frac{x_n}{y_n}, \quad y_{n+1} = \frac{x_n}{0.3x_n + 0.5y_n}$

Example 2.

Consider system (1), with $a = 1.7$, $b = 0.5$ and let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1) with $x_0 = 1$ and $y_0 = -0.8$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is unbounded (see Fig. 2).

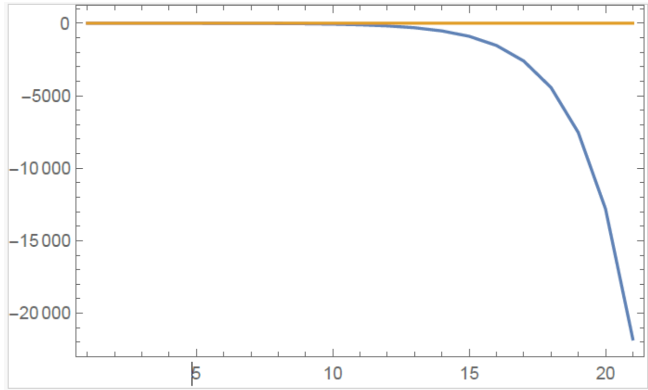

Fig. (2). $x_{n+1} = \frac{x_n}{y_n}, \quad y_{n+1} = \frac{x_n}{1.7x_n + 0.5y_n}$

Example 3.

Consider system (1), with $a = 1$, $b = 1.2$ and let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1) with $x_0 = -1.2$ and $y_0 = 2.8$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is unbounded (see Fig. 3).

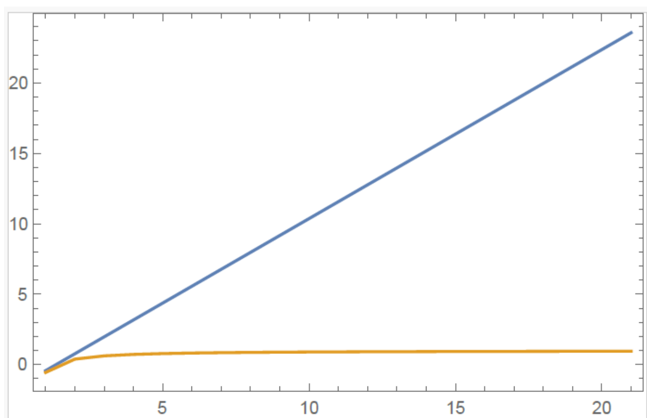

Fig. (3). $x_{n+1} = \frac{x_n}{y_n}, \quad y_{n+1} = \frac{x_n}{x_n + 1.2y_n}$

Conclusion

In this paper, we studied the admissible solutions of the system of difference equations

$$x_{n+1} = \frac{x_n}{y_n}, \qquad y_{n+1} = \frac{x_n}{ax_n + by_n}, \quad n = 0,1, \ldots,$$

where a, b are nonnegative real number such that $(a + b \neq 0)$ and the initial values x_0, y_0 are nonzero real numbers. We showed also that the equilibrium point $\left(\frac{b}{1-a}, 1\right)$ is globally asymptotically stable when $|a| < 1$.

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